Research Article

The Existence, Uniqueness, and Controllability of Neutral Stochastic Delay Partial Differential Equations Driven by Standard Brownian Motion and Fractional Brownian Motion

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We focus on a class of neutral stochastic delay partial differential equations perturbed by a standard Brownian motion and a fractional Brownian motion. Under some suitable assumptions, the existence, uniqueness, and controllability results for these equations are investigated by means of the Banach fixed point method. Moreover, an example is presented to illustrate our main results.

1. Introduction

Fractional Brownian motion (fBm for short) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $\{\beta^H(t), t \geq 0\}$ which is often used to model many complex phenomena in applications when the systems contain rough external forcing. When $H = 1/2$, the fBm is the standard Brownian motion which is a Markov process and martingale, and we can use the classical Itô theory to construct a stochastic integration with respect to the standard Brownian motion. However, when $H \neq 1/2$, the fBm is neither a Markov process nor a semimartingale, and the fBm behaviors become quite different. The classical techniques based on PDE cannot apply to in this context. For more details on fBm, we refer the readers to the articles [1–5].

Neutral differential equations (NDEs for short) are a family of differential equations depending on the past as well as the present state which involve derivatives with delays as well as the function itself. This family of equations has great applications in mathematical, electrical engineering, ecology, and other fields of science. In recent years, NDEs have received more and more attention; see, for example, [6–16] and the reference therein.

The observations of stock prices processes suggest that they are not self-similar. As a result, the mixed model describes the stock prices behavior in a better way. Since the pioneering paper by Cheridito [17], mixed stochastic models

containing both a standard Brownian motion and fBm gained a lot of attention. The main reason for this is that they allow us to model systems driven by white noise and fractional Gaussian noise. For more details on this topic see [18–21].

The controllability problem is one of the most attractive and important problems of differential equations because of their great significance and applications in physics, population dynamics, engineering, mathematical biology, and other areas of science. Controllability for stochastic models only containing standard Brownian motion has been investigated very well (see [7, 12, 16, 22]). Recently, controllability for stochastic systems only driven by fBm has gained a lot of attention; we refer to [6, 9, 11]. Up to now, there is no paper which considers the controllability for stochastic systems driven by standard Brownian motion and fBm.

In this paper, we study the existence, uniqueness, and controllability results for a class of neutral stochastic delay partial differential equations (NSDPDEs for short) driven by a standard Brownian motion and an fBm in the abstract form

$$\begin{aligned} d[x(t) + h(t, x(t-r(t)))] \\ = [Ax(t) + f(t, x(t-\rho(t))) + Bu(t)] dt \\ + g(t, x(t-\eta(t))) dW(t) + \sigma(t) dB_Q^H(t), \quad (1) \\ t \geq 0, \\ x(s) = \varphi(s) \in \mathcal{C}([- \tau, 0]; \mathcal{L}^2(\Omega; X)), \quad -\tau \leq s \leq 0, \end{aligned}$$

where A is the infinitesimal generator of an analytic semi-group $\{S(t), t \geq 0\}$ of bounded linear operators in a real separable Hilbert space X . The delay functions $r, \rho, \eta : [0, +\infty) \rightarrow [0, \tau]$ ($\tau \geq 0$) are continuous, and $f, h : [0, +\infty) \times X \rightarrow X, g : [0, +\infty) \times X \rightarrow \mathcal{L}_{Q^{(1)}}^0(Y_1; X)$, and $\sigma : [0, +\infty) \rightarrow \mathcal{L}_{Q^{(2)}}^0(Y_2; X)$ are Borel measurable satisfying appropriate conditions. The control mapping $u(\cdot)$ takes values in $\mathcal{L}^2([0, T]; U)$, and the Hilbert space of admissible control mappings for a real separable Hilbert U , B is a bounded linear operator from U to X . $\{W(t), t \geq 0\}$ denotes a standard Brownian motion in a real separable Hilbert space Y_1 . $\{B_Q^H(t), t \geq 0\}$ denotes a fractional Brownian motion in a real separable Hilbert space Y_2 with Hurst parameter $H \in (1/2, 1)$. Let $\mathcal{L}_{Q^{(i)}}^0(Y_i; X)$ ($i = 1, 2$) be the space of all $Q^{(i)}$ -Hilbert-Schmidt operators from Y_i into X .

Recently, Boufoussi and Hajji in [8] considered the existence and uniqueness problems of a class of neutral stochastic delay differential equations; that is, $B \equiv 0$ and $g \equiv 0$ in (1) by means of the Banach fixed point theory. Very recently, Liu and Luo in [13] studied the existence and uniqueness problems of a wide class of neutral stochastic delay partial differential equations, that is, $B \equiv 0$ in (1) by means of the Banach fixed point theory. However, they all required $1/2 < \beta < 1$ which is a stronger condition than ours (see Section 3).

In this paper, based on the above papers, we also use the Banach fixed point theory to consider a class of neutral stochastic delay partial differential equations, which is more general than ones in [8, 13]. Some conditions on the existence, uniqueness, and controllability of (1) are obtained. The related known results in Boufoussi and Hajji [8] and Liu and Luo in [13] are improved and generalized.

The contents of this paper are as follows. In Section 2, some notions, definitions, and lemmas which will be needed throughout this paper are introduced. In Section 3, the existence and uniqueness of mild solutions of (1) are proved. In Section 4, the controllability of (1) is investigated by means of the Banach fixed point theory. In Section 5, an example is given to illustrate our main results. At last, in Section 6, our conclusion is presented.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying standard assumptions; that is, the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets. Let $\mathcal{C}([-\tau, T]; \mathcal{L}^2(\Omega; X))$ be the family of all continuous functions from $[-\tau, T]$ into $\mathcal{L}^2(\Omega; X)$. Let X and Y_i , $i = 1, 2$, be three real separable Hilbert spaces. We denote by $\mathcal{L}(Y_i; X)$ the space of all bounded linear operators from Y_i to X , $i = 1, 2$. We suppose that $\{e_n^{(i)}\}_{n \in \mathbb{N}^+}$ is a complete orthonormal basis in Y_i . Let $Q^{(i)} \in \mathcal{L}(Y_i; X)$ be an operator defined by $Q^{(i)}e_n^{(i)} = \lambda_n^{(i)}e_n^{(i)}$ with finite trace $\text{tr}Q^{(i)} = \sum_{n=1}^{\infty} \lambda_n^{(i)} < \infty$, where $\{\lambda_n^{(i)}\}_{n \in \mathbb{N}^+}$ are nonnegative real numbers. Then there exists a real-valued sequence $\{\omega_n(t)\}_{n \in \mathbb{N}^+}$ of one-dimensional standard Brownian

motions mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n^{(1)}} \omega_n(t) e_n^{(1)}, \quad t \geq 0. \quad (2)$$

Consider that a Y_2 -valued stochastic process $B_Q^H(t)$ is defined by the formal infinite sum (see [2]).

$$B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n^{(2)}} \beta_n^H(t) e_n^{(2)}, \quad t \geq 0, \quad (3)$$

where the sequence $\{\beta_n^H(t)\}_{n \in \mathbb{N}^+}$ is mutually independent scalar fBms with Hurst parameter $H \in (1/2, 1)$.

Let $\mathcal{L}_{Q^{(i)}}^0(Y_i; X)$ be the space of all $Q^{(i)}$ -Hilbert-Schmidt operators from Y_i to X , $i = 1, 2$. Now we show the following definitions.

Definition 1. Let $\xi_i \in \mathcal{L}(Y_i; X)$ and define

$$\|\xi_i\|_{\mathcal{L}_{Q^{(i)}}^0} := \text{tr}(\xi_i Q^{(i)} \xi_i^*) = \left\{ \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n^{(i)}} \xi_i e_n^{(i)} \right\|_X^2 \right\}^{1/2}. \quad (4)$$

If $\|\xi_i\|_{\mathcal{L}_{Q^{(i)}}^0} < \infty$, then ξ_i is called a $Q^{(i)}$ -Hilbert-Schmidt operator and the space $\mathcal{L}_{Q^{(i)}}^0 := \mathcal{L}_{Q^{(i)}}^0(Y_i; X)$ is a real separable Hilbert space equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_{Q^{(i)}}^0} = \sum_{n=1}^{\infty} \langle \varphi e_n^{(i)}, \psi e_n^{(i)} \rangle$, $i = 1, 2$.

Let $T > 0$, $\beta = \{\beta(t); t \in [0, T]\}$ be a Wiener process and $\{\beta^H(t), t \in [0, T]\}$ be the one-dimensional fBm with Hurst parameter $H \in (1/2, 1)$. β^H has the following stochastic integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) d\beta(s), \quad (5)$$

$K_H(t, s)$ is the kernel defined as

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du \quad (6)$$

for $t > s$, where $c_H = (H(2H-1)/B(2-2H, H-1/2))^{1/2}$, and $B(\cdot, \cdot)$ denotes the Beta function. Set $K_H(t, s) = 0$ for $t \leq s$. Let $\mathcal{H}$ denote the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of the set of indicator functions $\{I_{[0,t]}, t \in [0, T]\}$ with respect to the inner product.

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s). \quad (7)$$

Then the mapping $I_{[0,t]} \rightarrow \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{\text{span}}^{\mathcal{L}^2(\Omega)} \{\beta^H, t \in [0, T]\}$. The image of an element $\varphi \in \mathcal{H}$ by this isometry is called the Wiener integral of φ with respect to β^H (see [23]).

We recall that for $\psi, \varphi \in \mathcal{H}$ their inner product in $\mathcal{H}$ is given by

$$\begin{aligned} \langle \psi, \varphi \rangle_{\mathcal{H}} &= H(2H-1) \int_0^T \int_0^T \psi(s) \varphi(t) |t-s|^{2H-2} ds dt. \end{aligned} \quad (8)$$

K_H^* is an operator from $\mathcal{H}$ to $\mathcal{L}^2([0, T])$ defined as

$$(K_H^* \varphi)(s) = \int_s^t \varphi(r) \frac{\partial K_H(r, s)}{\partial r} dr. \quad (9)$$

Then

$$(K_H^* I_{[0,t]})(s) = K_H(t, s) I_{[0,t]}(s) \quad (10)$$

and K_H^* is an isometry between the set of indicator functions $\{I_{[0,t]}, t \in [0, T]\}$ and $\mathcal{L}^2([0, T])$ that can be extended to $\mathcal{H}$ (see [1]).

Let $\psi(s)$ for $s \in [0, T]$ be a mapping with values in $\mathcal{L}_2^0(Y_2; X)$. The stochastic integral of ψ with respect to B_Q^H is defined by

$$\begin{aligned} \int_0^t \psi(s) dB_Q^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n^{(2)}} \psi(s) e_n^{(2)} d\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n^{(2)}} K_H^*(\psi e_n^{(2)})(s) d\beta_n(s), \end{aligned} \quad (11)$$

where β_n is the standard Brownian motion which was introduced in (5).

In order to set existence, uniqueness, and controllability problems, we need the following lemmas which are Lemma 2 in [2] and Lemma 7.7 in [24], respectively.

Lemma 2 (see [2]). For any $\varphi : [0, T] \mapsto \mathcal{L}_{Q^{(2)}}^0(Y_2; X)$ and $0 \leq a \leq b \leq T$, if $\sum_{n=1}^{\infty} \sqrt{\lambda_n^{(2)}} < \infty$ and the series $\sum_{n=1}^{\infty} \sqrt{\lambda_n^{(2)}} \varphi(t) e_n^{(2)}$ is uniformly convergent for $t \in [0, T]$, then one has

$$\begin{aligned} \mathbb{E} \left\| \int_a^b \varphi(s) dB_Q^H(s) \right\|_X^2 \\ \leq c_H H (2H - 1) (b - a)^{2H-1} \int_a^b \|\varphi(s)\|_{\mathcal{L}_{Q^{(2)}}^0}^2 ds, \end{aligned} \quad (12)$$

where $H \in (1/2, 1)$ and $c_H = \sqrt{H(2H-1)/B(2-2H, H-1/2)}$.

Lemma 3 (see [24]). For any $r \geq 1$ and for arbitrary predictable process $\Phi(\cdot) \in \mathcal{L}_{Q^{(1)}}^0(Y_1; X)$, one has

$$\begin{aligned} \sup_{s \in [0, t]} \mathbb{E} \left| \int_0^s \Phi(\sigma) dW(\sigma) \right|^{2r} \\ \leq (r(2r-1))^r \left(\int_0^t \left(\mathbb{E} \|\Phi(s)\|_{\mathcal{L}_{Q^{(1)}}^0}^{2r} \right)^{1/r} ds \right)^r \end{aligned} \quad (13)$$

$t \in [0, T].$

We assume that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A and the analytic semigroup $\{S(t), t \geq 0\}$ in X is uniformly bounded; that is, $\|S(t)\|_X \leq M$ for some positive constant $M \geq 1$. Then, for $\alpha \in (0, 1]$, it is possible under some circumstance to define the fractional power $(-A)^\alpha$ as a

closed linear operator with domain $D((-A)^\alpha)$. Moreover, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$\|v\|_\alpha = \|(-A)^\alpha v\|_X, \quad v \in D((-A)^\alpha) \quad (14)$$

defines a norm on $D((-A)^\alpha)$.

Let H_α denote the Banach space $D((-A)^\alpha)$ equipped with the norm $\|\cdot\|_\alpha$, then the following properties are well known (see [25]).

Lemma 4 (see [25]). Suppose that the preceding conditions are satisfied.

(1) If $0 < \beta \leq \alpha$, then the injection $H_\alpha \hookrightarrow H_\beta$ is continuous.

(2) For every $0 < \beta \leq 1$, there exists $M_\beta > 0$ such that

$$\|(-A)^\beta S(t)\| \leq M_\beta t^{-\beta}, \quad 0 \leq t \leq T. \quad (15)$$

3. Existence and Uniqueness of Mild Solution

In this section, the Banach fixed point theory is used to investigate the existence and uniqueness problems of (1). Now, we introduce the definition of mild solution of (1).

Definition 5. An X -valued stochastic process $x(t)$ is called a mild solution of (1) if $x(\cdot) \in C([- \tau, T]; \mathcal{L}^2(\Omega; X))$ and $x(t) = \varphi(t)$ for $t \in [- \tau, 0]$ and for $t \in [0, T]$ satisfies

$$\begin{aligned} x(t) &= S(t) [\varphi(0) + h(0, x(-r(0)))] \\ &\quad - h(t, x(t-r(t))) \\ &\quad - \int_0^t AS(t-s) h(s, x(s-r(s))) ds \\ &\quad + \int_0^t S(t-s) [f(s, x(s-\rho(s))) + Bu(s)] ds \\ &\quad + \int_0^t S(t-s) g(s, x(s-\eta(s))) dW(s) \\ &\quad + \int_0^t S(t-s) \sigma(s) dB_Q^H(s) \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (16)$$

In order to set the existence and uniqueness problems, we assume that the following conditions hold:

(C.1) A is the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ of bounded linear operators in X and there exists a positive constant $M \geq 1$ such that $\|S(t)\|_X \leq M$ for $0 \leq t \leq T$.

(C.2) The mappings $f : [0, +\infty) \times X \rightarrow X$ and $g : [0, +\infty) \times X \rightarrow \mathcal{L}_{Q^{(1)}}^0(Y_1; X)$ satisfy the following conditions: for any $x, y \in X$ and $t \in [0, T]$, there exist nonnegative constants L_f, C_f, L_g , and C_g , such that

$$\begin{aligned} (i) \quad &\|f(t, x) - f(t, y)\|_X \leq L_f \|x - y\|_X, \quad \|f(t, x)\|_X^2 \leq C_f (1 + \|x\|_X^2); \\ (ii) \quad &\|g(t, x) - g(t, y)\|_{\mathcal{L}_{Q^{(1)}}^0} \leq L_g \|x - y\|_X, \\ &\|g(t, x)\|_{\mathcal{L}_{Q^{(1)}}^0}^2 \leq C_g (1 + \|x\|_X^2). \end{aligned}$$

(C.3) The mapping $h(t, x) \in \mathcal{D}((-A)^\beta)$ satisfies the following conditions: for any $x, y \in X$ and $t \geq 0$, there exist nonnegative constants $L_h, C_h \geq 0$ and $\beta \in (0, 1)$ such that

- (i) $\|(-A)^\beta h(t, x) - (-A)^\beta h(t, y)\|_X \leq L_h \|x - y\|_X$;
- (ii) $\|(-A)^\beta h(t, x)\|_X^2 \leq C_h(1 + \|x\|_X^2)$;
- (iii) the constants L_h and β satisfy the following inequality $k := L_h \|(-A)^{-\beta}\| < 1$.

(C.4) The mapping $(-A)^\beta h$ is continuous in mean square: for all $x \in C([0, T]; \mathcal{L}^2(\Omega; X))$, $\lim_{t \rightarrow s} \mathbb{E} \|(-A)^\beta h(t, x(t)) - (-A)^\beta h(s, x(s))\|_X^2 = 0$.

(C.5) The mapping $\sigma : [0, T] \rightarrow \mathcal{L}_{Q^{(2)}}^0(Y_2; X)$ satisfies $\int_0^T \|\sigma(s)\|_{\mathcal{L}_{Q^{(2)}}^0}^2 ds < \infty, \forall T > 0$.

(C.6) The mapping $u(\cdot)$ takes values in $\mathcal{L}^2([0, T]; U)$ and B is a bounded linear operator from U to X .

Theorem 6. *Let conditions (C.1)–(C.6) be satisfied. Then system (1) has a unique mild solution on $[-\tau, T]$.*

Proof. Denote $\mathcal{B}_T := C([-\tau, T]; \mathcal{L}^2(\Omega; X))$ by subspace of all continuous functions from $[-\tau, T]$ into $\mathcal{L}^2(\Omega; X)$ equipped with the norm $\|\zeta\|_{\mathcal{B}_T} = \sup_{s \in [-\tau, T]} (\mathbb{E} \|\zeta(s)\|^2)^{1/2}$. Fix $T > 0$ and consider the space $S_T := \{x \in \mathcal{B}_T : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0]\}$. S_T is a closed subspace of $\mathcal{B}_T$ equipped with the norm $\|\cdot\|_{\mathcal{B}_T}$. Obviously, S_T and $\mathcal{B}_T$ are two Banach spaces. Define an operator $\pi : S_T \rightarrow S_T$ by $\pi(x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0, T]$,

$$\begin{aligned} \pi(x)(t) &= S(t) [\varphi(0) + h(0, x(-r(0)))] \\ &\quad - h(t, x(t-r(t))) \\ &\quad - \int_0^t AS(t-s) h(s, x(s-r(s))) ds \\ &\quad + \int_0^t S(t-s) [f(s, x(s-\rho(s))) + Bu(s)] ds \\ &\quad + \int_0^t S(t-s) g(s, x(s-\eta(s))) dW(s) \\ &\quad + \int_0^t S(t-s) \sigma(s) dB_Q^H(s). \end{aligned} \quad (17)$$

We first verify that π is continuous in mean square on $[0, T]$. Let $x \in S_T$, $0 \leq t < T$, and let ε be positive and sufficiently small (similar estimates hold for $\varepsilon < 0$), then

$$\begin{aligned} \mathbb{E} \|\pi(x)(t+\varepsilon) - \pi(x)(t)\|_X^2 &\leq 7\mathbb{E} \|(S(t+\varepsilon) - S(t)) \\ &\quad \cdot [\varphi(0) + h(0, x(-r(0)))]\|_X^2 \\ &\quad + 7\mathbb{E} \|h(t+\varepsilon, x(t+\varepsilon-r(t+\varepsilon))) \\ &\quad - h(t, x(t-r(t)))\|_X^2 \\ &\quad + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} AS(t+\varepsilon-s) h(s, x(s-r(s))) ds \right. \\ &\quad \left. - \int_0^t AS(t-s) h(s, x(s-r(s))) ds \right\|_X^2 \\ &\quad + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) f(s, x(s-\rho(s))) ds \right. \\ &\quad \left. - \int_0^t S(t-s) f(s, x(s-\rho(s))) ds \right\|_X^2 \\ &\quad + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) g(s, x(s-\eta(s))) dW(s) \right. \\ &\quad \left. - \int_0^t S(t-s) g(s, x(s-\eta(s))) dW(s) \right\|_X^2 \\ &\quad + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) \sigma(s) dB_Q^H(s) \right. \\ &\quad \left. - \int_0^t S(t-s) \sigma(s) dB_Q^H(s) \right\|_X^2 \\ &\quad + 7\mathbb{E} \|Bu(t+\varepsilon) - Bu(t)\|_X^2. \end{aligned}$$

$$\begin{aligned} &- h(t, x(t-r(t)))\|_X^2 \\ &+ 7\mathbb{E} \left\| \int_0^{t+\varepsilon} AS(t+\varepsilon-s) h(s, x(s-r(s))) ds \right. \\ &\quad \left. - \int_0^t AS(t-s) h(s, x(s-r(s))) ds \right\|_X^2 \\ &+ 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) f(s, x(s-\rho(s))) ds \right. \\ &\quad \left. - \int_0^t S(t-s) f(s, x(s-\rho(s))) ds \right\|_X^2 \\ &+ 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) g(s, x(s-\eta(s))) dW(s) \right. \\ &\quad \left. - \int_0^t S(t-s) g(s, x(s-\eta(s))) dW(s) \right\|_X^2 \\ &+ 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) \sigma(s) dB_Q^H(s) \right. \\ &\quad \left. - \int_0^t S(t-s) \sigma(s) dB_Q^H(s) \right\|_X^2 \\ &+ 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s) Bu(s) ds \right. \\ &\quad \left. - \int_0^t S(t-s) Bu(s) ds \right\|_X^2 := 7 \sum_{i=1}^7 \mathbb{E} \|K_i(\varepsilon)\|_X^2. \end{aligned} \quad (18)$$

It is easy to know that $\mathbb{E} \|K_i(\varepsilon)\|_X^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$, $i = 1, 2, 4, 7$. Further, by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \mathbb{E} \|K_3(\varepsilon)\|_X^2 &\leq 2\mathbb{E} \left\| \int_0^t (S(\varepsilon) - I) (-A)^{1-\beta} S(t-s) \right. \\ &\quad \cdot (-A)^\beta h(s, x(s-r(s))) ds \Big\|_X^2 + 2\mathbb{E} \left\| \int_t^{t+\varepsilon} (-A)^{1-\beta} \right. \\ &\quad \cdot S(t+\varepsilon-s) (-A)^\beta h(s, x(s-r(s))) ds \Big\|_X^2 \\ &:= 2\mathbb{E} \|K_{31}(\varepsilon)\|_X^2 + 2\mathbb{E} \|K_{32}(\varepsilon)\|_X^2. \end{aligned} \quad (19)$$

Applying conditions (C.1) and (C.3) to $\mathbb{E} \|K_{31}(\varepsilon)\|_X^2$, we can obtain

$$\begin{aligned} \mathbb{E} \|K_{31}(\varepsilon)\|_X^2 &\leq (S(\varepsilon) - I)^2 \\ &\quad \cdot M_{1-\beta}^2 \mathbb{E} \left\| \int_0^t \frac{1}{(t-s)^{1-\beta}} (-A)^\beta \right. \\ &\quad \cdot h(s, x(s-r(s))) ds \Big\|_X^2 \leq (S(\varepsilon) - I)^2 \\ &\quad \cdot M_{1-\beta}^2 T^\beta \beta^{-1} \int_0^t \frac{1}{(t-s)^{1-\beta}} \mathbb{E} \|(-A)^\beta \\ &\quad \cdot h(s, x(s-r(s)))\|_X^2 ds \leq (S(\varepsilon) - I)^2 \\ &\quad \cdot M_{1-\beta}^2 T^{2\beta} \beta^{-2} C_h \left(1 + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \right) \rightarrow 0 \end{aligned} \quad (20)$$

as $\varepsilon \rightarrow 0$. Applying condition (C.1) and (C.3) to $\mathbb{E}\|I_{32}(\varepsilon)\|_X^2$, we have

$$\begin{aligned} \mathbb{E}\|K_{32}(\varepsilon)\|_U^2 &\leq M_{1-\beta}^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} \frac{1}{(t+\varepsilon-s)^{1-\beta}} (-A)^\beta \right. \\ &\quad \cdot h(s, x(s-r(s))) ds \Big\|_X^2 \\ &\leq M_{1-\beta}^2 \varepsilon^\beta \beta^{-1} \int_t^{t+\varepsilon} \frac{1}{(t+\varepsilon-s)^{1-\beta}} \mathbb{E} \|(-A)^\beta \\ &\quad \cdot h(s, x(s-r(s)))\|_X^2 ds \leq M_{1-\beta}^2 \varepsilon^{2\beta} \beta^{-2} C_h \left(1 \right. \\ &\quad \left. + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \right) \rightarrow 0 \end{aligned} \quad (21)$$

as $\varepsilon \rightarrow 0$. That is to say, $\mathbb{E}\|K_3(\varepsilon)\|_U^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For the fifth term, by using Lemma 3, we can obtain

$$\begin{aligned} \mathbb{E}\|K_5(\varepsilon)\|_X^2 &\leq 2\mathbb{E} \left\| \int_0^t (S(\varepsilon) - I) S(t-s) \right. \\ &\quad \cdot g(s, x(s-\eta(s))) dW(s) \Big\|_X^2 \\ &\quad + 2\mathbb{E} \left\| \int_t^{t+\varepsilon} S(t+\varepsilon-s) \right. \\ &\quad \cdot g(s, x(s-\eta(s))) dW(s) \Big\|_X^2 \leq 2(S(\varepsilon) \\ &\quad - I)^2 \int_0^t \mathbb{E} \|S(t-s) g(s, x(s-\eta(s)))\|_{\mathcal{L}^0_{Q(1)}}^2 ds \\ &\quad + 2 \int_t^{t+\varepsilon} \mathbb{E} \|S(t+\varepsilon-s) g(s, x(s-\eta(s)))\|_{\mathcal{L}^0_{Q(1)}}^2 ds \rightarrow 0 \end{aligned} \quad (22)$$

as $\varepsilon \rightarrow 0$.

For the sixth term, we get

$$\begin{aligned} \mathbb{E}\|K_6(\varepsilon)\|_X^2 &\leq 2\mathbb{E} \left\| \int_0^t (S(t+\varepsilon-s) - S(t-s)) \sigma(s) dB_Q^H(s) \right\|_X^2 \\ &\quad + 2\mathbb{E} \left\| \int_t^{t+\varepsilon} S(t+\varepsilon-s) \sigma(s) dB_Q^H(s) \right\|_X^2 \\ &:= 2\mathbb{E}\|K_{61}(\varepsilon)\|_X^2 + 2\mathbb{E}\|K_{62}(\varepsilon)\|_X^2. \end{aligned} \quad (23)$$

Firstly, apply Lemma 2 to $\mathbb{E}\|K_{61}(\varepsilon)\|_X^2$, and we have

$$\begin{aligned} \mathbb{E}\|K_{61}(\varepsilon)\|_X^2 &\leq c_H H (2H-1) \\ &\quad \cdot T^{2H-1} \int_0^t \|S(t-s) (S(\varepsilon) - I) \sigma(s)\|_{\mathcal{L}^0_{Q(2)}}^2 ds \\ &\leq (S(\varepsilon) - I)^2 c_H H (2H-1) \\ &\quad \cdot T^{2H-1} M^2 \int_0^t \|\sigma(s)\|_{\mathcal{L}^0_{Q(2)}}^2 ds \rightarrow 0 \end{aligned} \quad (24)$$

as $\varepsilon \rightarrow 0$. For every s fixed, we get $S(\varepsilon)\sigma(s) \rightarrow \sigma(s)$ as $\varepsilon \rightarrow 0$ and $\|S(\varepsilon)\sigma(s)\|_{\mathcal{L}^0_{Q(2)}} \leq M\|\sigma(s)\|_{\mathcal{L}^0_{Q(2)}}$.

By Lemma 2 and the Lebesgue dominated convergence theorem, we can obtain

$$\begin{aligned} \mathbb{E}\|K_{62}(\varepsilon)\|_X^2 &\leq c_H H (2H-1) \varepsilon^{2H-1} M^2 \int_t^{t+\varepsilon} \|\sigma(s)\|_{\mathcal{L}^0_{Q(2)}}^2 ds \rightarrow 0 \end{aligned} \quad (25)$$

as $\varepsilon \rightarrow 0$. Therefore $\mathbb{E}\|K_6(\varepsilon)\|_X^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus, π is continuous in mean square on $[0, T]$. So we conclude that $\pi(S_T) \subset S_T$.

Moreover, we shall show that π is contractive in S_{T_1} with some $0 < T_1 \leq T$. Let $x, y \in S_T$ and for any fixed $t \in [0, T]$, we get

$$\begin{aligned} \mathbb{E}\|\pi(x)(t) - \pi(y)(t)\|_X^2 &\leq \frac{1}{k} \mathbb{E} \|h(t, x(t-r(t))) \\ &\quad - h(t, y(t-r(t)))\|_X^2 + \frac{3}{1-k} \mathbb{E} \left\| \int_0^t (-A) S(t-s) \right. \\ &\quad \cdot [h(s, x(s-r(s))) - h(s, y(s-r(s)))] ds \Big\|_X^2 \\ &\quad + \frac{3}{1-k} \mathbb{E} \left\| \int_0^t S(t-s) [f(s, x(s-\rho(s))) \right. \\ &\quad \left. - f(s, y(s-\rho(s)))] ds \right\|_X^2 + \frac{3}{1-k} \mathbb{E} \left\| \int_0^t S(t-s) \right. \\ &\quad \left. [g(s, x(s-\eta(s))) - g(s, y(s-\eta(s)))] dW(s) \right\|_X^2. \end{aligned} \quad (26)$$

By conditions (C.1)–(C.4) and Lemma 3, we have

$$\begin{aligned} \mathbb{E}\|\pi(x)(t) - \pi(y)(t)\|_X^2 &\leq k \mathbb{E} \|x(t-r(t)) - y(t-r(t))\|_X^2 \\ &\quad + \frac{3M_{1-\beta}^2 L_h^2 t^{2\beta}}{(1-k)\beta^2} \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 \\ &\quad + \frac{3M^2 L_f^2 t^2}{1-k} \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 \\ &\quad + \frac{3M^2 L_g^2 t}{1-k} \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2. \end{aligned} \quad (27)$$

Hence

$$\begin{aligned} \sup_{t \in [-\tau, T]} \mathbb{E}\|\pi(x)(t) - \pi(y)(t)\|_X^2 &\leq \gamma(t) \sup_{t \in [-\tau, T]} \mathbb{E} \|x(t) - y(t)\|_X^2, \end{aligned} \quad (28)$$

where

$$\gamma(t) = k + \frac{3M_{1-\beta}^2 L_h^2 t^{2\beta}}{(1-k)\beta^2} + \frac{3M^2 L_f^2 t^2}{1-k} + \frac{3M^2 L_g^2 t}{1-k}. \quad (29)$$

By condition (C.3), we have $\gamma(0) = k < 1$, then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ which shows that

π is a contraction mapping in S_{T_1} ; therefore π has a unique fixed point in S_{T_1} , which is a unique mild solution of (1) on $[-\tau, T_1]$. Repeat this procedure to extend the mild solution to $[\tau, T]$. The proof is completed. $\square$

Remark 7. Boufoussi and Hajji in [8] and Liu and Luo in [13], respectively, considered the existence and uniqueness of solutions for special cases of (1). In (20) and (21), we use the Hölder inequality and linear growth condition to obtain the continuity of $K_3(\varepsilon)$ and obtain a weaker result than those of [8, 13]. The necessary conditions both in [8, 13] are

$$1/2 < \beta < 1. \quad (30)$$

However, our condition is

$$0 < \beta < 1. \quad (31)$$

In this sense, this paper improves and generalizes the results in [8, 13].

4. Controllability Result

In this section, we focus on the controllability problem of (1). Now we introduce the concept of controllability of neutral stochastic delay partial differential equations.

Definition 8. System (1) is said to be controllable on the finite interval $[-\tau, T]$, if, for each initial stochastic process φ defined on $[-\tau, 0]$ and $x_1 \in X$, there exists a stochastic control $u \in \mathcal{L}^2([0, T]; U)$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that the mild solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$, where x_1 and T are the preassigned terminal state and time, respectively.

In order to set the controllability problem, we assume that the following conditions hold:

(C.7) The linear operator ω from U to X defined by $\omega u = \int_0^T S(T-s)Bu(s)ds$ has an inverse operator ω^{-1} with values in $\mathcal{L}^2([0, T]; U) \setminus \ker \omega$, where $\ker \omega = \{x \in \mathcal{L}^2([0, T]; U) : \omega x = 0\}$ (see [22]), and there exists a pair of finite positive constants M_B and M_ω such that $\|B\| \leq M_B$ and $\|\omega^{-1}\| \leq M_\omega$.

Theorem 9. Let conditions (C.1)–(C.7) be satisfied. Then, system (1) is controllable on $[-\tau, T]$.

Proof. Using the condition (C.7) for an arbitrary mapping $x(\cdot)$, define the stochastic control by

$$\begin{aligned} u(t) = & \omega^{-1} \left\{ x_1 - S(T) [\varphi(0) + h(0, x(-r(0)))] \right. \\ & + h(T, x(T-r(T))) \\ & + \int_0^T AS(T-s)h(s, x(s-r(s)))ds \\ & - \int_0^T S(T-s)f(s, x(s-\rho(s)))ds \\ & - \int_0^T S(T-s)g(s, x(s-\eta(s)))dW(s) \\ & \left. - \int_0^T S(T-s)\sigma(s)dB_Q^H(s) \right\} (t), \quad t \in [0, T]. \end{aligned} \quad (32)$$

Applying this control to the operator π . Obviously, to find a fixed point for the operator π is equivalent to prove the existence of mild solutions of equation. Clearly, $\pi(T) = x_1$, which implies that the stochastic control u steers the system from the initial state φ to x_1 in time T , provided we can find a fixed point of the operator π which means that the system is controllable on $[-\tau, T]$.

First, we shall prove that π is continuous in mean square on $[0, T]$. Let $x \in S_T$, $0 \leq t < T$, and ε be positive and sufficiently small (similar estimates hold for $\varepsilon < 0$), then

$$\begin{aligned} \mathbb{E} \|\pi(x)(t+\varepsilon) - \pi(x)(t)\|_X^2 \leq & 7\mathbb{E} \|(S(t+\varepsilon) - S(t)) \\ & \cdot [\varphi(0) + h(0, x(-r(0)))]\|_X^2 + 7\mathbb{E} \|h(t+\varepsilon, x(t \\ & + \varepsilon - r(t+\varepsilon))) - h(t, x(t-r(t)))\|_X^2 \\ & + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} AS(t+\varepsilon-s)h(s, x(s-r(s)))ds \right. \\ & \left. - \int_0^t AS(t-s)h(s, x(s-r(s)))ds \right\|_X^2 \\ & + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s)f(s, x(s-\rho(s)))ds \right. \\ & \left. - \int_0^t S(t-s)f(s, x(s-\rho(s)))ds \right\|_X^2 \\ & + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s)g(s, x(s-\eta(s)))dW(s) \right. \\ & \left. - \int_0^t S(t-s)g(s, x(s-\eta(s)))dW(s) \right\|_X^2 \\ & + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t+\varepsilon-s)\sigma(s)dB_Q^H(s) - \int_0^t S(t-s) \right. \\ & \cdot \sigma(s)dB_Q^H(s) \left\|_X^2 + 7\mathbb{E} \left\| \int_0^{t+\varepsilon} S(t \right. \right. \\ & + \varepsilon - v)B\omega^{-1} \left\{ x_1 - S(T) [\varphi(0) + h(0, x(-r(0)))] \right. \\ & + h(T, x(T-r(T))) \\ & + \int_0^T AS(T-s)h(s, x(s-r(s)))ds \\ & - \int_0^T S(T-s)f(s, x(s-\rho(s)))ds \\ & - \int_0^T S(T-s)g(s, x(s-\eta(s)))dW(s) \\ & \left. - \int_0^T S(T-s)\sigma(s)dB_Q^H(s) \right\} dv - \int_0^t S(t \\ & - v)B\omega^{-1} \left\{ x_1 - S(T) [\varphi(0) + h(0, x(-r(0)))] \right. \\ & + h(T, x(T-r(T))) \\ & + \int_0^T AS(T-s)h(s, x(s-r(s)))ds \\ & \left. - \int_0^T S(T-s)f(s, x(s-\rho(s)))ds \right. \end{aligned}$$

$$\begin{aligned}
& - \int_0^T S(T-s) g(s, x(s-\eta(s))) dW(s) \\
& - \int_0^T S(T-s) \sigma(s) dB_Q^H(s) \Big\} dv \Big\|_X^2 \\
& := 7 \sum_{i=1}^7 \mathbb{E} \|K_i(\varepsilon)\|_X^2.
\end{aligned} \tag{33}$$

From Section 3, we can obtain that $\mathbb{E} \|K_i(\varepsilon)\|_X^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$, $i = 1, 2, \dots, 6$. Now we estimate the term $\mathbb{E} \|K_7(\varepsilon)\|_X^2$, and we have

$$\begin{aligned}
\mathbb{E} \|K_7(\varepsilon)\|_X^2 & \leq 2 \mathbb{E} \left\| \int_t^{t+\varepsilon} S(t+\varepsilon-v) B \omega^{-1} \left\{ x_1 \right. \right. \\
& \quad - S(T) [\varphi(0) + h(0, x(-r(0)))] \\
& \quad + h(T, x(T-r(T))) \\
& \quad + \int_0^T AS(T-s) h(s, x(s-r(s))) ds \\
& \quad - \int_0^T S(T-s) f(s, x(s-\rho(s))) ds \\
& \quad - \int_0^T S(T-s) g(s, x(s-\eta(s))) dW(s) \\
& \quad \left. \left. - \int_0^T S(T-s) \sigma(s) dB_Q^H(s) \right\} dv \right\|_X^2 \\
& \quad + 2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-s) - S(t-s)) B \omega^{-1} \left\{ x_1 \right. \right. \\
& \quad - S(T) [\varphi(0) + h(0, x(-r(0)))] \\
& \quad + h(T, x(T-r(T))) \\
& \quad + \int_0^T AS(T-s) h(s, x(s-r(s))) ds \\
& \quad - \int_0^T S(T-s) f(s, x(s-\rho(s))) ds \\
& \quad - \int_0^T S(T-s) g(s, x(s-\eta(s))) dW(s) \\
& \quad \left. \left. - \int_0^T S(T-s) \sigma(s) dB_Q^H(s) \right\} dv \right\|_X^2 \\
& := 2 \mathbb{E} \|K_{71}(\varepsilon)\|_X^2 + 2 \mathbb{E} \|K_{72}(\varepsilon)\|_X^2.
\end{aligned} \tag{34}$$

Applying conditions (C.1)–(C.7) and Lemmas 2–4 to $\mathbb{E} \|K_{71}(\varepsilon)\|_X^2$, we get

$$\begin{aligned}
\mathbb{E} \|K_{71}(\varepsilon)\|_X^2 & \leq 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} x_1 dv \right\|_X^2 \\
& \quad + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} M[x(0)] \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + h(0, x(-r(0))) \right\} dv \Big\|_X^2 \\
& + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} h(T, x(T-r(T))) dv \right\|_X^2 \\
& + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} \int_0^T (-A)^{1-\beta} S(T-s) (-A)^\beta \right. \\
& \quad \cdot h(s, x(s-r(s))) ds dv \Big\|_X^2 \\
& + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} \int_0^T S(T-s) \right. \\
& \quad \cdot f(s, x(s-\rho(s))) ds dv \Big\|_X^2 \\
& + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} \int_0^T S(T-s) \right. \\
& \quad \cdot g(s, x(s-\eta(s))) dW(s) dv \Big\|_X^2 \\
& + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_t^{t+\varepsilon} \int_0^T S(T-s) \right. \\
& \quad \cdot \sigma(s) dB_Q^H(s) dv \Big\|_X^2 \\
& \leq 7M^2 M_B^2 M_\omega^2 \varepsilon^2 \mathbb{E} \|x_1\|_X^2 + 7M^4 M_B^2 M_\omega^2 \varepsilon^2 \mathbb{E} \|x(0) \\
& \quad + h(0, x(-r(0)))\|_X^2 + 7M^2 M_B^2 M_\omega^2 C_h \|(-A)^{-\beta}\|^2 \\
& \quad \cdot \varepsilon^2 \left(1 + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \right) \\
& \quad + 7M^2 M_B^2 M_\omega^2 M_{1-\beta}^2 C_h \beta^{-2} T^{2\beta} \varepsilon^2 \left(1 \right. \\
& \quad \left. + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \right) + 7M^4 M_B^2 M_\omega^2 C_f T^2 \varepsilon^2 \left(1 \right. \\
& \quad \left. + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \right) + 7M^4 M_B^2 M_\omega^2 C_g T \varepsilon^2 \left(1 \right. \\
& \quad \left. + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \right) + 7M^4 M_B^2 M_\omega^2 c_H H (2H-1) \\
& \quad \cdot T^{2H-1} \varepsilon^2 \int_0^T \|\sigma(s)\|_{\mathcal{L}^0_{Q(2)}}^2 ds.
\end{aligned} \tag{35}$$

Then $\mathbb{E} \|K_{71}(\varepsilon)\|_X^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Using the similar technique to $\mathbb{E} \|K_{72}(\varepsilon)\|_X^2$, we have

$$\begin{aligned}
& \mathbb{E} \|K_{72}(\varepsilon)\|_X^2 \\
& \leq 7M^2 M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-v) - S(t-v)) x_1 dv \right\|_X^2
\end{aligned}$$

$$\begin{aligned}
& + 7M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) \right. \\
& \cdot M[x(0) + h(0, x(-r(0)))] d\nu \Big\|_X^2 \\
& + 7M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) \right. \\
& \cdot h(T, x(T-r(T))) d\nu \Big\|_X^2 \\
& + 7M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) \right. \\
& \cdot \int_0^T AS(T-s) h(s, x(s-r(s))) ds d\nu \Big\|_X^2 \\
& + 7M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) \right. \\
& \cdot \int_0^T S(T-s) f(s, x(s-\rho(s))) ds d\nu \Big\|_X^2 \\
& + 7M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) \right. \\
& \cdot \int_0^T S(T-s) g(s, x(s-\eta(s))) dW(s) d\nu \Big\|_X^2 \\
& + 7M_B^2 M_\omega^2 \mathbb{E} \left\| \int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) \right. \\
& \cdot \int_0^T S(T-s) \sigma(s) dB_Q^H(s) d\nu \Big\|_X^2 \\
& \leq \left\{ 7M_B^2 M_\omega^2 \mathbb{E} \|x_1\|_X^2 + 7M^2 M_B^2 M_\omega^2 \mathbb{E} \|x(0) \right. \\
& + h(0, x(-r(0)))\|_X^2 + 7M_B^2 M_\omega^2 C_h \|(-A)^{-\beta}\|^2 \left(1 \right. \\
& + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \Big) \\
& + 7M_B^2 M_\omega^2 M_{1-\beta}^2 C_h \beta^{-2} T^{2\beta} \left(1 \right. \\
& + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \Big) + 7M^2 M_B^2 M_\omega^2 C_f T^2 \left(1 \right. \\
& + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \Big) + 7M^2 M_B^2 M_\omega^2 C_g T \left(1 \right. \\
& + \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s)\|_X^2 \Big) + 7M^2 M_B^2 M_\omega^2 c_H H (2H-1) \\
& \cdot T^{2H-1} \int_0^T \|\sigma(s)\|_{\mathcal{L}^0_{Q(2)}}^2 ds \Big\} \\
& \times \left(\int_0^t (S(t+\varepsilon-\nu) - S(t-\nu)) d\nu \right)^2.
\end{aligned} \tag{36}$$

Then $\mathbb{E} \|K_{72}(\varepsilon)\|_X^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Thus, π is indeed continuous in mean square on $[0, T]$. So we conclude that $\pi(S_T) \subset S_T$.

Last, we will show that π is a contraction mapping in S_{T_1} with some $0 < T_1 \leq T$. Let $x, y \in S_T$, as proceeding as we did previously; we can obtain

$$\begin{aligned}
& \mathbb{E} \|\pi(x)(t) - \pi(y)(t)\|_X^2 \leq \frac{1}{k} \mathbb{E} \|h(t, x(t-r(t))) - h(t, y(t-r(t)))\|_X^2 \\
& + \frac{7}{1-k} \left\{ \mathbb{E} \left\| \int_0^t (-A)^{1-\beta} S(t-s) (-A)^\beta [h(s, x(s-r(s))) - h(s, y(s-r(s)))] ds \right\|_X^2 \right. \\
& + \mathbb{E} \left\| \int_0^t S(t-s) [f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))] ds \right\|_X^2 \\
& + \mathbb{E} \left\| \int_0^t S(t-s) [g(s, x(s-\eta(s))) - g(s, y(s-\eta(s)))] dW(s) \right\|_X^2 \\
& + \mathbb{E} \left\| \int_0^t S(t-\nu) B\omega^{-1} [h(T, x(T-r(T))) - h(T, y(T-r(T)))] d\nu \right\|_X^2 \\
& + \mathbb{E} \left\| \int_0^t S(t-\nu) B\omega^{-1} \int_0^T AS(T-s) [h(s, x(s-r(s))) - h(s, y(s-r(s)))] ds d\nu \right\|_X^2 \\
& + \mathbb{E} \left\| \int_0^t S(t-\nu) B\omega^{-1} \int_0^T S(T-s) [f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))] ds d\nu \right\|_X^2 \\
& + \mathbb{E} \left\| \int_0^t S(t-\nu) B\omega^{-1} \int_0^T S(T-s) [g(s, x(s-\eta(s))) - g(s, y(s-\eta(s)))] dW(s) d\nu \right\|_X^2 \Big\}.
\end{aligned} \tag{37}$$

Thus

$$\begin{aligned}
& \mathbb{E} \|\pi(x)(t) - \pi(y)(t)\|^2 \\
& \leq k \mathbb{E} \|x(t-r(t)) - y(t-r(t))\|_X^2 + \frac{7M_{1-\beta}^2 L_h^2 t^{2\beta}}{(1-k)\beta^2} \\
& \quad \cdot \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 + \frac{7M^2 L_f^2 t^2}{1-k} \\
& \quad \cdot \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 + \frac{7M^2 L_g^2 t}{1-k} \\
& \quad \cdot \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 \\
& \quad + \frac{7M^2 M_B^2 M_\omega^2 L_h^2 \|(-A)^{-\beta}\|^2 t^2}{1-k} \\
& \quad \cdot \mathbb{E} \|x(T-r(T)) - y(T-r(T))\|_X^2 \\
& \quad + \frac{7M^2 M_{1-\beta}^2 M_B^2 M_\omega^2 L_h^2 T^{2\beta} t^2}{(1-k)\beta^2} \\
& \quad \cdot \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 + \frac{7M^4 M_B^2 M_\omega^2 L_f^2 T^2 t^2}{1-k} \\
& \quad \cdot \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2 + \frac{7M^4 M_B^2 M_\omega^2 L_g^2 T t^2}{1-k} \\
& \quad \cdot \sup_{s \in [-\tau, T]} \mathbb{E} \|x(s) - y(s)\|_X^2.
\end{aligned} \tag{38}$$

Hence

$$\begin{aligned}
& \sup_{t \in [-\tau, T]} \mathbb{E} \|\pi(x)(t) - \pi(y)(t)\|_X^2 \\
& \leq b(t) \sup_{t \in [-\tau, T]} \mathbb{E} \|x(t) - y(t)\|_X^2,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
b(t) = & k + \frac{7M_{1-\beta}^2 L_h^2 t^{2\beta}}{(1-k)\beta^2} + \frac{7M^2 L_f^2 t^2}{1-k} + \frac{7M^2 L_g^2 t}{1-k} \\
& + \frac{7M^2 M_B^2 M_\omega^2 L_h^2 \|(-A)^{-\beta}\|^2 t^2}{1-k} \\
& + \frac{7M^2 M_{1-\beta}^2 M_B^2 M_\omega^2 L_h^2 T^{2\beta} t^2}{(1-k)\beta^2} \\
& + \frac{7M^4 M_B^2 M_\omega^2 L_f^2 T^2 t^2}{1-k} + \frac{7M^4 M_B^2 M_\omega^2 L_g^2 T t^2}{1-k}.
\end{aligned} \tag{40}$$

By condition (C.3), we know that $b(0) = k < 1$ then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ which shows that π is a contraction mapping in S_{T_1} . Then π has a unique fixed point in S_{T_1} , which is a unique mild solution of (1)

on $[-\tau, T_1]$. Repeating this procedure can be used to extend the mild solution to the entire interval $[-\tau, T]$. Obviously, $\pi(x)(T) = x_1$ which means that system (1) is controllable on $[-\tau, T]$. This completes the proof. $\square$

5. An Example

In this section, we provide an example to illustrate our main results. Consider the following NSDPDE with finite delays τ_1 , τ_2 , and τ_3 ($0 \leq \tau_i \leq \tau < \infty$, $i = 1, 2, 3$):

$$\begin{aligned}
& d[x(t, \xi) - H(t, x(t - \tau_1, \xi))] \\
& = \left[\frac{\partial^2}{\partial \xi^2} x(t, \xi) + \mu(t, \xi) + F(t, x(t - \tau_2, \xi)) \right] dt \\
& \quad + G(t, x(t - \tau_3, \xi)) dW(t) + \sigma(t) dB_Q^H(t), \tag{41}
\end{aligned}$$

$$0 \leq \xi \leq \pi, \quad t \in [0, T],$$

$$x(t, 0) = x(t, \pi) = 0, \quad t \in [0, T],$$

$$x(t, \xi) = \varphi(t, \xi), \quad t \in [-\tau, 0], \quad 0 \leq \xi \leq \pi,$$

where $W(t)$ is a cylindrical Brownian motion and $B_Q^H(t)$ is a cylindrical fractional Brownian motion in X .

Let $X = L^2([0, \pi])$ and define the operator $A : D(A) \subset X \rightarrow X$ by $A = \partial^2 / \partial \xi^2$ with the domain

$$\begin{aligned}
D(A) := & \{z \\
& \in X, z, z' \text{ are absolutely continuous on } z'' \\
& \in X, z(0) = z(\pi) = 0\}.
\end{aligned} \tag{42}$$

It is well known that an analytic semigroup $\{S(t)\}_{t \geq 0}$ is generated by the operator A in X satisfying $\|S(t)\|_X \leq e^{-t}$. Furthermore,

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, e_n \rangle e_n, \quad u \in D(A), \tag{43}$$

where $e_n(x) = \sqrt{2/\pi} \sin(nx)$ is a complete orthonormal set of eigenvectors of A . Then, the following properties hold:

- (i) If $y \in D(A)$, then $Ay = \sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n$.
- (ii) For each $y \in X$, $A^{-1/2}y = \sum_{n=1}^{\infty} (1/n) \langle y, e_n \rangle e_n$. In particular, $\|A^{-1/2}\|_X = 1$.
- (iii) The operator $A^{1/2}$ is given by

$$A^{1/2}y = \sum_{n=1}^{\infty} n \langle y, e_n \rangle e_n \tag{44}$$

on the space $D[A^{1/2}] = \{y(\cdot) \in X, \sum_{n=1}^{\infty} n \langle y, e_n \rangle e_n \Sigma\}$.

We assume that the following conditions hold:

- (1) Let $B : U \rightarrow X$ be a bounded linear operator defined by

$$Bu(t)(\xi) = \mu(t, \xi), \quad 0 \leq \xi \leq \pi, \quad u \in L^2([0, \pi], U). \tag{45}$$

(2) The operator $\omega : \mathbb{L}^2([0, \pi], U) \rightarrow X$ is given by

$$\omega u(\xi) = \int_0^T S(T-s) \mu(s, \xi) ds, \quad 0 \leq \xi \leq \pi, \quad (46)$$

has a bounded invertible operator ω^{-1} , and satisfies condition (C.7). For the construction of the operator ω and its inverse, see [26] for more details.

(3) There exist nonnegative constants $L_h < 1$, L_f , and L_g such that, for all $t \in [-\tau, T]$, $x, y \in X$, and $\xi \in [0, \pi]$,

$$\begin{aligned} & \|F(t, x(t, \xi)) - F(t, y(t, \xi))\|_X \\ & \leq L_f \|x(t, \xi) - y(t, \xi)\|_X, \\ & \|G(t, x(t, \xi)) - G(t, y(t, \xi))\|_X \\ & \leq L_g \|x(t, \xi) - y(t, \xi)\|_X, \\ & \|(-A)^{1/2} H(t, x(t, \xi)) - (-A)^{1/2} H(t, y(t, \xi))\|_X \\ & \leq L_h \|x(t, \xi) - y(t, \xi)\|_X. \end{aligned} \quad (47)$$

(4) There exist nonnegative constants C_f, C_g , and C_h , such that, for all $t \in [-\tau, T]$, $x \in X$, and $\xi \in [0, \pi]$,

$$\begin{aligned} & \|F(t, x(t, \xi))\|_X^2 \leq C_f (1 + \|x(t, \xi)\|_X^2), \\ & \|G(t, x(t, \xi))\|_X^2 \leq C_g (1 + \|x(t, \xi)\|_X^2), \\ & \|(-A)^{1/2} H(t, x(t, \xi))\|_X^2 \leq C_h (1 + \|x(t, \xi)\|_X^2). \end{aligned} \quad (48)$$

(5) The mapping $(-A)^{1/2} H$ is continuous in mean square on $[-\tau, T]$: for all $t \in [-\tau, T]$, $x \in X$, and $\xi \in [0, \pi]$, we have

$$\begin{aligned} & \lim_{t \rightarrow s} \mathbb{E} \|(-A)^{1/2} H(t, x(t, \xi)) \\ & - (-A)^{1/2} H(s, x(s, \xi))\|_X^2 = 0. \end{aligned} \quad (49)$$

(6) The mapping $\sigma : [0, T] \rightarrow \mathcal{L}_{Q^{(2)}}^0(Y_2; X)$ satisfies

$$\int_0^T \|\sigma(s)\|_{\mathcal{L}_{Q^{(2)}}^0}^2 ds < \infty, \quad \forall T > 0. \quad (50)$$

Define the operators F, G , and $H : \mathbb{R}^+ \times L^2([0, \pi]) \rightarrow L^2([0, \pi])$ by

$$\begin{aligned} & h(t, \varphi)(\xi) = H(t, \varphi(-\tau_1))(\xi) \\ & \quad \text{for } \xi \in [0, \pi], \varphi \in L^2([0, \pi]), \\ & f(t, \varphi)(\xi) = F(t, \varphi(-\tau_2))(\xi) \\ & \quad \text{for } \xi \in [0, \pi], \varphi \in L^2([0, \pi]), \\ & g(t, \varphi)(\xi) = G(t, \varphi(-\tau_3))(\xi) \\ & \quad \text{for } \xi \in [0, \pi], \varphi \in L^2([0, \pi]). \end{aligned} \quad (51)$$

If we put

$$\begin{aligned} & x(t)(\xi) = x(t, \xi), \quad t \in [0, T], \xi \in [0, \pi], \\ & x(t, \xi) = \varphi(t, \xi), \quad t \in [-\tau, 0], \xi \in [0, \pi], \end{aligned} \quad (52)$$

then problem (41) can be written in the abstract form

$$\begin{aligned} & d[x(t) + h(t, x(t - r(t)))] \\ & = [Ax(t) + f(t, x(t - \rho(t))) + Bu(t)] dt \\ & \quad + g(t, x(t - \eta(t))) dW(t) + \sigma(t) dB_Q^H(t), \end{aligned} \quad (53)$$

$$0 \leq t \leq T,$$

$$x(s) = \varphi(s), \quad -\tau \leq s \leq 0.$$

Thus all the conditions of Theorems 6 and 9 are fulfilled. Therefore, we conclude that the system (41) has a unique mild solution which is controllable on $[-\tau, T]$.

6. Conclusion

In this paper, by using the Banach fixed point theorem, we obtain some sufficient conditions ensuring the existence, uniqueness, and controllability of neutral stochastic delay partial differential equations driven by standard Brownian motion and fractional Brownian motion. In our next paper, we will explore the existence, uniqueness, and controllability problems with non-Lipschitz condition.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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