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Research Article

Dynamical Analysis via Möbius Conjugacy Map on a Uniparametric Family of Optimal Fourth-Order Multiple-Zero Solvers with Rational Weight Functions

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A triparametric family of fourth-order multiple-zero solvers have been proposed. In this paper, we select among them a uniparametric family of optimal fourth-order multiple-zero solvers with rational weight functions and pursue their dynamics by exploring the relevant parameter spaces and dynamical planes, by means of Möbius conjugacy map applied to a prototype polynomial of the form $(z-A)^m(z-B)^m$. The resulting dynamics is best illustrated through various stability surfaces and parameter spaces as well as dynamical planes.

1. Introduction

The root-finding problem [1] plays a significant role in many branches of computational sciences. It arises in various fields such as physics, biochemistry, applied engineering, and earth sciences, including industrial mathematics. A considerable number of literatures [2–8] can be found describing the dynamical behavior of iterative methods to locate the multiple root of a nonlinear equation under consideration.

Many of such existing literatures have studied and emphasized local convergence behavior of the iterative root-finders for nonlinear governing equations under consideration through the viewpoints of the relevant basins of attraction. It is, however, not only worthwhile but also important to investigate the convergence behavior of the root-finders in a global sense. Dealing with such a global convergence behavior certainly motivates our current investigation via the concept of the parameter space where the relevant dynamics of the iterative root-finders under their free critical points continuously evolves as the values of parameter vary along the axes of the extended complex plane.

Homeomorphic conjugacy maps will be introduced in Section 2 in order to better understand the dynamics of the given iterative zero solvers in view of the fact that a topologically conjugated dynamical system preserves its orbit behavior as well as fixed point properties. As a convenient homeomorphic conjugacy map, we will make use of Möbius conjugacy map that enables us to effectively treat the relevant dynamics, by observing that the inverse of Möbius conjugacy map is also of Möbius-type. Indeed, Theorem 3 will show the desired resulting conjugacy map to be studied under current investigation. Additional results on the dynamical behavior including parameter spaces and dynamical planes will be shown in later sections.

To proceed with our investigation, we will employ the triparametric family of fourth-order multiple-zero methods developed by Kim and Geum [9] and introduce a uniparametric family of fourth-order multiple-zero solvers with rational weight functions as follows:

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \left(a + \frac{b}{s} + \frac{cs}{1+\lambda s}\right)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $m \in \mathbb{N}$ is a multiplicity of the sought zero, $a = (1/8)m\{8 + (2+m)(-2m(1+m) - 3m(2+m)\kappa\lambda - (2+$

$m)^2\kappa^2\lambda^2)\}$, $b = (1/8)m(2+m)^3\kappa(1+\kappa\lambda)$, $c = m(m+(2+m)\kappa\lambda)^3/8\kappa$, $s = f'(y_n)/f'(x_n)$, and $\kappa = (m/(2+m))^m$ and $\lambda \in \mathbb{C}$ is a free parameter.

The aim of this paper is to investigate the complex dynamics on the Riemann sphere by analyzing the parameter spaces associated with the free critical points and the dynamical planes related to the uniparametric family of fourth-order multiple-zero solvers. Such investigation from a viewpoint of complex dynamics may have a drawback restricting us from treating the real dynamics for real nonlinear equations. Nonetheless, our primary motivation for analyzing the relevant complex dynamics lies in seeking the dynamical behavior of a family of iterative methods (1) via Möbius conjugacy map by presenting λ-parameter spaces and the corresponding dynamical planes.

The rest of this paper is made up of three sections. In Section 2, conjugacy maps along with the property of dynamical analysis for the aforementioned numerical methods are studied and the stability surfaces of the strange fixed points for the conjugacy map are displayed. Section 3 shows the relevant complex dynamics including the parameter spaces and the dynamical planes associated with the basins of attraction. In the last section, we draw a conclusion and suggest the future study by extending the current analysis.

2. Conjugacy Maps and Dynamics

A nonlinear equation (1) can be written in a generic form as a discrete dynamical system:

$$x_{n+1} = R_f(x_n), \tag{2}$$

where R_f is the iteration function. As a result, we obtain a complex discrete dynamical system:

$$z_{n+1} = R_f(z_n) = z_n - \frac{f(z_n)}{f'(z_n)} H_f(z_n), \tag{3}$$

where $H_f(z_n) = a + b/s + c/(1/s+\lambda)$, $y_n = z_n - (2m/(m+2))(f(z_n)/f'(z_n))$, and $s = f'(y_n)/f'(z_n)$.

The following definition and remark are useful to construct the conjugacy map and to investigate the relevant dynamics.

Definition 1. Let $F : X \to X$ and $G : Y \to Y$ be two functions (dynamical systems). We say that F and G are conjugate if there is a function $h : X \to Y$ such that $h \circ F = G \circ h$. Then the map h is called a conjugacy [10].

Remark 2. Note that a conjugacy indeed preserves the dynamical behavior between the two dynamical systems; for example, if F is conjugate to G via h and ξ is a fixed point of F, then $h(\xi)$ is a fixed point of G.

Furthermore, if h is a homeomorphism, that is, if F is topologically conjugate to G via h, and ζ is a fixed point of G, then $h^{-1}(\zeta)$ is a fixed point of F. Also, we find $G = h \circ F \circ h^{-1}$ and $G^n = (h\circ F\circ h^{-1})\circ(h\circ F\circ h^{-1})\cdots\circ(h\circ F\circ h^{-1}) = h\circ F^n\circ h^{-1}$. If F and G are invertible, then the topological conjugacy h maps an orbit

$$\ldots, F^{-2}(x), F^{-1}(x), x, F(x), F^2(x), \ldots, \tag{4}$$

of F onto an orbit

$$\ldots, G^{-2}(y), G^{-1}(y), y, G(y), G^2(y), \ldots, \tag{5}$$

of G, where $y = h(x)$ and the order of points is preserved. Hence, the orbits of the two maps behave similarly under homeomorphism h or h^{-1}.

Via Möbius conjugacy map $M(z) = (z-A)/(z-B)$ with $z, A \neq B$, $A, B \in \mathbb{C} \cup \{\infty\}$ considered by Blanchard [11], R_f in (3) is conjugated to J satisfying

$$J(z; A, B, \lambda) = \frac{H(z; A, B, \lambda)}{D(z; A, B, \lambda)}. \tag{6}$$

when applied to a quadratic polynomial $f(z) = [(z-A)(z-B)]^m$ raised to the power of m, where H and D are polynomials with no common factors whose coefficients are generally dependent upon parameters A, B, and λ. The following theorem shows that J is dependent only on λ but independent of parameters A and B.

Theorem 3. *Let $f(z) = [(z-A)(z-B)]^m$ with $m \in \mathbb{N}$ and $M(z) = (z-A)/(z-B)$, $A \neq B$, $A, B \in \mathbb{C} \cup \{\infty\}$. Then $R_f(z;\lambda)$ is conjugate to $J(z;\lambda)$ satisfying*

$$\begin{aligned} J(z;\lambda) &= z \cdot \frac{r_1 + r_2\sigma(\beta_1 + z\beta_2)\tau_1 + \sigma^2(\beta_1 + z\beta_2)^2\tau_2}{r_1 z + r_2\sigma(\beta_1 + z\beta_2)\omega_1 + \sigma^2(\beta_1 + z\beta_2)^2\omega_2}, \\ r_1 &= (2+m)^{1+4m}(1+z)^{4m}\kappa^2(1+\kappa\lambda), \\ r_2 &= -2(2+m)^{2m-1}(1+z)^{2m}\kappa, \\ \sigma &= (\beta_1\beta_2)^{m-1}, \\ \beta_1 &= 2+m+mz, \\ \beta_2 &= m+(2+m)z, \\ \tau_1 &= 4(z-\kappa\lambda) + m(2+m(3+m)(1+\kappa\lambda)), \\ \tau_2 &= m^3 + (m^3 - 4m - 8z)\kappa\lambda, \\ \omega_1 &= z\tau_1 + 4(1-z^2), \\ \omega_2 &= z\tau_2 + 8\kappa\lambda(-1+z^2), \\ \kappa &= \left(\frac{m}{m+2}\right)^m. \end{aligned} \tag{7, 8}$$

Proof. Since the inverse of $M(z)$ is easily found to be $M^{-1}(z) = (Bz-A)/(z-1)$, we find after a lengthy computation with the aid of Mathematica [12] symbolic capability:

$$J(z;\lambda) = M \circ R_f \circ M^{-1}(z) = z \cdot \frac{\mathscr{H}(z;\lambda)}{\mathscr{D}(z;\lambda)}, \tag{9}$$

where $\mathscr{H}(z;\lambda) = r_1 + r_2\sigma(\beta_1 + z\beta_2)\tau_1 + \sigma^2(\beta_1 + z\beta_2)^2\tau_2$ and $\mathscr{D}(z;\lambda) = r_1 z + r_2\sigma(\beta_1 + z\beta_2)\omega_1 + \sigma^2(\beta_1 + z\beta_2)^2\omega_2$ are polynomials of degree at most $4m+1$ in z with a single free parameter $\lambda \in \mathbb{C}$. This gives the desired result, completing the proof. □

The result of Theorem 3 enables us to discover that $z = 0$ (corresponding to fixed point A of R_f or root A of $f(z) = [(z-A)(z-B)]^m$) and $z = \infty$ (corresponding to fixed point B of R_f or root B of $f(z)$) are clearly two of their fixed points of the conjugate map $J(z;\lambda)$, regardless of λ-values. Besides, by direct computation, we find that $z = 1$ is a strange fixed point [13–15] of J (that is not a root of $f(z) = [(z-A)(z-B)]^m$) due to the fact that $J(1;\lambda) = 1$, regardless of λ-values.

We now seek further strange fixed points including $z = 1$ (corresponding to the original convergence to infinity in view of the fact that $M^{-1}(1) = \infty$ or $M(\infty) = 1$). To do so, we first investigate some properties of $J(z;\lambda) = z\cdot(\mathscr{H}(z;\lambda)/\mathscr{D}(z;\lambda))$ stated in the following theorem.

Theorem 4. *Let $\mathscr{H}(z;\lambda)$ and $\mathscr{D}(z;\lambda)$ be given by (9). Then the following hold.*

(a) The leading highest-order term of $\mathscr{H}(z;\lambda)$ is given by $-8m^{3m-2}(m+2)^m z^{4m+1}\lambda$, provided that $\lambda \neq 0$.

(b) $\mathscr{H}(z;\lambda)$ has a factor z^3, provided that $\lambda \neq -m(16 + 12m + 20m^2 + 15m^3)/(-16 + 16m + 52m^2 + 50m^3 + 15m^4)\kappa$.

(c) $\mathscr{H}(1;\lambda) = \mathscr{D}(1;\lambda)$, and $\mathscr{H}(1;\lambda)/\mathscr{D}(1;\lambda) = 1$, where $\lambda \neq -(m^3(m+1)^{4m-2} + w_2(m+2)^{2m-1}(m^3+3m^2+2m+4) + w_1)/\kappa((m+1)^{4m-2}(m^3-4m-8) + w_2(m-1)(m+2)^{2m+1} + w_1)$ with $w_1 = (m+2)^{4m+1}\kappa^2$ and $w_2 = 2(m+1)^{2m-1}\kappa$.

(d) $J(z;\lambda)$ approaches ∞ as z tends to ∞, provided that $\lambda \neq -((m+2)/m)^{m-1}$.

Proof. After a lengthy computation and careful algebraic treatments with the aid of Mathematica, (a), (c) follow without difficulty. For the proof of (b), we directly compute the values of $\mathscr{H}(0;\lambda) = \mathscr{H}'(0;\lambda) = \mathscr{H}''(0;\lambda) = 0$ and $\mathscr{H}'''(0;\lambda) \neq 0$. The proof of (d) follows from the fact that $J(\infty;\lambda) = \infty$, by using (a) along with a highest-order term of $\mathscr{D}(z;\lambda)$ having degree at most $4m+1$. □

We will find out the fixed points of the iteration function $J(z;\lambda)$. Let $\phi(z;\lambda) = J(z;\lambda) - z$, whose zeros are the desired fixed points of J. From (b) and(c) of Theorem 4, we find that $z = 0$ and $z = 1$ are the roots of ϕ. Hence the expression of $\phi(z;\lambda)$ will take the following form:

$$\phi(z;\lambda) = z(z-1)\cdot\frac{\Psi(z;\lambda)}{T(z;\lambda)}, \tag{10}$$

where

$$\begin{aligned}\Psi(z;\lambda) &= -(2+m)\beta_1{}^2\beta_2{}^2\left[r_1 + (\beta_1 + z\beta_2)\sigma\{r_2\nu_1 + (\beta_1 + z\beta_2)\nu_2\sigma\}\right],\\ \nu_1 &= -4(1+\kappa\lambda) + m(2 + m(3+m)(1+\kappa\lambda)),\\ \nu_2 &= m^3 + \left(8 - 4m + m^3\right)\kappa\lambda\end{aligned} \tag{11}$$

and $\beta_1, \beta_2, r_1, r_2, \sigma, \kappa$ are given in Theorem 3.

As a result, $z = 0$, $z = 1$, and $z = \infty$ are the fixed points of J. Among these fixed points, $z = 1$ is a strange fixed point that is not root A or B. Further strange fixed points are calculated from the roots of $\Psi(z;\lambda)$. The following theorem describes some properties of $\phi(z;\lambda)$.

Theorem 5. *Let $\phi(z;\lambda)$ be given by (10). Then the following hold.*

(a) $\Psi(1/z;\lambda) = (1/z^{4(m+1)})\Psi(z;\lambda)$ for $z \neq 0$, regardless of λ-values.

(b) $\Psi(z;\lambda)$ has double roots at $z = -(m+2)/m$ and $z = -m/(m+2)$, that is, has a factor $(z+(m+2)/m)^2(z+m/(m+2))^2$, provided that $\lambda \neq -1/\kappa$ for $m \neq 1$ and $\lambda \neq -13/23$ for $m = 1$.

(c) $T(1/z;\lambda) = (1/z^{4m+5})\widetilde{T}(z;\rho)$ for $z \neq 0$, regardless of λ-values, where $\widetilde{T}(z;\lambda) = (T(z;\rho) - \delta_0)/z$, $\delta_0 = 4(m+2)(z^2-1)\beta_1^2\beta_2^2(\beta_1 + z\beta_2)\sigma\{-r_2 + 2(\beta_1 + z\beta_2)\kappa\lambda\sigma\}$.

(d) $T(z;\lambda)$ has also double roots at $z = -(m+2)/m$ and $z = -m/(m+2)$, that is, has a common factor $(z+(m+2)/m)^2(z+m/(m+2))^2$ as shown in $\Psi(z;\lambda)$, provided that $\lambda \neq -1/\kappa$ for $m \neq 1$ and $\lambda \neq -7/5$ for $m = 1$.

(e) $\phi(1/z;\lambda) = -((z-1)/z)\cdot(\Psi(z;\lambda)/\widetilde{T}(z;\lambda))$, for $z \neq 0$, regardless of λ-values.

Proof. Via careful algebraic treatments and symbolic computation with the aid of Mathematica, (a), (c), (e) follow without difficulty. For the proof of (b), we directly compute the values of $\Psi(-(m+2)/m;\lambda) = \Psi'(-(m+2)/m;\lambda) = 0$ and

$$\Psi''\left(-\frac{m+2}{m};\lambda\right) = \begin{cases} -2^{5+4m}(1+m)^2(2+m)^2\kappa^{-2}(1+\kappa\lambda) \neq 0, & \text{for } m \neq 1,\ \lambda \neq -\dfrac{1}{\kappa},\\ -18432(13+23\lambda) \neq 0, & \text{for } m = 1,\ \lambda \neq -\dfrac{13}{23}.\end{cases} \tag{12}$$

In view of the relations, $\Psi(z) = z^{4m+4}\Psi(1/z)$, $\Psi'(z) = z^{4m+2}[4(m+1)z\Psi(1/z) - \Psi'(1/z)]$, $\Psi''(z) = z^{4m}[4(4m^2 + 7m + 3)z^2\Psi(1/z) - 2(4m+3)z\Psi'(1/z) + \Psi''(1/z)]$. We find $\Psi(-m/(m+2);\lambda) = \Psi'(-m/(m+2);\lambda) = 0$ and $\Psi''(-m/(m+2);\lambda) = \kappa^4\Psi''(-(m+2)/m;\lambda) \neq 0$ for $m = 1$ with $\lambda \neq -13/23$ and $m \neq 1$ with $\lambda \neq -1/\kappa$. The proof of (d) follows from the fact that $T(-(m+2)/m;\lambda) = T'(-(m+2)/m;\lambda) = 0$ and

$$T''\left(-\frac{m+2}{m};\lambda\right)=\begin{cases}-2^{5+4m}m(1+m)^2(2+m)\kappa^2(1+\kappa\lambda)\neq 0, & \text{for } m\neq 1,\ \lambda\neq-\frac{1}{\kappa},\\ -18432(7+5\lambda)\neq 0, & \text{for } m=1,\ \lambda\neq-\frac{7}{5}.\end{cases}\tag{13}$$

We also find $T(-m/(m+2);\lambda)=T'(-m/(m+2);\lambda)=0$ and

$$T''\left(-\frac{m}{m+2};\lambda\right)=\begin{cases}-2^{5+4m}m^{-1}(1+m)^2(2+m)^3\kappa^{-2}(1+\kappa\lambda)\neq 0, & \text{for } m\neq 1,\ \lambda\neq-\frac{1}{\kappa},\\ -\frac{2048}{9}(15+29\lambda)\neq 0, & \text{for } m=1,\ \lambda\neq-\frac{15}{29}.\end{cases}\tag{14}$$

□

With the use of properties of $\phi(z;\lambda)$, we now consider some strange fixed points along with their stability for selected values of $m=1$ and $m=2$.

To continue our investigation of dynamics behind iterative map (3) applied to a quadratic polynomial raised to the power of m, $f(z)=p(z)=[(z-A)(z-B)]^m$, we will describe the fixed points of J in (9) and their stability. In view of the fact that $M(z)$ is a fixed point of J for a fixed point z of R_p with $M^{-1}(z)=(zB-A)/(z-1)$, we are interested in the explicit form of $\phi(z;\lambda)=J(z;\lambda)-z$ for $m\in\{1,2\}$ as follows:

$$\phi(z;\lambda)=\begin{cases}\dfrac{z(z-1)\Psi_1(z)}{q_1(z)}, & \text{if } m=1,\\ \dfrac{z(z-1)\Psi_2(z)}{q_2(z)}, & \text{if } m=2,\end{cases}\tag{15}$$

where we conveniently denote

$$\begin{aligned}\Psi_1(z)&=9(1+\lambda)+9z^4(1+\lambda)+3z(11+7\lambda)\\&\quad+3z^3(11+7\lambda)+z^2(54+34\lambda),\\ q_1(z)&=9(1+\lambda)+12z(2+\lambda)+z^2(21+13\lambda),\\ \Psi_2(z)&=4(2+\lambda)+4z^8(2+\lambda)+4z(17+8\lambda)\\&\quad+4z^7(17+8\lambda)+3z^4(236+97\lambda)\\&\quad+z^2(256+113\lambda)+z^6(256+113\lambda)\\&\quad+z^3(548+230\lambda)+z^5(548+230\lambda),\\ q_2(z)&=4(2+\lambda)+10z^5(12+5\lambda)+4z(15+7\lambda)\\&\quad+z^6(20+9\lambda)+6z^3(52+21\lambda)\\&\quad+z^2(188+81\lambda)+z^4(280+111\lambda).\end{aligned}\tag{16}$$

This enables us to discover that $z=0$ (corresponding to fixed point A of R_p or root A of $p(z)$) and $z=\infty$ (corresponding to fixed point B of R_p or root B of $p(z)$) are clearly two of their fixed points regardless of m. To find further strange fixed points, we solve $\phi(z;\lambda)=0$ in (15) for z with typical values of $m\in\{1,2\}$.

We now investigate further strange fixed points including $z=1$ (corresponding to the original convergence to infinity in view of the fact that $M^{-1}(1)=\infty$ or $M(\infty)=1$). By direct computation, we will describe the roots of $\phi(z;\lambda)=0$ for $m\in\{1,2\}$. To this end, we first check the existence of λ-values for common factors (divisors) of $\Psi_i(z)$ and $q_i(z)$. Besides, $q_i(z)$ will be checked if it has a divisor $(z-1)$ or z. The following theorem best describes relevant properties of such existence as well as explicit strange fixed points.

Theorem 6. *Let $m=1$ in (15). Then the following hold.*

(a) If $\lambda=-3$, then $\phi(z;\lambda)=z(z-1)(z^2+z+1)$ and the strange fixed points z are given by $z=1$ and $z=(-1\pm\sqrt{3}i)/2$.

(b) If $\lambda=-1$, then $\phi(z;\lambda)=z(z-1)(3+5z+3z^2)/(3+2z)$ and the strange fixed points are given by $z=1$ and $z=(1/6)(-5-\pm i\sqrt{11})$.

(c) If $\lambda=-27/17$, then $\phi(z;\lambda)=z(15+z+z^2+15z^4)/(15+z)$ and the strange fixed points are given by $z=\pm1$ and $z=(1/60)(-1-\sqrt{1801}\pm i\sqrt{1798-2\sqrt{1801}})$, $z=(1/60)(-1+\sqrt{1801}\pm i\sqrt{1798+2\sqrt{1801}})$.

(d) Let $\lambda\notin\{-3,-1,-27/17\}$. Then $\Psi_1(1/z)=z^{-4}\Psi_1(z)$ holds for $z\neq0$. Hence, if $z\neq0$ is a root of $\Psi_1(z;\lambda)$, then so is $1/z$.

Proof. (a)–(c) Suppose that $\Psi_1(z)=0$ and $q_1(z)=0$ for some values of z. Observe that parameter λ exists in a linear fashion in all coefficients of both polynomials. By eliminating λ from the two polynomials, we obtain the relation: $G(z)=z(z-1)(z^2+z+1)=0$. Hence, any root of G is a candidate for a common divisor of $\Psi_1(z)$ and $q_1(z)$. Substituting all the roots of G into $\Psi_1(z)=0$ and $q_1(z)=0$, we find required relations for λ and, solving them for λ, we find $\lambda=-3,-1$. The remaining part of the proof is straightforward. (d) If z is a divisor of $q_1(z)$, then $q_1(0)=9(1+\lambda)=0$ yielding $\lambda=-1$, which is already handled in (b). If $(z-1)$ is a divisor of $q_1(z)$,

then $q_1(1) = 2(27 + 17\lambda) = 0$, yielding $\lambda = -27/17$. Then remaining proof is trivial. (e) By direct substitution, we find $\Psi_1(1/z) = z^{-4}\Psi_1(z)$ without difficulty. Hence $\Psi_1(1/z) = 0$ if and only if $\Psi_1(z) = 0$ for $z \neq 0$. This completes the proof. □

Theorem 7. *Let $m = 2$ in (15). Then the following hold.*

(a) If $\lambda = -4$, then $\phi(z;\lambda) = -(-1+z)z(1+z)^3/(1+2z)$ and the strange fixed points are given by $z = 0$ and $z = \pm 1$.

(b) If $\lambda = -2$, then $\phi(z;\lambda) = -(-1+z)z(2+9z+15z^2+9z^3+2z^4)/(2+7z+7z^2+z^3)$ and the strange fixed points are given by $z = 1$, $z = -1.80198 \pm 0.880308i$, and $z = -0.448023 \pm 0.21887i$.

(c) If $\lambda = -988/409$, then $\phi(z;\lambda) = z(170+951z+1735z^2+777z^3-516z^4+777z^5+1735z^6+951z^7+170z^8)/(-170-951z-1735z^2-955z^3+258z^4+178z^5)$ and the strange fixed points are given by $z = -2.35462$, $z = -0.424696$, $z = -1.68698 \pm 0.790067i$, $z = -0.486146 \pm 0.227678i$, and $z = 0.765729 \pm 0.643163i$.

(d) Let $\lambda \notin \{-4, -2, -988/409\}$. Then $\Psi_2(1/z) = z^{-8}\Psi_2(z)$ holds for $z \neq 0$. Hence, if $z \neq 0$ is a root of $\Psi_2(z;\lambda)$, then so is $1/z$.

Proof. The proofs immediately follow from the same argument as used in the proofs of Theorem 6. □

As a result of Theorem 5(a), we find the fixed points of $J(z:\lambda)$, that is, the roots of $\phi(z;\lambda)$ explicitly as stated in the following corollary.

Corollary 8. *Let $z \notin \{0, 1\}$ be a root of $\phi(z;\lambda)$, that is, a root of $\Psi_i(z;\lambda)$ for $i = 1, 2$ in (15). Suppose $\Psi_i(z;\lambda)$ and $q_i(z)$ have no common factors for some suitable λ-values. Then the roots of $\phi(z;\lambda)$ for $1 \le i \le 2$ are explicitly given by the following.*

(a) The four roots of $\Psi_1(z;\lambda)$ are explicitly found to be

$$\begin{aligned} z_1^{(1)} &= \frac{11 + 7\lambda - \sqrt{-59 - 126\lambda - 51\lambda^2}}{6 + 6\lambda}, \\ z_2^{(1)} &= \frac{1}{z_1^{(1)}}, \\ z_3^{(1)} &= \frac{11 + 7\lambda + \sqrt{-59 - 126\lambda - 51\lambda^2}}{6 + 6\lambda}, \\ z_4^{(1)} &= \frac{1}{z_3^{(1)}}. \end{aligned} \tag{17}$$

(b) The eight roots of $\Psi_2(z;\lambda)$ are explicitly found to be

$$\begin{aligned} z_1^{(2)} &= \frac{1}{2}\left(-s_1 - \sqrt{-4 + s_1^2}\right), \\ z_2^{(2)} &= \frac{1}{z_1^{(2)}}, \\ z_3^{(2)} &= \frac{1}{2}\left(-s_2 - \sqrt{-4 + s_2^2}\right), \\ z_4^{(2)} &= \frac{1}{z_3^{(2)}}, \\ z_5^{(2)} &= \frac{1}{2}\left(-u_1 - \sqrt{-4 + u_1^2}\right), \\ z_6^{(2)} &= \frac{1}{z_5^{(2)}}, \\ z_7^{(2)} &= \frac{1}{2}\left(-u_2 - \sqrt{-4 + u_2^2}\right), \\ z_8^{(2)} &= \frac{1}{z_7^{(2)}}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} s_1 &= \frac{1}{2}\left(c_1 - \sqrt{8 + c_1^2 - 4c_2}\right), \\ s_2 &= \frac{1}{2}\left(c_1 + \sqrt{8 + c_1^2 - 4c_2}\right), \\ u_1 &= \frac{1}{2}\left(d_1 - \sqrt{8 + d_1^2 - 4d_2}\right), \\ u_2 &= \frac{1}{2}\left(d_1 + \sqrt{8 + d_1^2 - 4d_2}\right), \\ c_1 &= \frac{17 + 8\lambda + \sqrt{-7 - \lambda(6 + \lambda)}}{2(2 + \lambda)}, \\ c_2 &= \frac{3\left(9 + 4\lambda + \sqrt{-7 - \lambda(6 + \lambda)}\right)}{2(2 + \lambda)}, \\ d_1 &= \frac{17 + 8\lambda - \sqrt{-7 - \lambda(6 + \lambda)}}{2(2 + \lambda)}, \\ d_2 &= -\frac{3\left(-9 - 4\lambda + \sqrt{-7 - \lambda(6 + \lambda)}\right)}{2(2 + \lambda)}. \end{aligned} \tag{19}$$

Proof. Since z is a root of $\Psi_i(z;\lambda)$ for $1 \le i \le 2$, so is $1/z$ from the result of Theorem 5(a). For the proof of (a), thus $\Psi_1(z;\lambda)$ can be written as a product of two factors:

$$\Psi_1(z;\lambda) = \left(z^2 + az + 1\right)\left(z^2 + bz + 1\right). \tag{20}$$

By expanding the right side of the above equation and comparing the coefficients of the first and second-order terms, we find two relations:

$$\begin{aligned} a + b &= \frac{11 + 7\lambda}{3(1 + \lambda)}, \\ 2 + ab &= \frac{54 + 34\lambda}{9(1 + \lambda)}, \end{aligned} \tag{21}$$

which gives the desired values of c, d. Then the four roots can be found explicitly from $z^2 + az + 1 = 0$ or $z^2 + bz + 1 = 0$. Similarly for the proof of (b), $\Psi_2(z;\lambda)$ can be written as a product of four factors:

$$\Psi_2(z;\lambda) = \left(z^2 + s_1 z + 1\right)\left(z^2 + s_2 z + 1\right) \cdot \left(z^2 + u_1 z + 1\right)\left(z^2 + u_2 z + 1\right). \tag{22}$$

By the same argument as used in the proof of (a), the desired result follows. This completes the proof. □

We find that $J'(z;\lambda)$ can be reduced to a fraction of a common denominator as follows:

$$J'(z;\lambda) = \frac{\Gamma(z;\lambda)}{\mathscr{D}^2(z;\lambda)}, \tag{23}$$

where $\Gamma(z)$ is a polynomial of degree at most $4m+1$ defined by

$$\begin{aligned}\Gamma(z) = {} & \mathscr{D}\cdot\mathscr{H} - z\Big(2\mathscr{D}\cdot\left(4a_2\left(\theta_2+\theta_2 z+\theta_3\kappa\lambda\right)+a_1\theta_2(1+z)\tau_1 - 2m\left(\theta_1+a_2\theta_2\tau_1\right)-a_1\theta_3\tau_1\right)+\mathscr{H}\cdot\Big(\theta_1(1+z+4mz) \\ & -2\theta_2\Big(a_2 m^3(1+z)(1+\kappa\lambda) \\ & + m\left(\beta_1+z\beta_2\right)\left((1+z)\beta_1\beta_2(2+3m(1+\kappa\lambda))+\left((m-1)(1+z)\beta_2+\beta_1\left(1+m+z+mz+2\beta_2\right)\right)\omega_1\right) \\ & + 2(1+z)\beta_1\left(-2\beta_2\left(\beta_1+z\beta_2\right)\kappa\lambda-\left(\beta_1+(z-\beta_2)\beta_2\right)\omega_1\right)\Big)+\theta_3\left(a_2\left(m^3+m\left(-4+m^2\right)\kappa\lambda\right)+2a_1\omega_2\right)\Big)\Big)\end{aligned} \tag{24}$$

with $a_1 = r_2\tau_1 + 2(\beta_1 + z\beta_2)\sigma\tau_2$, $a_2 = \beta_1\beta_2(\beta_1 + z\beta_2)$, $\theta_1 = (m+2)^{4m+1}(1+z)^{4m-1}\kappa^2(1+\kappa\lambda)$, $\theta_2 = (m+2)^{2m-1}(1+z)^{2m-1}(\beta_1\beta_2)^{m-2}\kappa$ and $\theta_3 = (\beta_1\beta_2)^{2m-3}(\beta_1+z\beta_2)$; $\mathscr{D}(z;\lambda)$ and $\mathscr{H}(z;\lambda)$ are described earlier in Theorem 6.

We are now ready to determine the stability of the fixed points. In particular, it is necessary to compute the derivative of J from Theorem 19:

$$J'(z;\lambda) = \begin{cases} \dfrac{4z^3 Q_1(z)}{w_1(z)^2}, & \text{if } m = 1, \\ \dfrac{2z^3 Q_2(z)}{w_2(z)^2}, & \text{if } m = 2, \end{cases} \tag{25}$$

where

$$\begin{aligned} Q_1(z) = {} & 36z\left(18+23\lambda+7\lambda^2\right) + 36z^3\left(18+23\lambda+7\lambda^2\right) + 9\left(21+34\lambda+13\lambda^2\right) \\ & + 9z^4\left(21+34\lambda+13\lambda^2\right) + 2z^2\left(459+546\lambda+175\lambda^2\right), \\ Q_2(z) = {} & 8\left(40+38\lambda+9\lambda^2\right) + 8z^{12}\left(40+38\lambda+9\lambda^2\right) + z\left(4200+3850\lambda+878\lambda^2\right) \\ & + z^{11}\left(4200+3850\lambda+878\lambda^2\right) + z^2\left(24880+22056\lambda+4861\lambda^2\right) \\ & + z^{10}\left(24880+22056\lambda+4861\lambda^2\right) + 2z^3\left(43848+37745\lambda+8088\lambda^2\right) \\ & + 2z^9\left(43848+37745\lambda+8088\lambda^2\right) + 6z^5\left(55820+46235\lambda+9581\lambda^2\right) \\ & + 6z^7\left(55820+46235\lambda+9581\lambda^2\right) + 2z^4\left(102448+86152\lambda+18081\lambda^2\right) \\ & + 2z^8\left(102448+86152\lambda+18081\lambda^2\right) + z^6\left(393440+324176\lambda+66891\lambda^2\right), \\ w_1(z) = {} & 9(1+\lambda) + 12z(2+\lambda) + z^2(21+13\lambda), \\ w_2(z) = {} & 4(2+\lambda) + 4z(15+7\lambda) + z^2(188+81\lambda) \\ & + 6z^3(52+21\lambda) + z^4(280+111\lambda) \\ & + 10z^5(12+5\lambda) + z^6(20+9\lambda). \end{aligned} \tag{26}$$

We first check the existence of λ-values for common factors (divisors) of $Q_i(z)$ and $w_i(z)$. Besides, $w_i(z)$ will be checked if it has divisors z, z^2 and z^3. The following theorem best describes relevant properties of such existence as well as explicit strange fixed points.

Theorem 9. *Let $m = 1$ in (25). Then the following hold.*

(a) If $\lambda = -3$, then $J'(z;\lambda) = 4z^3$.

(b) If $\lambda = -1$, then $J'(z;\lambda) = 2z^2(9+22z+9z^2)/(3+2z)^2$.

(c) If $\lambda = -3/5$, then $J'(z;\lambda) = 12z^3(11z^2+34z+11)/(11z+3)^2$.

(d) If $\lambda = -1/3$, then $J'(z;\lambda) = 4z^3(3z+5)(15z^2+26z+15)/(5z+3)^3$.

(e) If $\lambda = -5/3$, then $J'(z;\lambda) = 4z^3(3z-1)(3z^2-14z+3)/(z-3)^3$.

(f) If $\lambda = -27/17$, then $J'(z;\lambda) = -12z^3(5+94z+5z^2)/(z+15)^2$.

(g) Let $\lambda \notin \{-3,-1,-3/5,-1/3,-5/3,-27/17\}$. Let z be a fixed point of $J(z;\lambda)$ satisfying $\phi(z;\lambda) = 0$. Then $J'(z;\lambda) = J'(1/z;\lambda)$ holds for $z \neq 0$.

Table 1: Stability check from $|J'(\zeta;\lambda)|$ of strange fixed points ζ for special λ-values with $1 \le m \le 2$.

m	λ	ζ $\lvert J'(\zeta;\rho)\rvert$:**t**[*]						no. of ζ
1	$-\frac{3}{5}$	-3.89179	-0.256951	$0.241037 \pm 0.970516i$				4
		20.212:**r**	20.212:**r**	3.2324:**r**				
	-3	1	$\pm\frac{1+\sqrt{3}}{2}$					3
		4:**r**	4:**r**					
	-1	1	$-0.833333 \pm 0.552771i$					3
		3.2:**r**	4.66667:**r**					
	$-\frac{1}{3}$	-1	-2.46956	-0.40493	$0.270579 \pm 0.962698i$			5
		4:**r**	7.50978:**r**	7.50978:**r**	3.24022:**r**			
	$-\frac{5}{3}$	-1	$0.89538 \pm 0.445302i$	$-0.0620469 \pm 0.998073i$				5
		5:**r**	7.45743:**r**	5.54257:**r**				
	$-\frac{27}{17}$	-1	1	$-0.0333333 \pm 0.999444i$				4
		5.14286:**r**	4.875:**r**	4.99556:**r**				
2	-2	-1	-2.23964	-0.4465	$0.0930703 \pm 0.99566i$			5
		1:**r**	9.33031:**r**	15.5842:**r**	3.44158:**r**			
	-4	-2.29663	-0.435421	$0.366025 \pm 0.930605i$				4
		10.4641:**p**	10.4641:**r**	3.5359:**p**				
	$-\frac{988}{409}$	-1	1	$\frac{1}{5}(-7 \pm 2\sqrt{6})$	-2.19404	-0.455779	$-0.0721469 \pm 0.997394i$	8
		3.04938:**r**	5.82133:**r**	49.2133:**r**	8.72642:**r**	8.72642:**r**	3.70561:**r**	

[*] $|J'(\zeta;\rho)|$:**t** implies that ζ is **a**ttractive, **p**arabolic, and **r**epulsive, if **t** = **a** ($|J'| < 1$), **t** = **p** ($|J'| = 1$), and **t** = **r** ($|J'| > 1$), respectively.

Proof. The proofs of (a)–(f) immediately follow from the same argument as used in the proofs of Theorem 19. Eliminating λ from the two polynomials $Q_i(z)$ and $w_i(z)$ plays a key role in obtaining the relation: $G(z) = (-3+z)(-1+z)z(1+z)(3+5z)(3+2z+3z^2) = 0$, whose roots enable us to deduce some desired ρ-values. Additional requirement that z, z^2, z^3 are candidates for common divisors of $Q_i(z)$ and $w_i(z)$ gives only $\lambda = -1$. For the proof of (g), we use Theorem 19 to find $J'(z;\lambda) - J'(1/z;\lambda) = -(4Q_1(z)(z-1)\Delta_1/q_1(z)q_1(1/z)^2z^8)\phi(z;\lambda) = 0$, where $\Delta_1 = 9(1+\lambda) + 9z^4(1+\lambda) + 3z(11+7\lambda) + 3z^3(11+7\lambda) + z^2(54+34\lambda)$. This completes the proof. □

Theorem 10. *Let $m = 2$ in (25). Then the following hold.*

(a) If $\lambda = -2$, then $J'(z;\lambda) = 2z^2(3+35z+112z^2+159z^3+112z^4+35z^5+3z^6)/(2+7z+7z^2+z^3)^2$.

(b) If $\lambda = -4$, then $J'(z;\lambda) = z^3(8+17z+8z^2)/(1+2z)^2$.

(c) If $\lambda = -988/409$, then $J'(z;\lambda) = z^3\sigma_1/(170+951z+1735z^2+955z^3-258z^4-178z^5)^2$, where $\sigma_1 = 121040+727134z+624992z^2-5092795z^3-17328190z^4-24412987z^5-17328190z^6-5092795z^7+624992z^8+727134z^9+121040z^{10}$.

(d) Let $\lambda \notin \{-2,-4,-988/409\}$. Let z be a fixed point of $J(z;\lambda)$ satisfying $\phi(z;\lambda) = 0$. Then $J'(z;\lambda) = J'(1/z;\lambda)$ holds for $z \neq 0$.

Proof. The proofs of (a)–(c) immediately follow from the same argument as used in the proofs of Theorem 19. (d) We use Theorem 19 to find $J'(z;\lambda) - J'(1/z;\lambda) = -(2Q_2(z)(z-1)\Delta_2/q_2(z)q_2(1/z)^2z^{16})\phi(z;\lambda) = 0$, where $\Delta_2 = 4(2+\lambda) + 4z^8(2+\lambda)+4z(17+8\lambda)+4z^7(17+8\lambda)+3z^4(236+97\lambda)+z^2(256+113\lambda)+z^6(256+113\lambda)+z^3(548+230\lambda)+z^5(548+230\lambda)$. This completes the proof. □

Table 1 summarizes the stability results for the strange fixed points ζ of J for special λ-values with $m \in \{1,2\}$.

We are ready to discuss the stability of the fixed points described in Theorems 6 and 7 in terms of parameter λ.

Theorem 11. *Let $m = 1$ and $\lambda \notin \{-3,-1,-3/5,-1/3,-5/3,-27/17\}$. Then the following hold.*

(a) The strange fixed point $z = 1$ becomes an attractor, parabolic (indifferent, neutral) point, and a repulser, respectively, when $|32(3+2\lambda)/(27+17\lambda)| < 1$, $|32(3+2\lambda)/(27+17\lambda)| = 1$, and $|32(3+2\lambda)/(27+17\lambda)| > 1$.

(b) The strange fixed point $z = 1$ is a superattractor if $\lambda = -3/2$.

Proof. (a) From the case of $m = 1$ in (25), we find $J'(1;\lambda) = 32((3+2\lambda)/(27+17\lambda))$. Solving $|J'(1;\lambda)| = 32|(3+2\lambda)/(27+17\lambda)| = 1$ for λ, we obtain circle $(x+5685/3807)^2 + y^2 = (32/1269)^2$ in the cross-sectional λ-parameter plane for $z = 1$ to be a parabolic point, where $x = \mathrm{Re}(\lambda)$ and $y = \mathrm{Im}(\lambda)$. (b) Solving $|J'(1;\lambda)| = 0$ easily yields $\lambda = -3/2$. □

Theorem 12. *Let $m = 2$ and $\lambda \notin \{-2,-4,-988/409\}$. Then the following hold.*

(a) The strange fixed point $z = 1$ is a parabolic (neutral, indifferent) point, respectively, when $|54(64+27\lambda)/(988+409\lambda)| < 1$, $|54(64+27\lambda)/(988+409\lambda)| = 1$, and $|54(64+27\lambda)/(988+409\lambda)| > 1$.

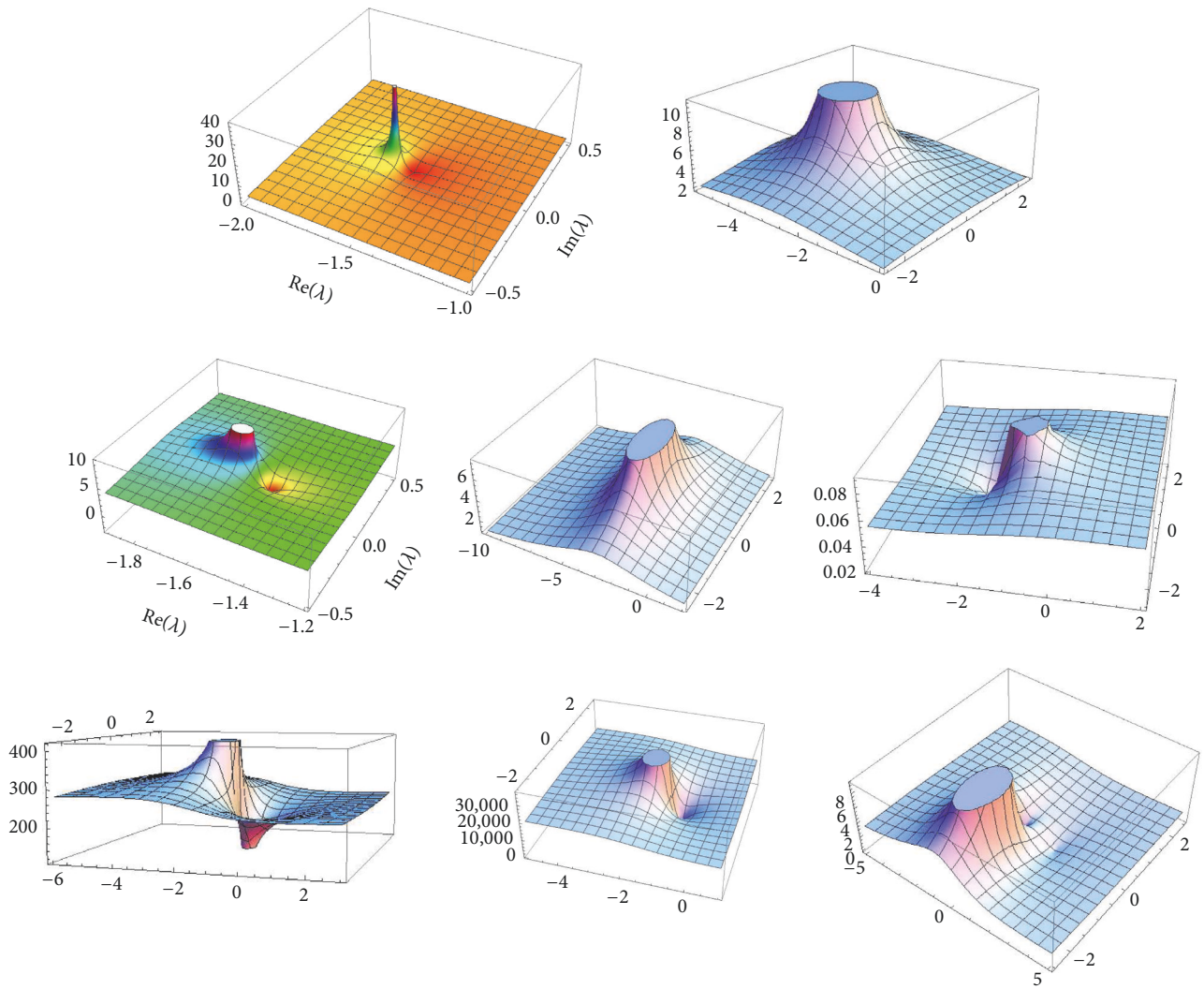


Figure 1: Stability surfaces of the strange fixed points $J(z;\rho)$ for $m = 1$.

(b) The strange fixed point $z = 1$ is a superattractor if $\lambda = -64/27$.

Proof. From the case of $m = 2$ in (25), we find $J'(1;\lambda) = 54((64 + 27\lambda)/(988 + 409\lambda))$. Solving $|J'(1;\lambda)| = 54|(64 + 27\lambda)/(988 + 409\lambda)| = 1$ for λ, we obtain an ellipse $(x + 4634756/1958483)^2/(27000/1958483)^2 + y^2/(5400\sqrt{5/898177930147})^2 = 1$ in the cross-sectional λ-parameter plane for $z = 1$ to be a parabolic point, where $x = \mathrm{Re}(\lambda)$ and $y = \mathrm{Im}(\lambda)$. (b) Solving $|J'(1;\lambda)| = 0$ easily yields $\lambda = -64/27$. □

We now proceed to discuss the stability of the strange fixed points ζ for conjugate map $J(z;\rho)$ with $m \in \{1,2\}$ using $J'(\zeta;\rho)$. As a consequence of Theorems 9(g) and 10(d) together with Corollary 8, the stability can be stated at most five strange fixed points including $z = 1$. Then the stability of these fixed points can be best described by illustrative conical surfaces shown in Figures 1-2. The top row of each figure refers to a stability surface for strange fixed point $z = 1$. The stability surfaces for the remaining fixed points z_i $(1 \le i \le 8)$ are displayed in order from top to bottom and from left to right in each case of $m = 1$ and $m = 2$. The underlying theory is clearly verified via cross-sectional views of the stability surfaces with λ-parameter domains.

The critical points of the iterative method are given by the roots of $J'(z,\lambda) = 0$. Clearly, $z = 0$ and $z = \infty$ are critical points associated with the roots a and b of the polynomial $(z-a)(z-b)$. The critical points that are not related to any roots of the polynomial $(z-a)(z-b)$ as free critical points.

Remark 13. In view of (25), the following hold.

(i) When $m = 1$ and $(1+\lambda)(3+\lambda)(1+3\lambda)(5+3\lambda)(3+5\lambda)(27+17\lambda) \neq 0$, four roots ξ of $Q_1(\xi)$ are the free critical points given by

$$\xi = \frac{1}{2}\left(-a \pm \sqrt{-4+a^2}\right), \tag{27}$$

where $a = 2(54 + 69\lambda + 21\lambda^2 \pm \sqrt{(3+\lambda)(27+105\lambda+160\lambda^2+64\lambda^3)})/3(1+\lambda)(21+13\lambda)$.

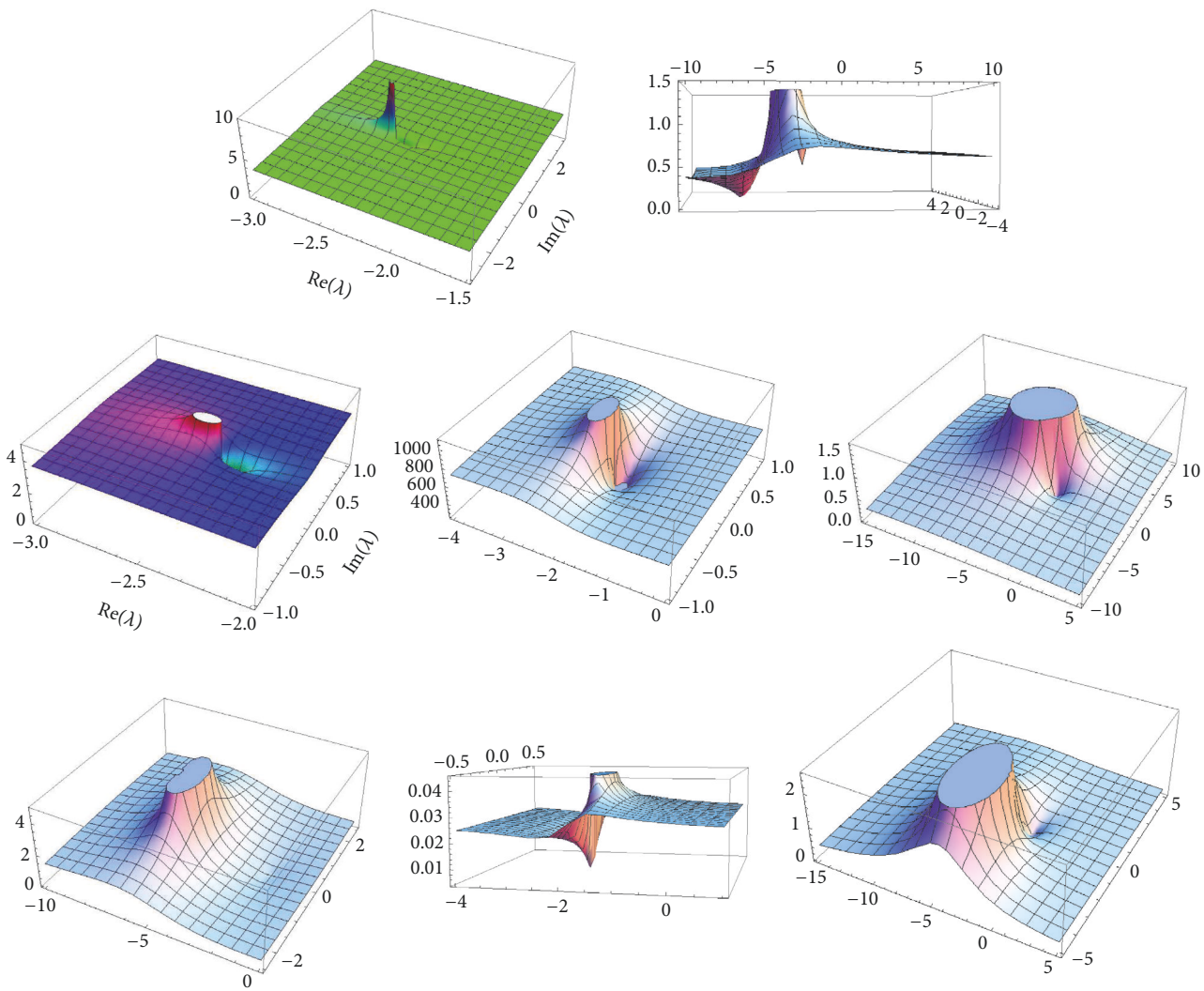


Figure 2: Stability surfaces of the strange fixed points $J(z;\rho)$ for $m = 2$.

(ii) When $m = 2$ and $(\lambda+2)(\lambda+4)(409\lambda+988)(9\lambda+20) \neq 0$, 12 roots ξ of $Q_2(\xi)$ can be found numerically for a given λ.

For Remark 13(i), we further find that since z is a root of $Q_1(z)$, so is $1/z$ from Theorem 19. Hence, there exist two constants $a_1(x)$ and $a_2(x)$ such that $Q_1(z) = 36(18 + 23\lambda + 7\lambda^2)(z^2 + a_1 z + 1)(z^2 + a_2 z + 1)$. For convenience, we define a function $\psi(t) = (-t + \sqrt{t^2 - 4})/2$. Then for λ satisfying $(1 + \lambda)(3 + \lambda)(1 + 3\lambda)(5 + 3\lambda)(3 + 5\lambda)(27 + 17\lambda)(21 + 13\lambda) \neq 0$, 4 roots $(\xi_1, \xi_2, \xi_3, \xi_4)$ of $\theta_1(\xi) = 0$ are the free critical points of $J(\xi, \lambda)$, $\xi_1 = \psi(a_1)$, $\xi_2 = \psi(a_2)$, $\xi_3 = 1/\xi_1$, $\xi_4 = 1/\xi_2$, where

$$a_1 = \frac{2\left(54 + 69\lambda + 21\lambda^2 - \sqrt{(3+\lambda)(27 + 105\lambda + 160\lambda^2 + 64\lambda^3)}\right)}{3(1+\lambda)(21+13\lambda)},$$
$$a_2 = \frac{2\left(54 + 69\lambda + 21\lambda^2 + \sqrt{(3+\lambda)(27 + 105\lambda + 160\lambda^2 + 64\lambda^3)}\right)}{3(1+\lambda)(21+13\lambda)}. \tag{28}$$

For Remark 13(ii), we also find that if z is a root of $Q_2(z)$, then so is $1/z$ from Theorem 19. Hence there exist six constants $a_1(\lambda), a_2(\lambda), \ldots, a_6(\lambda)$ such that

$$Q_2(z) = \prod_{i=1}^{6}\left(1 + a_i z + z^2\right). \tag{29}$$

With the use of function $\psi(t)$ introduced earlier, all 12 roots of $Q_2(\xi) = 0$ can be written as

$$\xi_i = \psi(a_i),$$
$$\xi_{i+6} = \frac{1}{\xi_i}, \quad 1 \le i \le 6. \tag{30}$$

Then desired among six constants $a_1, a_2, \ldots, a_6$ can be found by comparing coefficients of six terms up to order 6 with those of $Q_2(z)$ in (25); eliminating $a_1, a_2, \ldots, a_6$ from these relations, we find

$$\begin{aligned}
&32768 + 23072\lambda + 4145\lambda^2 + 8{a_1}^6\left(40 + 38\lambda + 9\lambda^2\right)\\
&\quad - 2{a_1}^5\left(2100 + 1925\lambda + 439\lambda^2\right)\\
&\quad + {a_1}^4\left(22960 + 20232\lambda + 4429\lambda^2\right)\\
&\quad - 2{a_1}^3\left(33348 + 28120\lambda + 5893\lambda^2\right)\\
&\quad - 2a_1\left(46416 + 35095\lambda + 6674\lambda^2\right)\\
&\quad + 2{a_1}^2\left(54128 + 43408\lambda + 8683\lambda^2\right) = 0,
\end{aligned} \tag{31}$$

which constitutes a sextic equation in a_1. For simplicity, we denote the left side of (31) by $Y(a_1;\lambda)$. Then the six roots $a_1, a_2, \ldots, a_6$ of $Y(a;\lambda) = 0$ are found to the desired six constants. By solving $Y(a;\lambda) = 0$ numerically for a given λ, we can determine the desired six constants $a_1, a_2, \ldots, a_6$. Having found such $a_1, a_2, \ldots, a_6$, the six critical points ξ_i can be given by $\xi_i = \psi(a_i)$ for $1 \le i \le 6$.

3. Parameter Spaces and Basins of Attraction

It is interesting to study the relevant complex dynamics from the viewpoint of parameter spaces and dynamical planes.

3.1. Parameter Spaces. It is our further interest to investigate the relevant complex dynamics from the viewpoint of parameter spaces and dynamical planes. The following lemma will be used for us to claim the favorable properties of symmetry on both parameter spaces and dynamical planes.

Lemma 14. *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by $F(\lambda, z) = P_0(z) + P_1(z)\cdot\lambda + P_2(z)\cdot\lambda^2$, where $P_i(z)$ $(0 \le i \le 2)$ are complex polynomials with real coefficients. Suppose $F(\lambda, z) = 0$. Let $\overline{z}$ denote a complex conjugate of z. Then the following hold.*

(a) $\lambda(\overline{z}) = \overline{\lambda(z)}$.

(b) If $z(\lambda)$ is a root of F, then so is $\overline{z}(\overline{\lambda})$.

Proof. (a) By directly solving $F(\lambda, z) = 0$ for λ by means of z, we find

$$\lambda = \lambda(z) = \frac{-P_1(z) \pm \sqrt{P_1(z)^2 - 4P_1(z)P_2(z)}}{2P_1(z)}. \tag{32}$$

Since $P_i(z)$ $(0 \le i \le 2)$ are complex polynomials with real coefficients, we get $P_i(\overline{z}) = \overline{P_i(z)}$.

In view of (32),

$$\begin{aligned}
\lambda(\overline{z}) &= \frac{-P_1(\overline{z}) \pm \sqrt{P_1(\overline{z})^2 - 4P_1(\overline{z})P_2(\overline{z})}}{2P_1(\overline{z})}\\
&= \frac{-\overline{P_1(z)} \pm \sqrt{\overline{P_1(z)}^2 - 4\overline{P_1(z)}\cdot\overline{P_2(z)}}}{2\overline{P_1(z)}}.
\end{aligned} \tag{33}$$

On the other hand, we also have

$$\overline{F(\lambda, z)} = 0 = \overline{P_0(z)} + \overline{P_1(z)}\cdot\overline{\lambda} + \overline{P_2(z)}\cdot\overline{\lambda}^2, \tag{34}$$

implying $\overline{\lambda(z)} = \lambda(\overline{z})$.

(b) Let $z(\lambda)$ be a root of $F(\lambda, z)$. Then

$$\begin{aligned}
F(\lambda, z) = 0 &= \overline{F(\lambda, z)}\\
&= \overline{P_0(z) + P_1(z)\cdot\lambda + P_2(z)\cdot\lambda^2}\\
&= \overline{P_0(z)} + \overline{P_1(z)}\cdot\overline{\lambda} + \overline{P_2(z)}\cdot\overline{\lambda}^2\\
&= P_0(\overline{z}) + P_1(\overline{z})\cdot\overline{\lambda} + P_2(\overline{z})\cdot\overline{\lambda}^2 = F\left(\overline{\lambda}, \overline{z}\right),
\end{aligned} \tag{35}$$

stating that $\overline{z}(\overline{\lambda})$ is also a root of F. □

Theorem 15. *Let $z(\lambda)$ be a free critical point of $J(z;\lambda)$ dependent upon parameter λ given by a root of $Q(z) = 0$ described in (15). Then the corresponding parameter space is symmetric with respect to its horizontal axis.*

Proof. It suffices to show the claim for $m = 1$. We find that such $z(\lambda)$ is a root of $F(\lambda, z) = Q_1(z) = 27(1+z)^2(7+10z+7z^2) + 6(51+138z+182z^2+138z^3+51z^4)\lambda + (3+2z+3z^2)(39+58z+39z^2)\lambda^2$ as shown in (15) for a given λ. Then $\overline{z}(\overline{\lambda})$ is also a root of Q (i.e., a free critical point of J) at $\overline{\lambda}$ from Lemma 14. For such a free critical point $z(\lambda)$, consider conjugated map $J(z;\lambda)$ from (9):

$$\begin{aligned}
&J(z;\lambda)\\
&\quad = z^4\frac{\left[3\left(7 + 8z + 3z^2\right) + \left(13 + 12z + 9z^2\right)\lambda\right]}{3\left(3 + 8z + 7z^2\right) + \left(9 + 12z + 13z^2\right)\lambda}.
\end{aligned} \tag{36}$$

This expression allows us to get

$$\begin{aligned}
|J(z;\lambda)| &= |J(z(\lambda);\lambda)| = \left|\overline{J(z(\lambda);\lambda)}\right|\\
&= \left|J\left(\overline{z(\lambda)};\overline{\lambda}\right)\right| = \left|J\left(\overline{z}\left(\overline{\lambda}\right);\overline{\lambda}\right)\right|,
\end{aligned} \tag{37}$$

which states that the magnitude of the orbit of free critical point z at λ is that same as that of the orbit of free critical point $\overline{z}$ at $\overline{\lambda}$ and hence implies that the parameter space associated with map $J(z;\lambda)$ is symmetric with respect to its horizontal axis. The proofs for $m \ge 2$ are similar. □

Theorem 16. *Given a parameter $\lambda \in \mathbb{R}$, let $z(\lambda)$ be a starting point of iterative map $J(z;\lambda)$ described in (15). Then the corresponding dynamical plane is symmetric with respect to its horizontal axis.*

Proof. We first note that $\overline{\lambda} = \lambda$ and similarly follow the proof Theorem 15. We now consider $J(z;\lambda)$:

$$|J(z;\lambda)| = \left|\overline{J(z;\lambda)}\right| = \left|J\left(\overline{z};\overline{\lambda}\right)\right| = |J(\overline{z};\lambda)|, \tag{38}$$

which states that the magnitude of the orbit of $z(\lambda)$ is the same as that of the orbit of $\overline{z(\lambda)}$ and hence implies that the dynamical plane associated with map $J(z;\lambda)$ is symmetric with respect to its horizontal axis if λ is real. □

We shall describe further properties on parameter spaces and dynamical planes associated with conjugated map $J(z;\lambda)$. By direct computation from $J(z;\lambda)$ in (9), we obtain the following lemma.

Lemma 17. *$J(z;\lambda) = 1/J(1/z;\lambda)$ for any $\lambda \in \mathbb{C}$ and $z \neq 0$.*

Corollary 18. *Let $q \in \mathbb{N}$ be given. If $z \neq 0$ is a q-periodic point of J, then so is $1/z$.*

Proof. As a result of Lemma 17 and via induction on q, we find that $J^q(z;\lambda) = 1/J^q(1/z;\lambda)$ holds for any integer $q \geq 1$ and for any $\lambda \in \mathbb{C}$ and $z \neq 0$. Hence $J^q(1/z;\lambda) = 1/J^q(z;\lambda) = 1/z$, stating that $1/z$ is also a q-periodic point of J. □

Theorem 19. *Let z be a critical point. Then the following holds.*
(i) $J'(z;\lambda) = J'(1/z;\lambda)$.
(ii) If $z \neq 0$ is a critical point, then so is $1/z$.

Proof. (i) Since $J'(1/z;\lambda) = \Gamma(z)/\mathscr{H}^2(z;\lambda)$ and z is a critical point satisfying $\Gamma(z) = 0$, we find that

$$\begin{aligned} &J'\left(\frac{1}{z};\lambda\right) - J'(z;\lambda) \\ &\quad = \Gamma(z)\left(\frac{1}{\mathscr{H}^2(z;\lambda)} - \frac{1}{\mathscr{D}^2(z;\lambda)}\right) = 0, \end{aligned} \tag{39}$$

using (23).

(ii) Hence $J'(z;\lambda) = 0$ implies $J'(1/z;\lambda) = 0$. This completes the proof. □

From (25), we first conveniently let $k = 4m - 2$ $(1 \leq m \leq 2)$ and denote the $2k$ branches of the free critical points $\xi_1, \xi_2, \ldots, \xi_k, 1/\xi_1, 1/\xi_2, \ldots, 1/\xi_k$ of $J(z;\lambda)$ by $\mathrm{cp}_1(\lambda), \mathrm{cp}_2(\lambda), \ldots, \mathrm{cp}_{2k}(\lambda)$ in order. Indeed, we find that, for $\lambda \notin \{-3, -1, -3/5, -1/3, -5/3, -27/17\}$ if $m = 1$ and for $\lambda \notin \{-2, -4, -988/409\}$ if $m = 2$,

$$\begin{aligned} \mathrm{cp}_j(\lambda) &= \psi(a_j), \quad \text{for } j = 1, 2, \ldots, k \\ \mathrm{cp}_j(\lambda) &= \frac{1}{\mathrm{cp}_{j-k}(\lambda)}, \quad \text{for } j = k+1, \ldots, 2k, \end{aligned} \tag{40}$$

where $\psi(t) = (-t - \sqrt{t^2 - 4})/2$; a_j $(1 \leq j \leq k)$ are described after Remark 13. For instance, when $m = 1$, we consider the orbit behavior of two branches $\mathrm{cp}_1(\lambda) = \xi_1$ and $\mathrm{cp}_2(\lambda) = \xi_2$ of the free critical points under the action of $J(z;\lambda)$. The orbit behavior of other two branches $\mathrm{cp}_3(\lambda) = 1/\mathrm{cp}_1(\lambda)$ and $\mathrm{cp}_4(\lambda) = 1/\mathrm{cp}_2(\lambda)$ can be similarly described and hence its investigation will be omitted here.

The following remark is useful in understanding the orbit behavior of two branches that are reciprocals of each other among the four free critical points of $J(z;\lambda)$.

Remark 20. In view of Lemma 17 and Corollary 18, we find that $J^n(\mathrm{cp}_j;\lambda) = 1/J^n(\mathrm{cp}_{j+2};\lambda)$ for $j = 1, 2$ holds for any integer $n \geq 1$ and $\lambda \notin \{-3, -1, -3/5, -1/3, -5/3, -27/17\}$. Hence, the critical orbit of one branch behaves in quite the same way as the other branch does in the following sense.

(1) If the critical orbit of one branch converges to a q-cycle with an integer $q \geq 1$, then so does the other branch.

(2) If the critical orbit of one branch is divergent but bounded, then so is the other branch.

(3) If the critical orbit of one branch converges to ∞, then the other branch converges to 0, and vice versa.

Therefore, we find that the λ-parameter space associated with one branch has same components [16] and boundaries in a neighborhood of which different anomalies occur as the other branch does. It is interesting to observe that the component associated with fixed point ∞ shares its boundary with the component associated with fixed point 0, in each branch.

The case when $m = 2$ can be similarly treated.

Without loss of generality, we limit ourselves to considering the case when $m = 1$, since similar treatment applies to the case when $m = 2$. In view of the above remark, it suffices to consider two branches cp_1 and cp_2 for their orbit behavior. Now, we are going to look for the best members of the family by means of the parameter space associated with the free critical points $\mathrm{cp}_1(\lambda)$ or $\mathrm{cp}_2(\lambda)$.

Let $\mathscr{P} = \{\lambda \in \mathbb{C}$: a critical orbit of z under $J(z;\lambda)$ tends to a number $\sigma_p \in \overline{\mathbb{C}}\}$. Similarly, we let $\mathscr{D} = \{z \in \mathbb{C}$: an orbit of z under $J(z;\lambda)$ tends to a number $\sigma_d \in \overline{\mathbb{C}}\}$. We call $\mathscr{P}$ and $\mathscr{D}$, respectively, as the parameter space and the dynamical plane (showing long-term orbit behavior) associated with $J(z;\lambda)$. If the number σ_p or σ_d is a finite constant, then there exist finite periods in the orbit. Otherwise, the orbit is nonperiodic but bounded or goes to infinity. One should find that $z = \infty$ can be treated as a fixed point in the dynamics on the Riemann sphere.

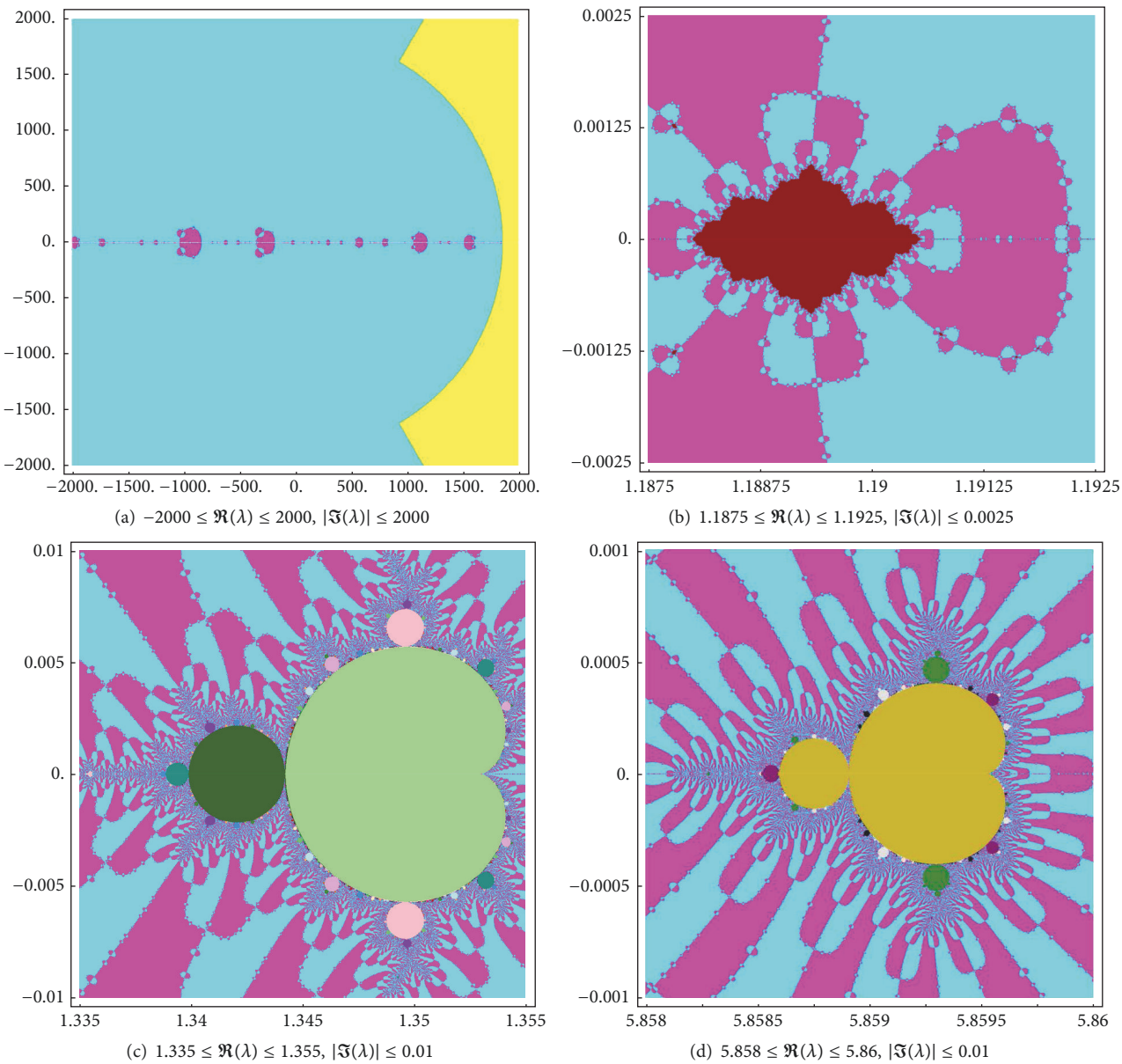
We now conveniently introduce a systematic scheme coloring a point $\lambda \in \mathscr{P}$ or point $z \in \mathscr{D}$ based on the period of the orbit of z under the action of $J(z;\lambda)$ as follows. Let t be a point in $\mathscr{P}$ or $\mathscr{D}$. Then t is painted in specified color C_q if t induces a q-periodic orbit with $q \in \mathbb{N} \cup \{0\}$ under the action of $J(z;\lambda)$. We accept the desired q-periodic convergence of an orbit associated with $\mathscr{P}$ or $\mathscr{D}$ after a maximum of 2000 iterations and with a tolerance of 10^{-6}. We further identify color C_q according to the scheme shown in Table 2.

In Figures 3-4, parameter spaces $\mathscr{P}$ associated with $\mathrm{cp}_j(\lambda)$, $(1 \leq j \leq 4)$ are shown. A point λ in $\mathscr{P}$ is painted according to the coloring scheme defined in Table 2. One should note that every point of the parameter space $\mathscr{P}$ whose color is not cyan (root $z = a$), magenta (root $z = b$), yellow, and red is not a good choice of λ in terms of relevant numerical behavior.

For convenience of analysis, we now let $\mathscr{P}_i$ denote the parameter space associated with branch cp_i for $1 \leq i \leq 4$.

Focusing the attention on the regions shown in Figures 3–14, it is evident that there exist members of the family with complicated behavior. We clearly see from Figure 11 that there exist components where q-periodic orbits are generated with $q \in \{2, 3, \ldots, 11\}$ budding from period-1 component (in red color) and 4-periodic orbits budding from period-2 component (in orange color). We further observe that q-periodic components with $q \in \{5, 7, 9, 11, \ldots\}$ can be evidently seen as indicated by arrow lines. It is interesting for us to watch these components budded along the boundary of the period-1 component in the manner of Farey sequence [17], according to the Schleicher's Algorithm [18].

We have painted the points both in the parameter space and in the dynamical plane with the same coloring scheme



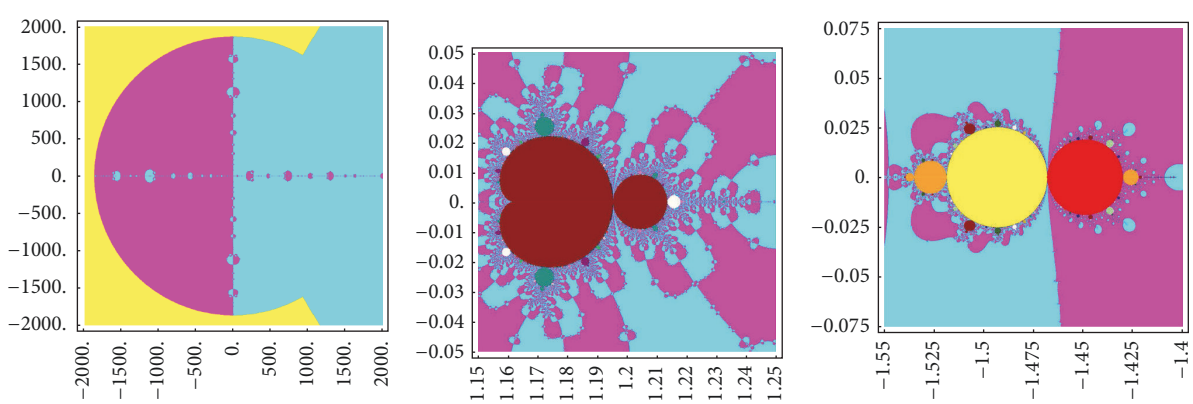
(a) $-2000 \leq \Re(\lambda) \leq 2000$, $|\Im(\lambda)| \leq 2000$

(b) $1.1875 \leq \Re(\lambda) \leq 1.1925$, $|\Im(\lambda)| \leq 0.0025$

(c) $1.335 \leq \Re(\lambda) \leq 1.355$, $|\Im(\lambda)| \leq 0.01$

(d) $5.858 \leq \Re(\lambda) \leq 5.86$, $|\Im(\lambda)| \leq 0.01$

Figure 3: Parameter spaces associated with free critical point cp_1 for $m = 1$.



(a) $-2000 \leq \Re(\lambda) \leq 2000$, $|\Im(\lambda)| \leq 2000$

(b) $1.15 \leq \Re(\lambda) \leq 1.25$, $|\Im(\lambda)| \leq 0.05$

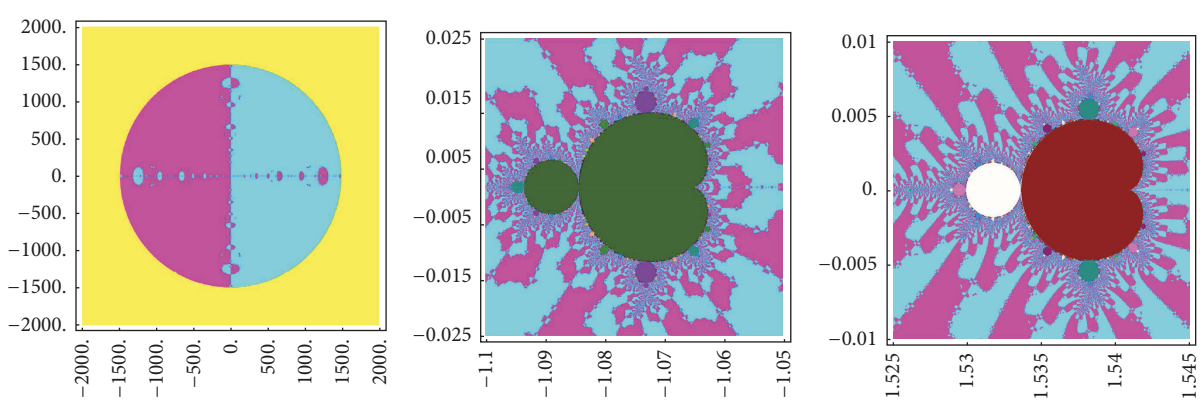
(c) $-1.55 \leq \Re(\lambda) \leq -1.4$, $|\Im(\lambda)| \leq 0.075$

Figure 4: Parameter spaces associated with free critical point cp_2 for $m = 1$.

Table 2: Stability check from $|J'(\zeta;\lambda)|$ of strange fixed points ζ for special λ-values with $1 \le m \le 2$.

q	C_q						
1	Magenta, for fixed point ∞ Cyan, for fixed point 0 Yellow, for fixed point 1 Red, for other strange fixed point						
$2 \le q \le 68$	C_2 orange	C_3 light green	C_4 dark red	C_5 dark blue	C_6 dark green	C_7 dark yellow	C_8 floral white
	C_9 light pink	C_{10} khaki	C_{11} dark orange	C_{12} turquoise	C_{13} lavender	C_{14} thistle	C_{15} plum
	C_{16} orchid	C_{17} medium orchid	C_{18} blue violet	C_{19} dark orchid	C_{20} purple	C_{21} powder blue	C_{22} sky blue
	C_{23} deep sky blue	C_{24} dodger blue	C_{25} royal blue	C_{26} medium spring green	C_{27} spring green	C_{28} medium sea green	C_{29} sea green
	C_{30} forest green	C_{31} olive drab	C_{32} bisque	C_{33} moccasin	C_{34} light salmon	C_{35} salmon	C_{36} light coral
	C_{37} Indian red	C_{38} brown	C_{39} fire brick	C_{40} peach puff	C_{41} wheat	C_{42} sandy brown	C_{43} tomato
	C_{44} orange red	C_{45} chocolate	C_{46} pink	C_{47} pale violet red	C_{48} deep pink	C_{49} violet red	C_{50} gainsboro
	C_{51} light gray	C_{52} dark gray	C_{53} gray	C_{54} charteruse	C_{55} electric indigo	C_{56} electric lime	C_{57} lime
	C_{58} silver	C_{59} teal	C_{60} pale turquoise	C_{61} sandy brown	C_{62} honeydew	C_{63} misty rose	C_{64} lemon chiffon
	C_{65} lavender blush	C_{66} gold	C_{67} crimson	C_{68} tan			
$q = 0^*$ or $q > 68$	black						

$^*q = 0$ implies that the orbit is nonperiodic but bounded.

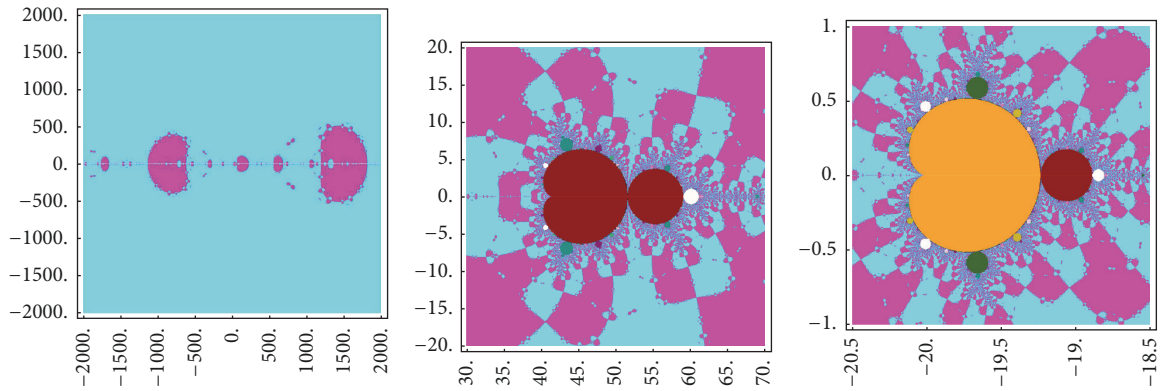


(a) $|\Re(\lambda)| \le 2000$, $|\Im(\lambda)| \le 2000$ (b) $-1.1 \le \Re(\lambda) \le -1.05$, $|\Im(\lambda)| \le 0.025$ (c) $1.525 \le \Re(\lambda) \le 1.545$, $|\Im(\lambda)| \le 0.01$

Figure 5: Parameter spaces associated with free critical point cp_1 for $m = 2$.

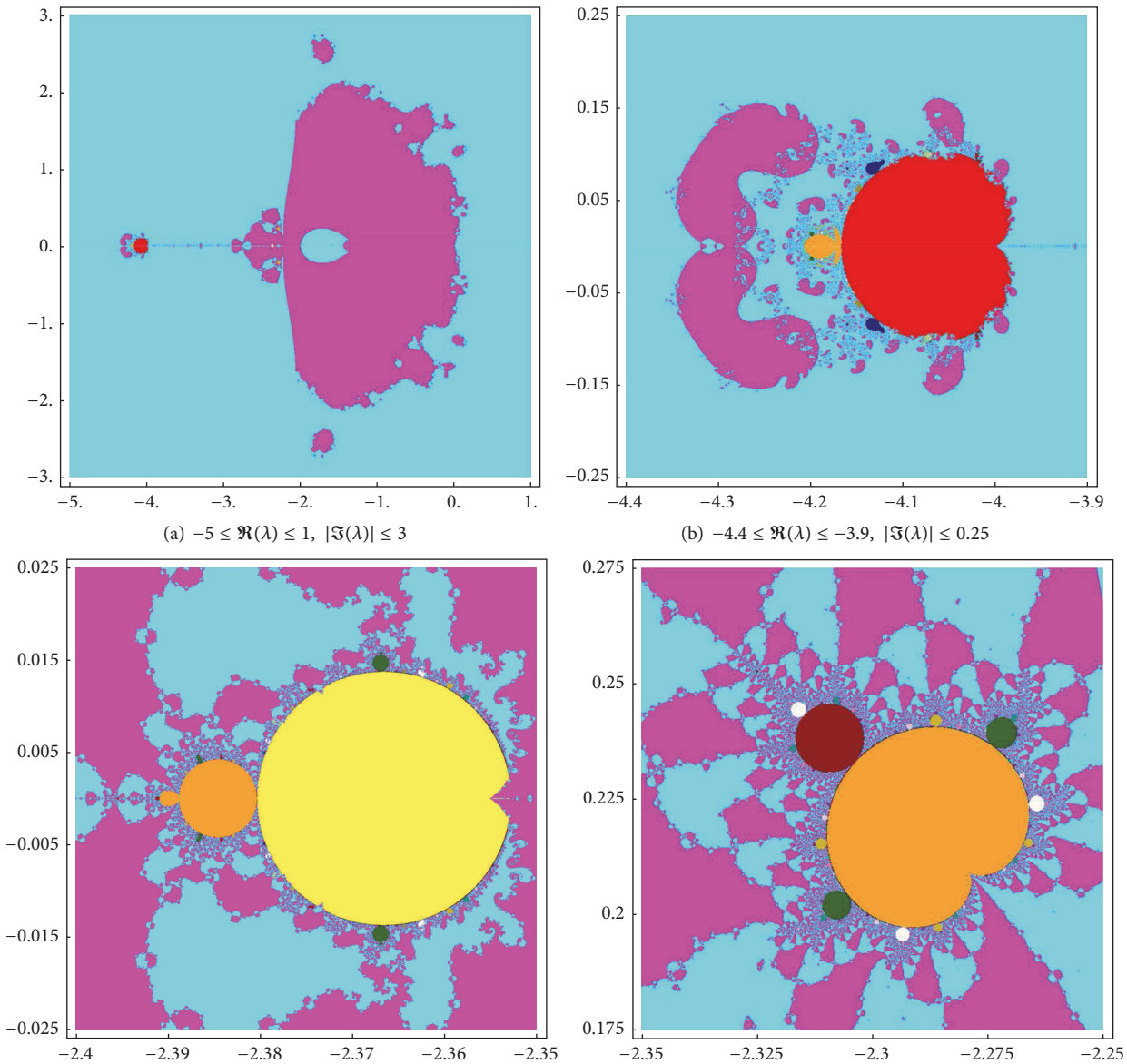
as defined in Table 2. In Figure 13, the dynamical planes of members of the family with regions of convergence to any of the strange fixed points are shown for the values of $\lambda \in \{1.8875, 1.35, 1.215, -1.525\}$, respectively. Some interesting cycles of finite period are clearly seen. In Figures 13, we observe the dynamical planes of a member of the family with regions of convergence to q-periodic cycles for $q \in \{4, 3, 8, 2\}$ in order from left to right and from top to bottom.

In summary, when $m = 1$, we have thus far observed that there exist regions of finite period for stable q-cycles with $q \ge 1$. We also observe the fascinating fractal boundaries between the basins of attraction associated with different cycles.



(a) $-2000 \le \Re(\lambda) \le 2000,\ |\Im(\lambda)| \le 2000$ (b) $30 \le \Re(\lambda) \le 70,\ |\Im(\lambda)| \le 20$ (c) $-20.5 \le \Re(\lambda) \le -18.5,\ |\Im(\lambda)| \le 1$

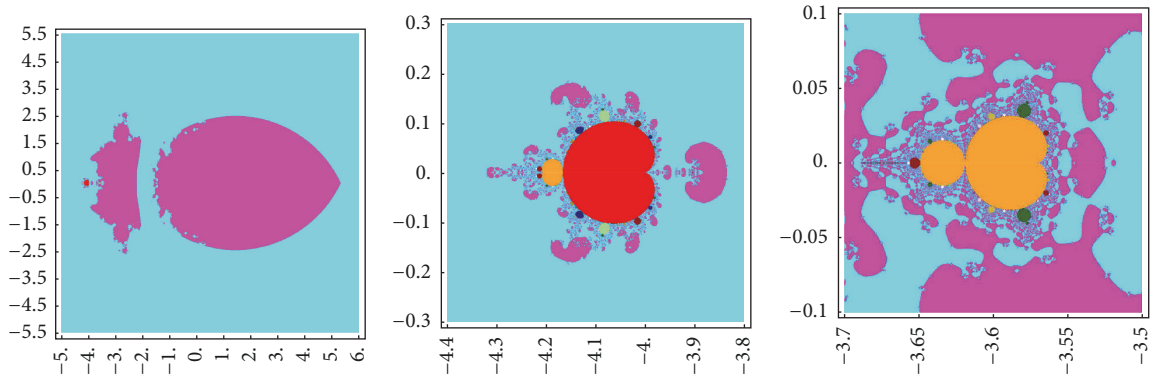
Figure 6: Parameter spaces associated with free critical point cp_2 for $m = 2$.



(a) $-5 \le \Re(\lambda) \le 1,\ |\Im(\lambda)| \le 3$ (b) $-4.4 \le \Re(\lambda) \le -3.9,\ |\Im(\lambda)| \le 0.25$

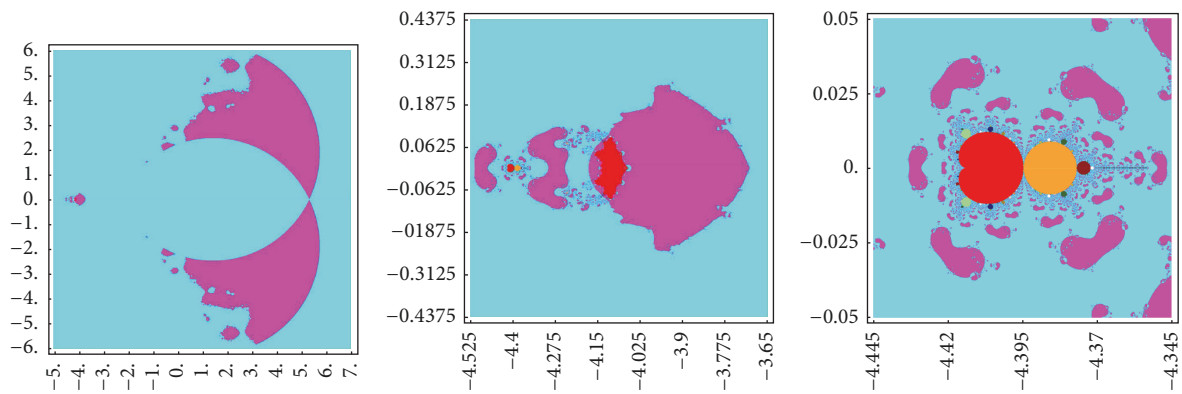
(c) $-2.4 \le \Re(\lambda) \le -2.35,\ |\Im(\lambda)| \le 0.025$ (d) $-2.35 \le \Re(\lambda) \le -2.25,\ 0.175 \le \Im(\lambda) \le 0.275$

Figure 7: Parameter spaces associated with free critical point cp_3 for $m = 2$.



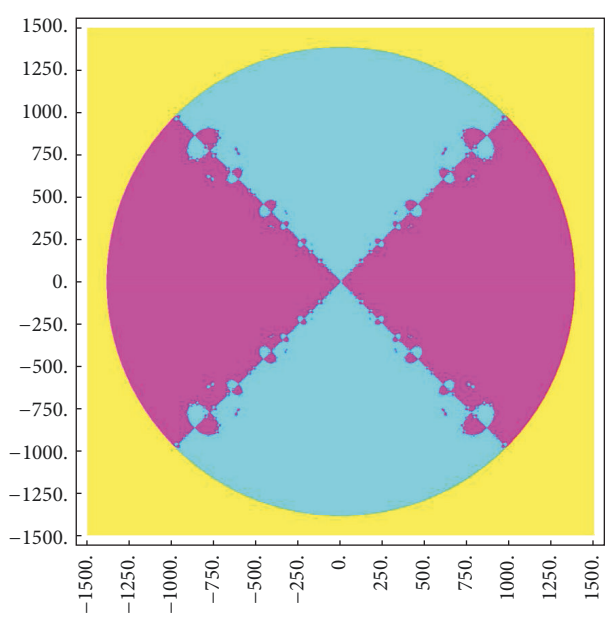
(a) $-5 \le \Re(\lambda) \le 6,\ |\Im(\lambda)| \le 5.5$ (b) $-4.4 \le \Re(\lambda) \le -3.8,\ |\Im(\lambda)| \le 0.3$ (c) $-3.7 \le \Re(\lambda) \le -3.5,\ |\Im(\lambda)| \le 0.1$

FIGURE 8: Parameter spaces associated with free critical point cp_4 for $m = 2$.



(a) $-5 \le \Re(\lambda) \le 7,\ |\Im(\lambda)| \le 6$ (b) $-4.525 \le \Re(\lambda) \le -3.65,\ |\Im(\lambda)| \le 0.4375$ (c) $-4.445 \le \Re(\lambda) \le -4.345,\ |\Im(\lambda)| \le 0.05$

FIGURE 9: Parameter spaces associated with free critical point cp_5 for $m = 2$.



$|\Re(\lambda)| \le 1500, |\Im(\lambda)| \le 1500.$

FIGURE 10: Parameter spaces associated with free critical point cp_6 for $m = 2$.

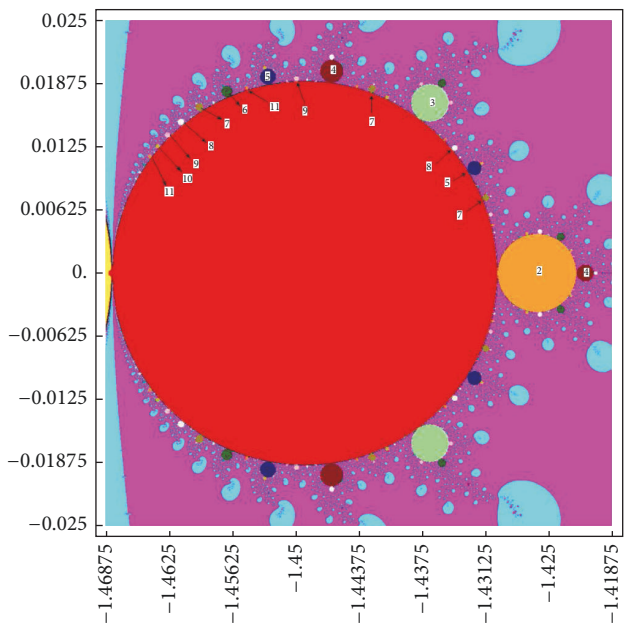


Figure 11: Farey-sequential components along the boundary of a red component for $m = 1$.

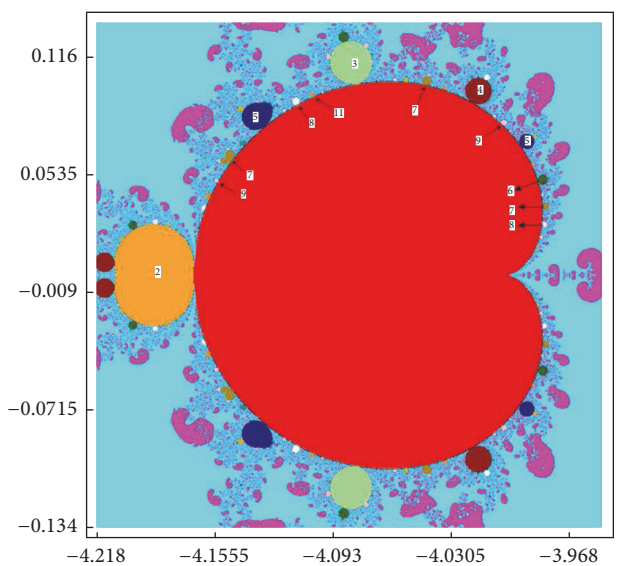


Figure 12: Farey-sequential components along the boundary of a red component for $m = 2$.

For the remaining case of $m = 2$, we have carried out similar analysis to explore the relevant dynamics described in subsequent paragraphs.

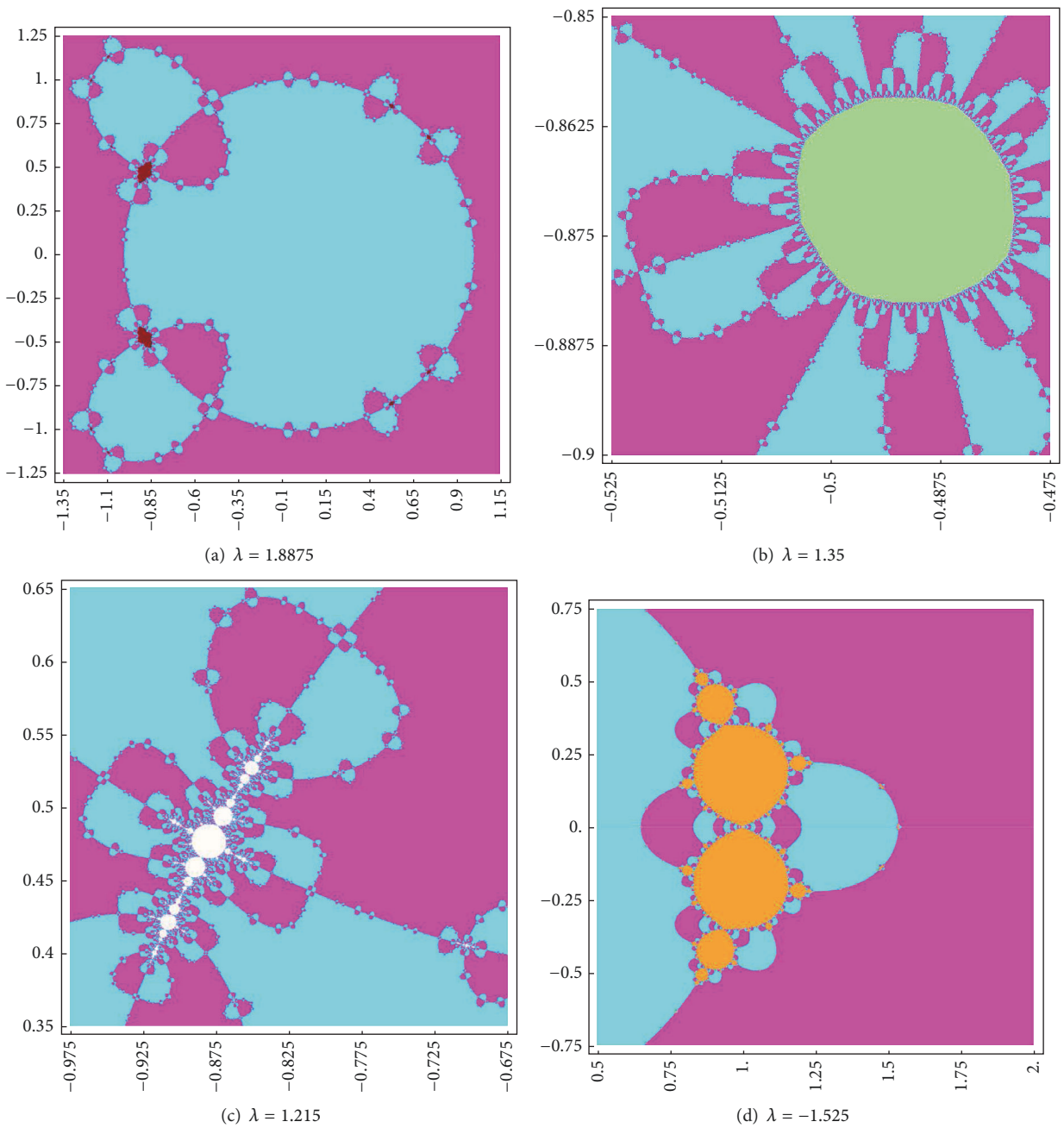
Figures 5–10 show the parameter spaces $\mathscr{P}_i$ associated with $\mathrm{cp}_i(\lambda)$, $(1 \leq i \leq 6)$ for $m = 2$. We evidently observe period-doubling components from Figures 5–9. For instance, Figures 5(c) and 6(c) exhibit components with periodic-doubling sequences $(4 \to 8 \to 16 \to 32 \to \cdots)$ and $(2 \to 4 \to 8 \to 16 \to \cdots)$, respectively.

It is interesting to note in Figure 7(c) that q-periodic components $(q \geq 1)$ arise along the boundary of a 1-periodic component in yellow where any λ-value can induce an orbit in the dynamical plane converging to a fixed point 1. The q-periodic components here are not likely to occur in the manner of Farey sequence. It is worthwhile to note that both Figures 7(d) and 8(c) contain Mandelbrot-like components. By magnifying Figure 8(b), we obtain Figure 12 leading to Farey-sequential components along the boundary of a red component for $m = 2$.

Figure 14 displays dynamical planes associated with $J(z;\lambda)$ when $m = 2$, with various values of λ inducing q-periodic cycles for $q \in \{6, 8, 16, 12, 4, 2, 1, 2, 1$ (fixed point 1), $3, 10, 5\}$ in order from left to right and from top to bottom.

4. Conclusion

Given multiplicity m, we have developed a uniparametric family of optimal fourth-order multiple-zero solvers with



(a) $\lambda = 1.8875$ (b) $\lambda = 1.35$

(c) $\lambda = 1.215$ (d) $\lambda = -1.525$

Figure 13: Dynamical planes associated with $J(z;\lambda)$ for $m = 1$.

rational weight functions and investigated their complex dynamics via Möbius conjugacy map applied to a polynomial of the form $f(z) = [(z - A)(z - B)]^m$ along with the stability analysis of strange fixed points. The relevant dynamical analysis has been carried out from the viewpoint of stability analysis and in terms of parameter spaces and dynamical planes associated with basins of attraction. In the current approach, although we have encountered technical difficulties in explicitly finding critical points from (25) when $m = 2$ in terms of parameter λ, a numerical approach developed based on Remark 13(ii) has been employed to resolve such technical difficulties. With the use of continuous dependence of zeros of a polynomial on its coefficients, that is, on λ-parameter values, it has been possible to construct numerical parameter spaces associated with twelve roots of $Q_2(z)$ in (25). Nevertheless, such continuous dependence on λ-parameter is not globally guaranteed since a numerical solution along one solution branch may trace along another nearby solution branch. This unfavorable result would be inevitable unless each solution branch is exactly found.

A future study dealing with the dynamics of other types of iterative methods will be primarily concerned with exact free critical points in order to construct favorable parameter spaces via global continuity of λ-parameter.

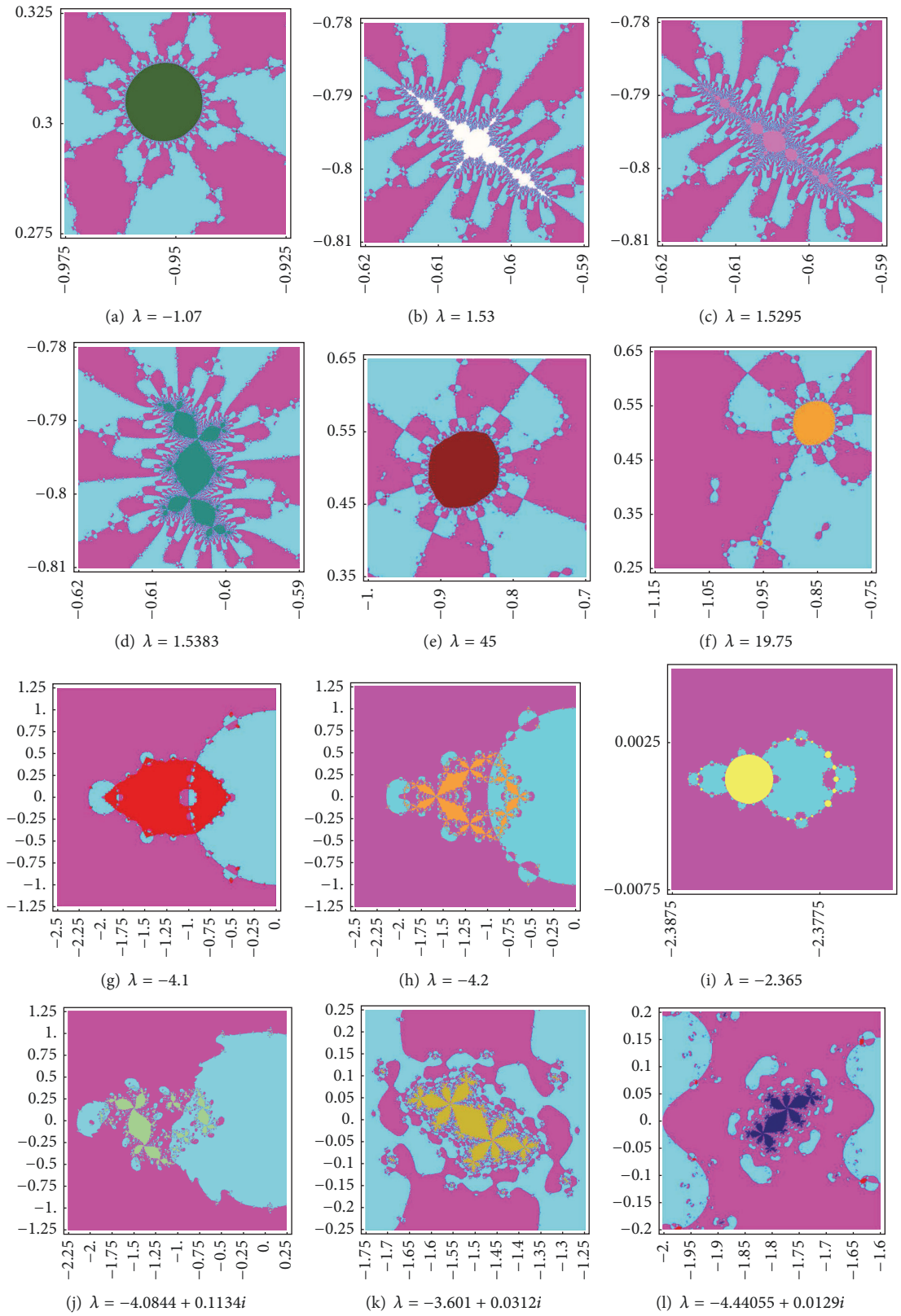


Figure 14: Dynamical planes associated with $J(z;\lambda)$ for $m = 2$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] A. M. Ostrowski, *Solution of Equations and Systems of Equations*, Pure and Applied Mathematics, Vol. IX. Academic Press, New York, NY, USA, 1960.

[2] J. A. Wright, J. H. Deane, M. Bartuccelli, and G. Gentile, "Basins of attraction in forced systems with time-varying dissipation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 29, no. 1-3, pp. 72–87, 2015.

[3] I. K. Argyros and A. A. Magreñan, "On the convergence of an optimal fourth-order family of methods and its dynamics," *Applied Mathematics and Computation*, vol. 252, pp. 336–346, 2015.

[4] F. Chicharro, A. Cordero, J. . Gutiérrez, and J. R. Torregrosa, "Complex dynamics of derivative-free methods for nonlinear equations," *Applied Mathematics and Computation*, vol. 219, no. 12, pp. 7023–7035, 2013.

[5] B. Neta and C. Chun, "Basins of attraction for several optimal fourth order methods for multiple roots," *Mathematics and Computers in Simulation*, vol. 103, pp. 39–59, 2014.

[6] S. Kumar, V. Kanwar, and S. Singh, "On some modified families of multipoint iterative methods for multiple roots of nonlinear equations," *Applied Mathematics and Computation*, vol. 218, no. 14, pp. 7382–7394, 2012.

[7] L. Shengguo, L. Xiangke, and C. Lizhi, "A new fourth-order iterative method for finding multiple roots of nonlinear equations," *Applied Mathematics and Computation*, vol. 215, no. 3, pp. 1288–1292, 2009.

[8] S. G. Li, L. Z. Cheng, and B. Neta, "Some fourth-order nonlinear solvers with closed formulae for multiple roots," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 126–135, 2010.

[9] Y. I. Kim and Y. H. Geum, "A triparametirc family of optimal fourth-order multiple-root finders and their dynamics," *Journal of Applied Mathematics*, vol. 2016, Article ID 8436759, 23 pages, 2016.

[10] A. F. Beardon, *Iteration of Rational Functions*, vol. 132 of *Graduate Texts in Mathematics*, Springer, New York, NY. USA, 1991.

[11] P. Blanchard, "The dynamics of Newton's method," in *Complex dynamical systems (Cincinnati, OH, 1994)*, vol. 49 of *Proc. Sympos. Appl. Math.*, pp. 139–154, Amer. Math. Soc., Providence, RI, 1994.

[12] S. Wolfram, *Assembly Automation*, Wolfram Media, 5th edition, 2003.

[13] R. Behl, A. Cordero, S. S. Motsa, and J. R. Torregrosa, "On developing fourth-order optimal families of methods for multiple roots and their dynamics," *Applied Mathematics and Computation*, vol. 265, pp. 520–532, 2015.

[14] A. A. Magrenan, A. Cordero, J. M. Gutierrez, and J. R. Torregrosa, "Real qualitative behavior of a fourth-order family of iterative methods by using the convergence plane," *Mathematics and Computers in Simulation*, vol. 105, pp. 49–61, 2014.

[15] A. Cordero, C. Jordan, and J. R. Torregrosa, "One-point Newton-type iterative methods: a unified point of view," *Journal of Computational and Applied Mathematics*, vol. 275, pp. 366–374, 2015.

[16] S. Lipschutz, "Theory and Problems of General Topology," *McGraw-Hill Inc*, 1965.

[17] J. Ainsworth, M. Dawson, J. Pianta, and J. Warwick, "The Farey Sequence," 2012, http://www.maths.ed.ac.uk/aar/fareyproject.pdf.

[18] R. L. Devaney, "Complex dynamical systems: the mathematics behind the Mandelbrot and Julia sets," in *Proceedings of the Symposia in Applied Mathematics*, R. L. Devaney, Ed., vol. 49, 1994.