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Research Article

Explicit Pricing Formulas for European Option with Asset Exposed to Double Defaults Risk

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We derive analytical formulas for European call and put options on underlying assets that are exposed to double defaults risks which include exogenous counterparty default risk and endogenous default risk. The endogenous default risk leads the asset price to drop to zero and the exogenous counterparty default risk induces a drop in the asset price, but the asset can still be traded after this default time. A novel technique is developed to evaluate the European call and put options by first conditioning on the predefault and the postdefault time and then obtaining the unconditional analytic formulas for their price. We also compare the pricing results of our model with default-free option model and counterparty default risk option model.

1. Introduction

Over the past few decades, academic researchers and market practitioners have developed and adopted different models and techniques for pricing European option. The path-breaking work on option valuation was done by Black and Scholes [1] and Merton [2]; their works assumed that the absence of arbitrage opportunities and the asset price dynamics are governed by a Geometric Brownian Motion (GBM). For the option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion, a review of the literature can refer to Elliott et al. [3] and the references therein.

In the financial market, a counterparty default usually has important influences in various contexts. In terms of credit spreads, one observes in general a positive jump of the default intensity which is called the contagious jump (see Jarrow and Yu [4]). In terms of asset values for a firm, the default of a counterparty will generally induces a drop of its value process (see El Karoui et al. [5]). Jiao and Pham analyzed the impact of the single exogenous counterparty risk and the multiple exogenous counterparty risk on the optimal investment problem; for more detail refer to [6, 7]. In this paper, we study the impact of the double defaults risk, which includes exogenous counterparty default risk and endogenous default risk, on option pricing problem. In particular, we focus on the pricing of European option with the underlying asset being subject to double defaults risk such that the instantaneous loss of the asset at the exogenous counterparty default time and the asset price instantaneous become to zero at endogenous default time.

The explicit valuation of European options with assets exposed to exogenous counterparty default risk was partly given by Ma et al. [8]. Yan derived analytical formulas for lookback and barrier options on underlying assets that are subject to an exogenous counterparty risk (see Yan [9]). However, the derivation of the analytic formula for pricing European call and put options under the double defaults risk model has not been done in the previous literature. The main difficulty lies in the derivation of the distribution of the stock price at expire time T under the double defaults and the continuous trading of the underlying asset after the exogenous counterparty default time. We use the conditional density approach of default, which is particularly suitable to study what goes on after the default and was adopted by Jiao and Pham [6] for the optimal investment problem, to derive the explicit distribution of the stock price at expire time T and then obtain the analytic formulas for valuation of the European call and put options. We also compare the pricing results of double defaults risk model with Black and Scholes

[1] default-free option model and Ma et al. [8] counterparty default risk option model.

The rest of this paper is organized as follows. In the next section, we introduce the financial model and change from the actual probability measure P to risk-neutral probability measure $\widetilde{P}$ in model of financial market. In Section 3, we derive the distribution of the stock price at expire time T and then the formula for pricing European call and put options. Conclusions are given in the final section.

2. Model Setting and Change of Measure

In this section, we consider a financial market model with a risk asset (stock) subject to double default risks. We denote the stock by $(S_t)_{t\in[0,T]}$; the dynamic of the stock is affected by not only the possibility of the exogenous counterparty default but also the possibility of the endogenous default. However, this stock still exists and can be traded after the exogenous counterparty default.

Assume $(\Omega, \mathscr{G}, \mathbf{P})$ is a complete probability space satisfying the usual conditions. Let $(W_t)_{t\in[0,T]}$ be a Brownian motion with horizon $T < \infty$ on the probability space $(\Omega, \mathscr{G}, \mathbf{P})$ and denote by $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ the natural filtration of W. Let τ_1 and τ_2 be both almost surely nonnegative random variables on $(\Omega, \mathscr{G}, \mathbf{P})$, representing the stock of the exogenous counterparty default time and the endogenous default time, respectively. Then $(\mathscr{H}_t^1)_{t\in[0,T]}$ is defined by $\mathscr{H}_t^1 := \sigma(H_u^1 : u \le t)$, where $H_t^1 := \mathbf{1}_{\{\tau_1 \le t\}}$ which equals 0 if $\tau_1 > t$ and 1 otherwise, and denote $\mathbb{H}^1 = (\mathscr{H}_t^1)_{t\in[0,T]}$. Similarly, $(\mathscr{H}_t^2)_{t\in[0,T]}$ is defined by $\mathscr{H}_t^2 := \sigma(H_u^2 : u \le t)$, where $H_t^2 := \mathbf{1}_{\{\tau_2 \le t\}}$, and denote $\mathbb{H}^2 = (\mathscr{H}_t^2)_{t\in[0,T]}$. Denote by $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ the progressively enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$, representing the structure of information available for the investors over $[0,T]$. The market model is given by the following SDE (stochastic differential equation):

$$\frac{dS_t}{S_{t-}} = \mu_t dt + \sigma_t dW_t - \gamma_t d\mathrm{H}_t^1 - dH_t^2, \quad 0 \le t \le T, \tag{1}$$

where μ_t, σ_t, and γ_t are $\mathbb{G}$-predictable processes. μ_t and σ_t are drift rate and volatility rate of the stock S, respectively, and γ_t is the (percentage) loss on the stock price induced by the defaults of the counterparty. At default time τ_1, the stock price S is reduced by a percentage of γ_t. However, the stock price S falls to zero at default time τ_2. Denote $\widehat{\tau}_1 \equiv \min(\tau_1, \tau_2)$ and $\widehat{\tau}_2 \equiv \max(\tau_1, \tau_2)$, according to Pham [10]; any $\mathbb{G}$-predictable process $Y = (Y_t)_{t\ge 0}$ can be represented as

$$\begin{aligned} Y_t = {} & Y_t^0 \mathbf{1}_{\{t \le \widehat{\tau}_1\}} + Y_t^{1,1}(\tau_1, \gamma_1)\mathbf{1}_{\{\tau_1 < t \le \tau_2\}} \\ & + Y_t^{1,2}(\tau_2, \gamma_2)\mathbf{1}_{\{\tau_2 < t \le \tau_1\}} + Y_t^2(\tau_1, \tau_2, \gamma_1, \gamma_2)\mathbf{1}_{\{t > \widehat{\tau}_2\}}, \end{aligned} \tag{2}$$

where Y_t^0 is $\mathbb{F}$-predictable and $Y_t^{1,1}$, $Y_t^{1,2}$ are measurable with respect to $\mathscr{F}_t \otimes \mathscr{B}(R_+) \otimes \mathscr{B}(L)$ and Y_t^2 is measurable with respect to $\mathscr{F}_t \otimes \mathscr{B}(R_+^2) \otimes \mathscr{B}(L^2)$ for all $t \in [0,T]$, and $L = (-\infty, 1)$ represents the possibility of the value of γ_t. Let us define the following (mutually exclusive and exhaustive) events ordering the default times:

$$\begin{aligned} A &= T < \tau_1 \le \tau_2, \\ B &= T < \tau_2 \le \tau_1, \\ C &= \tau_2 \le \tau_1 \le T, \\ D &= \tau_2 \le T < \tau_1, \\ E &= \tau_1 \le T < \tau_2, \\ F &= \tau_1 \le \tau_2 \le T. \end{aligned} \tag{3}$$

Therefore, the dynamic of stock price process (1) can be decomposed by the following four situations.

Situation i. If the stock is in absence of any default in the life of the option, i.e., the default times satisfy $A \cup B$, then we have

$$dS_t^F = S_t^F \left(\mu_t^F dt + \sigma_t^F dW_t\right), \quad 0 \le t \le T. \tag{4}$$

Situation ii. If the default times satisfy $C \cup D$, then we have

$$\begin{aligned} dS_t^F &= S_t^F \left(\mu_t^F dt + \sigma_t^F dW_t\right), \quad 0 \le t < \tau_2, \\ S_t^{d_2}(\tau_2) &= 0, \quad \tau_2 \le t \le T. \end{aligned} \tag{5}$$

Situation iii. If the stock has only exogenous counterparty default in the life of the option, i.e., the default times satisfy E, then we obtain

$$\begin{aligned} dS_t^F &= S_t^F \left(\mu_t^F dt + \sigma_t^F dW_t\right), \quad 0 \le t < \tau_1, \\ dS_t^{d_1}(\tau_1) &= S_t^{d_1}(\tau_1)\left(\mu_t^{d_1}(\tau_1) dt + \sigma_t^{d_1}(\tau_1) dW_t\right), \\ &\qquad \tau_1 < t \le T, \\ S_{\tau_1}^{d_1}(\tau_1) &= S_{\tau_1 -}^F \left(1 - \gamma_t^F\right). \end{aligned} \tag{6}$$

Situation iv. If the stock has both endogenous default and exogenous counterparty default in the life of the option, and the exogenous default time is earlier than the endogenous default time, i.e., the default times satisfy F, then we obtain

$$\begin{aligned} dS_t^F &= S_t^F \left(\mu_t^F dt + \sigma_t^F dW_t\right), \quad 0 \le t < \tau_1, \\ dS_t^{d_1}(\tau_1) &= S_t^{d_1}(\tau_1)\left(\mu_t^{d_1}(\tau_1) dt + \sigma_t^{d_1}(\tau_1) dW_t\right), \\ &\qquad \tau_1 < t < \tau_2, \\ S_t^{d_{1,2}}(\tau_1, \tau_2) &= 0, \quad \tau_2 \le t \le T, \\ S_{\tau_1}^{d_1}(\tau_1) &= S_{\tau_1 -}^F \left(1 - \gamma_t^F\right), \end{aligned} \tag{7}$$

where μ_t^F, σ_t^F, γ_t^F, S_t^F are $\mathbb{F}$-adapted process and $\mu_t^{d_1}(\tau_1)$, $\sigma_t^{d_1}(\tau_1)$, $S_t^{d_1}(\tau_1)$ are $\mathscr{F}_t \otimes \mathscr{B}([0,t))$-measurable functions for all $t \in [0,T]$. When the counterparty default, the drift, and volatility coefficient (μ, σ) of the stock price switch from (μ^F, σ^F) to $(\mu_t^{d_1}(\tau_1), \sigma_t^{d_1}(\tau_1))$, the postdefault coefficients may depend on the default time τ_1. However, when the stock itself defaults, the drift and diffusion coefficients (μ, σ) of the stock price switch from (μ^F, σ^F) to $(0, 0)$ due to the stock price identically vanishing. Here for simplicity we assume that

$$\begin{aligned}\mu_t^F &= \mu_1,\\ \sigma_t^F &= \sigma_1,\\ \gamma_t^F &= \gamma,\\ \mu_t^{d_1}(\tau_1) &= \mu_2,\\ \sigma_t^{d_1}(\tau_1) &= \sigma_2,\end{aligned} \tag{8}$$

where μ_1, σ_1 are nonnegative constants, $\mu_t^{d_1}(\tau_1)$, $\sigma_t^{d_1}(\tau_1)$ are only deterministic functions of $\tau_1 \in R_+$, in Jiao and Pham [6], for example, $\mu_t^{d_1}(\tau) = \mu^F \cdot (\tau/T)$, $\sigma_t^{d_1}(\tau) = \sigma^F \cdot (2 - \tau/T)$, $\tau \in [0, T]$, which have meaningful economic interpretation. And then we assume that the distribution of $\gamma(\gamma < 1)$ is fixed. Moreover γ, τ_1, τ_2, and W_t are independent and τ_1, τ_2 are all the exponential variables with parameters λ_1, λ_2, respectively. For more details, refer to Jiao and Pham [6].

According to Bielecki and Rutkowski [11] or Elliott et al. [12], we can decompose $\gamma H_t = M_t + A_t$, where M_t is a martingale and $A_t = \int_0^{t\wedge\tau} \lambda m ds$ is a bounded variation process and $m = E(\gamma)$. Thus we obtain that $M_t^1 = \gamma H_t^1 - A_t^1 = \gamma H_t^1 - \int_0^{t\wedge\tau_1} \lambda_1 m ds = \gamma H_t^1 - \int_0^t \lambda_1 m \mathbf{1}_{\{\tau_1 \geq s\}} ds$ and $M_t^2 = H_t^2 - \int_0^t \lambda_2 \mathbf{1}_{\{\tau_2 \geq s\}} ds$ are all martingale.

Assume that r is a risk-free interest rate; let $D(t) = e^{-rt}$. Since H_t^1 and H_t^2 are pure jump process and $D(t)$ is continuous, $[D, S](t) = 0$. We now use Ito's product rule for jump process and model (1) to obtain

$$\begin{aligned}D(t)S(t) &= S(0) + \int_0^t D(u-)\,\mathrm{d}S(u) + \int_0^t S(u-)\,\mathrm{d}D(u) + [D,S](t) = S(0)\\ &+ \int_0^t D(u-)S(u-)(\mu_1 - r)\,\mathrm{d}u\\ &+ \int_0^t D(u-)S(u-)\sigma_1\,\mathrm{d}W_u^1\\ &- \int_0^t D(u-)S(u-)\gamma\,\mathrm{d}H_u^1 - \int_0^t D(u-)S(u-)\,\mathrm{d}H_u^2\\ &= S(0)\\ &+ \int_0^t D(u)S(u)\left(\mu_t - r - \lambda_1 m \mathbf{1}_{\{t\leq\tau_1\}} - \lambda_2 \mathbf{1}_{\{t\leq\tau_2\}}\right)\mathrm{d}u\\ &+ \int_0^t D(u)S(u)\sigma_1\,\mathrm{d}W_u^1 - \int_0^t D(u-)S(u-)\,\mathrm{d}M_u^1\\ &- \int_0^t D(u-)S(u-)\,\mathrm{d}M_u^2,\end{aligned} \tag{9}$$

which in differential form is

$$\begin{aligned}d(D(t)S(t)) &= D(t-)\,dS(t) + S(t-)\,dD(t) + dD(t)\,dS(t) = D(t-)S(t-)\\ &\cdot \left[(\mu_1 - r)\,dt + \sigma_1 dW_t^1 - \gamma dH_t^1 - dH_t^2\right] = D(t)\\ &\cdot S(t)\\ &\cdot \left[\left(\mu_t - r - \lambda_1 m \mathbf{1}_{\{t\leq\tau_1\}} - \lambda_2 \mathbf{1}_{\{t\leq\tau_2\}}\right)dt + \sigma_1 dW_t^1\right]\\ &- D(t-)S(t-)\,dM_t^1 - D(t-)S(t-)\,dM_t^2.\end{aligned} \tag{10}$$

Since both M_t^1 and M_t^2 are martingales and $D(t-)S(t-)$ is left continuous, the second term and last term of (10) are all martingales. In order to make discounted stock price be martingale, we would like to rewrite (10) as

$$\begin{aligned}d(D(t)S(t)) &= \sigma_1 D(t)S(t)\,d\widetilde{W_t}^1\\ &- D(t-)S(t-)\,dM_t^1\\ &- D(t-)S(t-)\,dM_t^2,\end{aligned} \tag{11}$$

where $d\widetilde{W_t}^1 = ((\mu_t - r - \lambda_1 m \mathbf{1}_{\{t\leq\tau_1\}} - \lambda_2 \mathbf{1}_{\{t\leq\tau_2\}})/\sigma_1)dt + dW_t^1 := \theta_t dt + dW_t^1$.

Let us define the $\mathbb{G}$-adapted process

$$\theta_t = \frac{\mu_t - r - \lambda_1 m \mathbf{1}_{\{t\leq\tau_1\}} - \lambda_2 \mathbf{1}_{\{t\leq\tau_2\}}}{\sigma_t}, \quad 0 \leq t \leq T. \tag{12}$$

By assuming $\mathbb{E}[\int_0^T (1/2)|\theta_t|^2 \mathrm{d}t] < \infty$, we define a probability measure $\widetilde{P}$ which is equivalent to P on $(\Omega, \mathscr{G})$ with Radon-Nikodym density

$$\frac{d\widetilde{P}}{dP} = Z_T^{\mathbb{G}} = \exp\left\{-\int_0^T \theta_t \mathrm{d}W_t - \frac{1}{2}\int_0^T \theta_t^2 \mathrm{d}t\right\}, \tag{13}$$

under which, by Girsanov's theorem, $\widetilde{W_t} = W_t + \int_0^t \theta_u du$ is a $(\widetilde{P}, \mathscr{G})$-Brownian motion. And thus we can rewrite (1) as follows:

$$\begin{aligned}dS_t = S_{t-}&\left[\left(r + \lambda_1 m \mathbf{1}_{\{t\leq\tau_1\}} + \lambda_2 \mathbf{1}_{\{t\leq\tau_2\}}\right)dt + \sigma_t d\widetilde{W}_t\right.\\ &\left. - \gamma_t dH_t^1 - dH_t^2\right], \quad 0 \leq t \leq T.\end{aligned} \tag{14}$$

That is, by changing measure, the four situations to decompose the stock price S_t under the physical measure P can be transformed into the corresponding following four forms under the equivalent martingale measure $\widetilde{P}$.

Situation I. If the stock is in absence of any default in the life of the option, i.e., the default times satisfy $A \cup B$, then we have

$$dS_t^F = S_t^F\left[(r + \lambda_1 m + \lambda_2)\,dt + \sigma_1 d\widetilde{W_t}\right], \quad 0 \leq t \leq T. \tag{15}$$

Situation II. If the default times satisfy $C \cup D$, then we obtain

$$\begin{aligned} dS_t^F &= S_t^F\left[\left(r+\lambda_1 m+\lambda_2\right)dt+\sigma_1 d\widetilde{W}_t\right], \\ & \qquad 0 \le t < \tau_2, \\ S_t^{d_2}\left(\tau_2\right) &= 0, \quad \tau_2 \le t \le T. \end{aligned} \tag{16}$$

Situation III. If the stock has only exogenous counterparty default in the life of the option, i.e., the default times satisfy E, then we have

$$\begin{aligned} dS_t^F &= S_t^F\left[\left(r+\lambda_1 m+\lambda_2\right)dt+\sigma_1 d\widetilde{W}_t\right], \\ & \qquad 0 \le t < \tau_1, \\ dS_t^{d_1}\left(\tau_1\right) &= S_t^{d_1}\left(\tau_1\right)\left[\left(r+\lambda_2\right)dt+\sigma_2 d\widetilde{W}_t\right], \\ & \qquad \tau_1 < t \le T, \\ S_{\tau_1}^{d_1}\left(\tau_1\right) &= S_{\tau_1 -}^F\left(1-\gamma\right). \end{aligned} \tag{17}$$

Situation IV. If the stock has both endogenous default and exogenous counterparty default in the life of the option and the exogenous default time is earlier than the endogenous default time, i.e., the default times satisfy F, then we obtain

$$\begin{aligned} dS_t^F &= S_t^F\left[\left(r+\lambda_1 m+\lambda_2\right)dt+\sigma_1 d\widetilde{W}_t\right], \\ & \qquad 0 \le t < \tau_1, \\ dS_t^{d_1}\left(\tau_1\right) &= S_t^{d_1}\left(\tau_1\right)\left[\left(r+\lambda_2\right)dt+\sigma_2 d\widetilde{W}_t\right], \\ & \qquad \tau_1 < t < \tau_2, \\ S_t^{d_{1,2}}\left(\tau_1,\tau_2\right) &= 0, \quad \tau_2 \le t \le T, \\ S_{\tau_1}^{d_1}\left(\tau_1\right) &= S_{\tau_1 -}^F\left(1-\gamma\right). \end{aligned} \tag{18}$$

In practice, we may assume γ is a discrete random variable to simplify the computation; in what follows, we assume that γ takes value γ_i with probability p_i for $i = 1, 2, 3$, where $0 < \gamma_1 < 1$ (loss), $\gamma_2 = 0$ (no change), and $\gamma_3 < 0$ (gain).

3. Analytic Formula for Pricing European Options

Consider a European option, expiring at time T, with strike price K. In this section, we derive an analytical formula for pricing this option, whose payoff is the difference between the stock price at expiration T and the strike price K. Firstly, we need to compute the distribution function of random variable S_T and obtain the following lemma.

Lemma 1. *If the dynamic of stock price process follows model (1), then the distribution function of the stock price at expire time T is given by*

$$\begin{aligned} F(S) &= e^{-(\lambda_1+\lambda_2)T}\Phi\left(\frac{\ln\left(S/S_0\right)-a(T)}{b(T)}\right)+\left(1\right. \\ &\quad \left.-e^{-\lambda_2 T}\right)+e^{-\lambda_2 T}\sum_{i=1}^{3}p_i \\ &\quad \cdot\int_0^T \lambda_1 e^{-\lambda_1 t}\Phi\left[\frac{1}{b(t)}\left(\ln\left(\frac{S}{S_0\left(1-\gamma_i\right)}\right)\right.\right. \\ &\quad \left.\left.-a(t)\right)\right]\mathrm{d}t, \end{aligned} \tag{19}$$

with $a(t) = (r+\lambda_1 m+\lambda_2-\sigma_1^2/2)t + (r+\lambda_2-\sigma_2^2/2)(T-t)$, $b(t) = \sqrt{\sigma_1^2 t+\sigma_2^2(T-t)}$, and Φ being the standard normal distribution function.

Proof. Let $S \ge 0$; according to the definition of distribution function, we have

$$\begin{aligned} F(S) &= \widetilde{P}\left(S_T \le S\right) = \widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\right] \\ &= \widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{A\cup B\cup C\cup D\cup E\cup F\}}\right] \\ &= \widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{A\cup B\}}\right]+\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{C\cup D\}}\right] \\ &\quad +\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{E\}}\right]+\widetilde{E}\left[\mathbf{1}_{S_T\le S}\mathbf{1}_{\{F\}}\right]. \end{aligned} \tag{20}$$

If the default times satisfy Situation I, then the dynamic of stock price process satisfies the stochastic differential equation (15). By Ito's lemma and (15), we have

$$S_T^F = S_0 e^{(r+m\lambda_1+\lambda_2-\sigma_1^2/2)T+\sigma_1\widetilde{W}_T}, \tag{21}$$

and thus

$$\begin{aligned} &\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{A\cup B\}}\right] = \widetilde{E}\left[\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{\tau_1>T,\tau_2>T\}}\right]\mid\tau_1,\tau_2\right] \\ &\quad = \int_T^{+\infty}\lambda_2 e^{-\lambda_2 u}\int_T^{+\infty}\lambda_1 e^{-\lambda_1 t}\widetilde{E}\left[\mathbf{1}_{\{S_T^F\le S\}}\mid\tau_1=t,\tau_2\right. \\ &\quad \left.=u\right]\mathrm{d}t\mathrm{d}u = \int_T^{+\infty}\lambda_2 e^{-\lambda_2 u}\int_T^{+\infty}\lambda_1 e^{-\lambda_1 t}\widetilde{P}\left(S_T^F\right. \\ &\quad \left.\le S\right)\mathrm{d}t\mathrm{d}u \\ &\quad = e^{-(\lambda_1+\lambda_2)T}\Phi\left(\frac{\ln\left(S/S_0\right)-\left(r+\lambda_1 m+\lambda_2-\sigma_1^2/2\right)T}{\sigma_1\sqrt{T}}\right). \end{aligned} \tag{22}$$

If the stock has endogenous default in the life of the option, i.e., the default times satisfy Situations II and IV, then the price of the stock at expiration T is zero, and thus we obtain

$$\begin{aligned} &\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{C\cup D\}}\right] = \widetilde{E}\left[\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{\tau_1\le T,\tau_2\le T\}}\right]\mid\tau_1,\tau_2\right] \\ &\quad = \int_0^T\lambda_2 e^{-\lambda_2 u}\int_0^T\lambda_1 e^{-\lambda_1 t}\widetilde{E}\left[\mathbf{1}_{\{0\le S\}}\mid\tau_1=t,\tau_2\right. \\ &\quad \left.=u\right]\mathrm{d}t\mathrm{d}u = \int_0^T\lambda_2 e^{-\lambda_2 u}\mathrm{d}u\int_0^T\lambda_1 e^{-\lambda_1 t}\mathrm{d}t \\ &\quad = \left(1-e^{-\lambda_1 T}\right)\left(1-e^{-\lambda_2 T}\right), \end{aligned} \tag{23}$$

$$\widetilde{E}\left[\mathbf{1}_{S_T\le S}\mathbf{1}_{\{F\}}\right] = \widetilde{E}\left[\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{\tau_2\le T<\tau_1\}}\right] \mid \tau_1,\tau_2\right] = \int_0^T \lambda_2 e^{-\lambda_2 u}\int_T^{+\infty}\lambda_1 e^{-\lambda_1 t}\widetilde{E}\left[\mathbf{1}_{\{0\le S\}} \mid \tau_1 = t,\tau_2 = u\right] \mathrm{d}t\mathrm{d}u = \int_0^T \lambda_2 e^{-\lambda_2 u}\mathrm{d}u \cdot \int_T^{+\infty}\lambda_1 e^{-\lambda_1 t}\mathrm{d}t = e^{-\lambda_1 T}\left(1-e^{-\lambda_2 T}\right). \quad (24)$$

If the default times satisfy Situation III, then the dynamic of stock price process satisfies the stochastic differential equation (17). By Ito's lemma and (17), we can obtain

$$S_T^{d_1}(\tau_1) = S_{\tau_1}^{d_1}(\tau_1)\, e^{(r+\lambda_2-\sigma_2^2/2)(T-\tau_1)+\sigma_2(\widetilde{W}_T-\widetilde{W}_{\tau_1})} = S_0(1-\gamma) \cdot e^{(r+\lambda_1 m+\lambda_2-\sigma_1^2/2)\tau_1+(r+\lambda_2-\sigma_2^2/2)(T-\tau_1)+\sigma_1\widetilde{W}_{\tau_1}+\sigma_2(\widetilde{W}_T-\widetilde{W}_{\tau_1})}, \quad (25)$$

and thus

$$\begin{aligned}\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{E\}}\right] &= \widetilde{E}\left[\widetilde{E}\left[\mathbf{1}_{\{S_T\le S\}}\mathbf{1}_{\{\tau_1\le T<\tau_2\}}\right] \mid \tau_1,\tau_2\right] = \int_T^{+\infty}\lambda_2 e^{-\lambda_2 u}\int_0^T \lambda_1 e^{-\lambda_1 t}\widetilde{E}\left[\mathbf{1}_{\{S_T^{d_1}(\tau_1)\le S\}} \mid \tau_1 = t,\tau_2 = u\right]\mathrm{d}t\mathrm{d}u \\ &= \int_T^{+\infty}\lambda_2 e^{-\lambda_2 u}\int_0^T \lambda_1 e^{-\lambda_1 t}\widetilde{P}\left(S_T^{d_1}(\tau_1)\le S\right)\mathrm{d}t\mathrm{d}u = \int_T^{+\infty}\lambda_2 e^{-\lambda_2 u}\int_0^T \lambda_1 e^{-\lambda_1 t}\widetilde{E}\left[\widetilde{P}\left(S_T^{d_1}(\tau_1)\le S \mid \gamma\right)\right]\mathrm{d}t\mathrm{d}u \\ &= \int_T^{+\infty}\lambda_2 e^{-\lambda_2 u}\int_0^T \lambda_1 e^{-\lambda_1 t}\widetilde{E}\left[\Phi\left(\frac{1}{b(t)}\left(\ln\left(\frac{S}{S_0(1-\gamma)}\right)-a(t)\right)\right) \mid \gamma\right]\mathrm{d}t\mathrm{d}u \\ &= e^{-\lambda_2 T}\sum_{i=1}^{3}p_i\int_0^T \lambda_1 e^{-\lambda_1 t}\Phi\left[\frac{1}{b(t)}\left(\ln\left(\frac{S}{S_0(1-\gamma_i)}\right)-a(t)\right)\right]\mathrm{d}t,\end{aligned} \quad (26)$$

where $a(t) = (r+\lambda_1 m+\lambda_2-\sigma_1^2/2)t + (r+\lambda_2-\sigma_2^2/2)(T-t)$, $b(t) = \sqrt{\sigma_1^2 t+\sigma_2^2(T-t)}$, and Φ being the standard normal distribution function. Therefore, combining equalities (22)–(26), we can obtain that the distribution function of S_T under model (1) is (19). □

Combining the distribution function of the stock price at expire time T (19) and the following identity

$$\int_0^\infty (S-K)^+ d\Phi\left(\frac{1}{B}\left(\ln\left(\frac{S}{C}\right)-A\right)\right) = Ce^{A+B^2/2}\Phi(x_0+B) - K\Phi(x_0), \quad (27)$$

where A is a constant, B, C, K are all positive constants, and $x_0 = (1/B)(A-\ln(K/C))$, we can compute the value of a call option at time 0 under model (1), summarizing to the following theorem.

Theorem 2. *The risk-neutral price of the European call option at time 0 under model (1) is given by*

$$\begin{aligned}C(0,S_0) &= S_0 e^{-(1-m)\lambda_1 T}\Phi(\delta_0+b(T)) - Ke^{-(r+\lambda_1+\lambda_2)T}\Phi(\delta_0) + e^{-(r+\lambda_2)T}\sum_{i=1}^{3}p_i \\ &\quad\cdot\int_0^T \lambda_1 e^{-\lambda_1 t}\left[S_0(1-\gamma_i)e^{a(t)+b(t)^2/2}\Phi(\delta_i(t)+b(t)) - K\Phi(\delta_i(t))\right]\mathrm{d}t,\end{aligned} \quad (28)$$

with $\delta_0 = (1/b(T))(a(T)-\ln(K/S_0))$ *and* $\delta_i(t) = (1/b(t))(a(t)-\ln(K/S_0(1-\gamma_i)))$ *for* $i = 1,2,3$.

Proof. According to the distribution function of S_T and noticing the identity $\int_0^{+\infty}(S-K)^+\mathrm{d}(1-e^{-\lambda_2 T}) = 0$, we have

$$\begin{aligned}C(0,S_0) &= e^{-rT}\widetilde{E}\left[(S_T-K)^+\right] = e^{-rT}\int_0^{+\infty}(S-K)^+\,\mathrm{d}F(S) \\ &= e^{-rT}\int_0^{+\infty}(S-K)^+\,\mathrm{d}\left[e^{-(\lambda_1+\lambda_2)T}\Phi\left(\frac{\ln(S/S_0)-a(T)}{b(T)}\right)\right] \\ &\quad + e^{-rT}\int_0^{+\infty}(S-K)^+\cdot\mathrm{d}\left\{e^{-\lambda_2 T}\sum_{i=1}^{3}p_i\int_0^T \lambda_1 e^{-\lambda_1 t}\Phi\left[\frac{1}{b(t)}\left(\ln\left(\frac{S}{S_0(1-\gamma_i)}\right)-a(t)\right)\right]\mathrm{d}t\right\} \equiv I_1+I_2.\end{aligned} \quad (29)$$

It follows from formula (27) that the first term in (29) is

$$I_1 = S_0 e^{-(1-m)\lambda_1 T}\Phi(\delta_0 + b(T)) - Ke^{-(r+\lambda_1+\lambda_2)T}\Phi(\delta_0), \tag{30}$$

and the second term in (29) is

$$I_2 = e^{-(r+\lambda_2)T}\sum_{i=1}^{3} p_i \int_0^T \lambda_1 e^{-\lambda_1 t}\left[\int_0^{+\infty}(S-K)^+ d\Phi\left(\frac{1}{b(t)}\left(\ln\left(\frac{S}{S_0(1-\gamma_i)}\right)-a(t)\right)\right)\right]\mathrm{d}t = e^{-(r+\lambda_2)T}\sum_{i=1}^{3} p_i \cdot \int_0^T \lambda_1 e^{-\lambda_1 t}\left[S_0(1-\gamma_i)e^{a(t)+b^2(t)/2}\Phi(\delta_i + b(t)) - Ke^{-(r+\lambda_1+\lambda_2)T}\Phi(\delta_i)\right]\mathrm{d}t, \tag{31}$$

where $\delta_0 = (1/b(T))(a(T) - \ln(K/S_0))$ and $\delta_i(t) = (1/b(t))(a(t) - \ln(K/S_0(1-\gamma_i)))$ for $i = 1, 2, 3$. By (30) and (31), we can obtain (28), immediately. Thus the proof of Theorem 2 is completed. □

Remark 3. (1) If $\lambda_2 = 0$, then the risk-neutral price of the European call option at time 0 under model (1) becomes

$$\begin{aligned} C(0, S_0) &= S_0 e^{-(1-m)\lambda_1 T}\Phi(\widetilde{\delta}_0 + b(T)) \\ &\quad - Ke^{-(r+\lambda_1)T}\Phi(\widetilde{\delta}_0) + e^{-rT}\sum_{i=1}^{3} p_i \\ &\quad \cdot \int_0^T \lambda_1 e^{-\lambda_1 t}\left[S_0(1-\gamma_i)\right. \\ &\quad \left.\cdot e^{a(t)+b(t)^2/2}\Phi(\widetilde{\delta}_i(t) + b(t)) - K\Phi(\widetilde{\delta}_i(t))\right]\mathrm{d}t, \end{aligned} \tag{32}$$

where $\widetilde{\delta}_0 = (1/b(T))(a(T) - \ln(K/S_0))$ and $\widetilde{\delta}_i(t) = (1/b(t))(a(t) - \ln(K/S_0(1-\gamma_i)))$, with $a(t) = (r + \lambda_1 m - \sigma_1^2/2)t + (r - \sigma_2^2/2)(T-t)$ and $b(t) = \sqrt{\sigma_1^2 t + \sigma_2^2(T-t)}$. It can be seen from Figure 1 that the price of call option decreases along with the reduction of default intensity λ_2. Figure 1 also shows that the value of call option at time 0 under this model is the same as the one at time 0 with the stock exposed to counterparty risk (refer to Ma et al. [8]); i.e., the double defaults risk model becomes counterparty default risk model when $\lambda_2 = 0$. This is because the default intensity $\lambda_2 = 0$ means that endogenous default risk has not occurred during the life of the option.

(2) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then the risk-neutral price of the European call option at time 0 under model (1) becomes

$$C(0, S_0) = S_0\Phi(\delta^+) - Ke^{-rT}\Phi(\delta^-), \tag{33}$$

with $\delta^+ = ((r + \sigma_1^2/2)T - \ln(K/S_0))/\sigma_1\sqrt{T}$ and $\delta^- = ((r - \sigma_1^2/2)T - \ln(K/S_0))/\sigma_1\sqrt{T}$. In the same way, it follows from Figure 2 that the double defaults risk model becomes standard Black-Scholes model when $\lambda_1 = 0$ and $\lambda_2 = 0$.

Theorem 4. *The risk-neutral price of the European put option at time 0 under model (1) can be obtained by put-call parity as follows:*

$$\begin{aligned} P(0, S_0) &= C(0, S_0) + Ke^{-rT} - S_0 e^{-(1-m)\lambda_1 T} \\ &\quad - S_0\lambda_1 e^{-(r+\lambda_2)T}\sum_{i=1}^{3} p_i(1-\gamma_i) \\ &\quad \cdot \left(\int_0^T e^{a(t)+b^2(t)/2-\lambda_1 t}\mathrm{d}t\right). \end{aligned} \tag{34}$$

Proof. According to the following identity,

$$(K - S_T)^+ - (S_T - K)^+ = K - S_T, \tag{35}$$

we have

$$\begin{aligned} P(0, S_0) &= e^{-rT}\widetilde{E}\left[(K - S_T)^+\right] = e^{-rT}\widetilde{E}\left[(S_T - K)^+\right] \\ &\quad + e^{-rT}\widetilde{E}\left[(K - S_T)\right] = e^{-rT}\widetilde{E}\left[(K - S_T)\right] + Ke^{-rT} \\ &\quad - e^{-rT}\int_0^{+\infty}(S-K)\,\mathrm{d}F(S) = C(0, S_0) + Ke^{-rT} \\ &\quad - e^{-(r+\lambda_1+\lambda_2)T}\int_0^{+\infty} S\mathrm{d}\left[\Phi\left(\frac{\ln(S/S_0) - a(T)}{b(T)}\right)\right] \\ &\quad - e^{-(r+\lambda_2)T}\int_0^{+\infty} S\mathrm{d}\left[\sum_{i=1}^{3} p_i\right. \\ &\quad \left.\cdot \int_0^T \lambda_1 e^{-\lambda_1 t}\Phi\left(\frac{1}{b(t)}\left(\ln\left(\frac{S}{S_0(1-\gamma_i)}\right) - a(t)\right)\right)\mathrm{d}t\right] \\ &\equiv C(0, S_0) + Ke^{-rT} - I_3 - I_4. \end{aligned} \tag{36}$$

Let $(\ln(S/S_0) - a(T))/b(T) = t$, then the third term in (36) can be calculated as

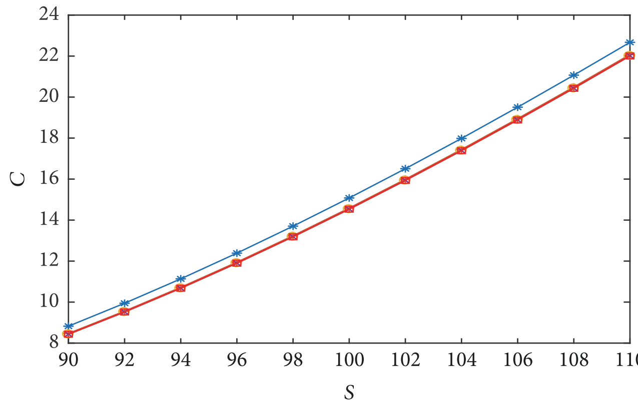

FIGURE 1: Call option prices versus stock prices for double defaults and counterparty default models with $\gamma_1 = 0.5$, $\gamma_2 = 0$, $\gamma_3 = -0.2$, $p_1 = 0.3$, $p_2 = 0.5$, $p_3 = 0.2$, $T = 1$, $r = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, and $K = 100$.

$$\begin{aligned} I_3 &= e^{-(r+\lambda_1+\lambda_2)T} \int_{-\infty}^{+\infty} S_0 e^{b(T)t+a(t)} \mathrm{d}\Phi(t) \\ &= e^{-(r+\lambda_1+\lambda_2)T} \int_{-\infty}^{+\infty} S_0 e^{b(T)t+a(t)} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \mathrm{d}t \\ &= e^{-(r+\lambda_1+\lambda_2)T} S_0 e^{a(T)+b(T)^2/2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(t-b(T))^2/2} \mathrm{d}t \\ &= e^{-(r+\lambda_1+\lambda_2)T} S_0 e^{(r+\lambda_1 m+\lambda_2-\sigma_1^2/2)T+(\sigma_1^2/2)T} \\ &= S_0 e^{-(1-m)\lambda_1 T}, \end{aligned} \tag{37}$$

and the last term in (36) can be calculated as

$$\begin{aligned} I_4 &= e^{-(r+\lambda_2)T} \sum_{i=1}^{3} p_i \int_0^T \lambda_1 e^{-\lambda_1 t} \left[\int_0^{+\infty} S \mathrm{d}\Phi \left(\frac{1}{b(t)} \left(\ln \left(\frac{S}{S_0(1-\gamma_i)} \right) - a(t) \right) \right) \right] \mathrm{d}t \\ &= e^{-(r+\lambda_2)T} \sum_{i=1}^{3} p_i \int_0^T \lambda_1 e^{-\lambda_1 t} \left[S_0 (1-\gamma_i) e^{a(t)+b^2(t)/2} \right] \mathrm{d}t = S_0 \lambda_1 e^{-(r+\lambda_2)T} \sum_{i=1}^{3} p_i (1-\gamma_i) \left(\int_0^T e^{a(t)+b^2(t)/2-\lambda_1 t} \mathrm{d}t \right). \end{aligned} \tag{38}$$

Incorporating (37) and (38) into (36), we obtain formula (34). Thus the proof of this theorem is completed. □

4. Conclusions

In this paper, we have derived explicit analytical formulas for the price of European call and put options when the underlying asset is subject to double defaults risks. The external counterparty default risk induces a drop in the price of the stock, and the stock price drops to zero when the stock itself defaults. Double defaults risks cause difficulty in deriving the distribution function of the stock price at expire time T. The conditional density approach (see Jiao and Pham [6]) is utilized to overcome the difficulty and derive the formulas for the price of European option. Many questions remain in option pricing with double defaults risks, for example, the prices of path-dependent options whose payoffs depend on the path of the underlying asset under this model and analytic formulas for the options with two underlying assets exposed to loop contagion risk. We leave these and other questions to the future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

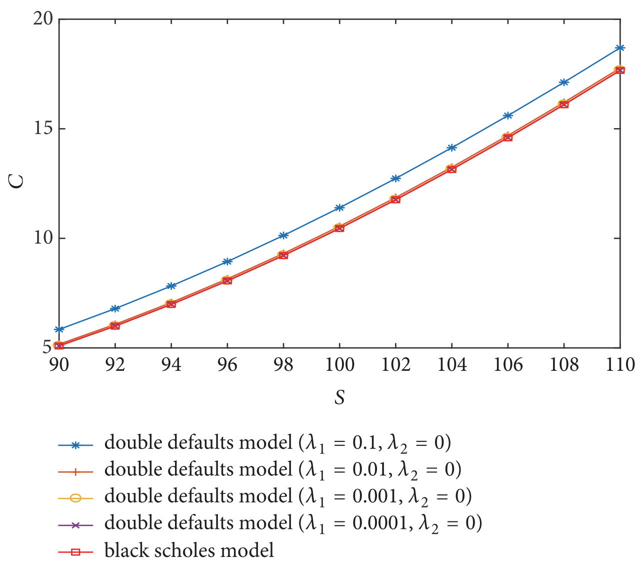

Figure 2: Call option prices versus stock prices for double defaults and Black-Scholes models with $\gamma_1 = 0.5$, $\gamma_2 = 0$, $\gamma_3 = -0.2$, $p_1 = 0.3$, $p_2 = 0.5$, $p_3 = 0.2$, $T = 1$, $r = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, and $K = 100$.

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