Research Article

Eventually Periodic Solutions of a Max-Type System of Difference Equations of Higher Order

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We study in this paper the following max-type system of difference equations of higher order: $x_n = \max\{A, y_{n-k}/x_{n-1}\}$ and $y_n = \max\{B, x_{n-k}/y_{n-1}\}$, $n \in \{0, 1, 2, \dots\}$, where $A \geq B > 0$, $k \geq 1$, and the initial conditions $x_{-k}, y_{-k}, x_{-k+1}, y_{-k+1}, \dots, x_{-1}, y_{-1} \in (0, +\infty)$. We show that (1) if $AB > 1$, then every solution of the above system is periodic with period 2 eventually. (2) If $AB = 1 > B$, then every solution of the above system is periodic with period $2k$ or 2 eventually. (3) If $A = B = 1$ or $AB < 1$, then the above system has a solution which is not periodic eventually.

1. Introduction

Our purpose in this paper is to study eventual periodicity of the following max-type system of difference equations of higher order:

$$\begin{aligned} x_n &= \max \left\{ A, \frac{y_{n-k}}{x_{n-1}} \right\}, \\ y_n &= \max \left\{ B, \frac{x_{n-k}}{y_{n-1}} \right\}, \end{aligned} \quad (1)$$

$$n \in \mathbb{N}_0 \equiv \{0, 1, \dots\}$$

with

$$A \geq B > 0, \quad (2)$$

where $k \in \mathbb{N} \equiv \{1, 2, \dots\}$ and the initial conditions $x_{-k}, y_{-k}, x_{-k+1}, y_{-k+1}, \dots, x_{-1}, y_{-1} \in \mathbb{R}_+ \equiv (0, +\infty)$.

Recently, there has been a great interest in studying max-type systems of difference equations. For example, Fotiadis and Papaschinopoulos [1] studied the following max-type system of difference equations:

$$\begin{aligned} x_n &= \max \left\{ A, \frac{y_{n-1}}{x_{n-2}} \right\}, \\ y_n &= \max \left\{ B, \frac{x_{n-1}}{y_{n-2}} \right\}, \end{aligned} \quad (3)$$

$$n \in \mathbb{N}_0,$$

and they showed that every positive solution of (3) is eventually periodic.

In 2012, Stević [2] obtained in an elegant way the general solution to the following max-type system of difference equations:

$$\begin{aligned} x_{n+1} &= \max \left\{ \frac{A}{x_n}, \frac{y_n}{x_n} \right\}, \\ y_{n+1} &= \max \left\{ \frac{A}{y_n}, \frac{x_n}{y_n} \right\}, \end{aligned} \quad (4)$$

$$n \in \mathbb{N}_0,$$

for the case $x_0, y_0 \geq A > 0$ and $y_0/x_0 \geq \max\{A, 1/A\}$.

In [3], we have recently studied the following max-type system of difference equations:

$$\begin{aligned} x_n &= \max \left\{ \frac{1}{x_{n-m}}, \min \left\{ 1, \frac{A}{y_{n-r}} \right\} \right\}, \\ y_n &= \max \left\{ \frac{1}{y_{n-m}}, \min \left\{ 1, \frac{B}{x_{n-t}} \right\} \right\}, \end{aligned} \quad (5)$$

$$n \in \mathbb{N}_0,$$

where $A, B \in \mathbb{R}_+$, $m, r, t \in \mathbb{N}$, and the initial conditions $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \dots, x_{-1}, y_{-1} \in \mathbb{R}_+$ with $d = \max\{m, r, t\}$,

and we showed that every positive solution of (5) is eventually periodic with period $2m$.

When $m = r = t = 1$ and $A = B$, (5) reduces to the max-type system of difference equations

$$\begin{aligned} x_n &= \max \left\{ \frac{1}{x_{n-1}}, \min \left\{ 1, \frac{A}{y_{n-1}} \right\} \right\}, \\ y_n &= \max \left\{ \frac{1}{y_{n-1}}, \min \left\{ 1, \frac{A}{x_{n-1}} \right\} \right\}, \end{aligned} \quad (6)$$

$n \in \mathbb{N}_0$.

Yazlik et al. [4] in 2015 obtained in an elegant way the general solution of (6).

In 2014, Stević et al. [5] investigated the behavior of positive solutions of the following max-type system of difference equations:

$$\begin{aligned} x_n &= \max \left\{ B, \frac{y_{n-1}^p}{z_{n-2}^p} \right\}, \\ y_n &= \max \left\{ B, \frac{z_{n-1}^p}{x_{n-2}^p} \right\}, \\ z_n &= \max \left\{ B, \frac{x_{n-1}^p}{y_{n-2}^p} \right\}, \end{aligned} \quad (7)$$

$n \in \mathbb{N}_0$,

where $B, p \in \mathbb{R}_+$. It is proved that system (7) is permanent when $p \in (0, 4)$.

For some other results on solutions of many max-type difference equations and systems, such as eventual periodicity, the boundedness character, and attractivity, see, for example, [6–20] and the related references therein.

2. Main Results and Proofs

In this section, we study the eventual periodicity of solutions of system (1). Let $\{(x_n, y_n)\}_{n \geq -k}$ be a solution of (1) with the initial conditions $x_{-k}, y_{-k}, x_{-k+1}, y_{-k+1}, \dots, x_{-1}, y_{-1} \in \mathbb{R}_+$. From (1), it immediately follows that, for any $n \in \mathbb{N}_0$,

$$x_n \geq A, \quad (8)$$

$$y_n \geq B. \quad (9)$$

Lemma 1. *If $x_n = A$ eventually, then y_n is a periodic sequence with period 2 eventually. If $y_n = B$ eventually, then x_n is a periodic sequence with period 2 eventually.*

Proof. Assume that $x_n = A$ eventually. By (1), we see

$$y_n = \max \left\{ B, \frac{A}{y_{n-1}} \right\} \quad \text{eventually}, \quad (10)$$

which implies

$$y_n y_{n-1} \geq A \quad \text{eventually}. \quad (11)$$

We have from (9), (10), and (11) that

$$B \leq y_n = \max \left\{ B, \frac{A y_{n-2}}{y_{n-1} y_{n-2}} \right\} \leq \max \{B, y_{n-2}\} = y_{n-2} \quad (12)$$

eventually.

Then, for any $i \in \{0, 1\}$, y_{2n+i} is eventually nonincreasing.

We claim that, for every $i \in \{0, 1\}$, y_{2n+i} is a constant sequence eventually. Assume on the contrary that, for some $i \in \{0, 1\}$, y_{2n+i} is not a constant sequence eventually. Then, by (10) and (12), we see that there exists a sequence of positive integers $k_1 < k_2 < \dots$ such that, for any $n \in \mathbb{N}$,

$$B < y_{2k_{n+1}+i} = \frac{A}{y_{2k_{n+1}+i-1}} < y_{2k_n+i} = \frac{A}{y_{2k_n+i-1}}, \quad (13)$$

which implies $y_{2k_{n+1}+i-1} > y_{2k_n+i-1}$ for any $n \in \mathbb{N}$. This is a contradiction. Thus, y_n is a periodic sequence with period 2 eventually. The other case is treated similarly/dually, so we omit the detail. The proof is complete. $\square$

Lemma 2. *If $AB \geq 1$, then $x_{2(n+1)k+i} \leq x_{2nk+i}$ for any $n \geq k+1$ and $i \in \mathbb{N}_0$.*

Proof. Assume that $AB \geq 1$. Then, for $2(n+1)k+i \geq k+1$, by (1) and (8), we have

$$\begin{aligned} A &\leq x_{2(n+1)k+i} \\ &= \max \left\{ A, \frac{B}{x_{2(n+1)k+i-1}}, \frac{x_{2nk+i}}{x_{2(n+1)k+i-1} y_{2(n+1)k+i-1}} \right\} \end{aligned} \quad (14)$$

and by (2), (8), and (9), we have

$$\frac{B}{x_{2(n+1)k+i-1}} \leq \frac{B}{A} \leq 1 \leq A, \quad (15)$$

$$x_{2(n+1)k+i-1} y_{2(n+1)k+i-1} \geq AB \geq 1.$$

Thus,

$$x_{2(n+1)k+i} \leq \max \{A, x_{2nk+i}\} = x_{2nk+i} \quad \text{eventually}. \quad (16)$$

The proof is complete. $\square$

Theorem 3. *Let $AB > 1$. Then, $x_n = A$ eventually and y_n is a periodic sequence with period 2 eventually.*

Proof. Assume that $AB > 1$. For any $i \in \{0, 1, \dots, 2k-1\}$ and $n \in \mathbb{N}_0$, let $\lim_{n \rightarrow \infty} x_{2nk+i} = A_i$ since x_{2nk+i} is eventually nonincreasing. Thus, by using (15) and by letting $n \rightarrow \infty$ in (14), it is obtained that $x_n = A$ eventually. By Lemma 1, we see that y_n is a periodic sequence with period 2 eventually. The proof is complete. $\square$

Now, we assume $AB = 1$, and by Lemma 2, we can assume that, for any $i \in \mathbb{N}_0$,

$$\lim_{n \rightarrow \infty} x_{2nk+i} = A_i. \quad (17)$$

Then, $A_i \geq A$ and $A_{2k+i} = A_i$ for any $i \in \mathbb{N}_0$.

Lemma 4. If $AB = 1$ and $A_i > A$ for some $i \in \mathbb{N}_0$, then, for any $s \in \mathbb{N}$, $x_{2nk+s+i}$ and $y_{2nk-k+s+i}$ are constant sequences eventually.

Proof. Assume that $AB = 1$ and $A_i > A$ for some $i \in \mathbb{N}_0$. Since $A_i > A$, it follows from (14), (15), and Lemma 2 that

$$\begin{aligned} x_{2nk+i} &= \max \left\{ A, \frac{x_{2(n-1)k+i}}{x_{2nk-1+i}y_{2nk-k-1+i}} \right\} \\ &= \frac{x_{2(n-1)k+i}}{x_{2nk-1+i}y_{2nk-k-1+i}} \quad \text{eventually.} \end{aligned} \quad (18)$$

By (9), (17), and (18), we have

$$\begin{aligned} B &\leq \lim_{n \rightarrow \infty} y_{2nk-k-1+i} = \lim_{n \rightarrow \infty} \frac{x_{2(n-1)k+i}}{x_{2nk+i}x_{2nk-1+i}} = \frac{1}{A_{-1+i+2k}} \\ &\leq \frac{1}{A} = B. \end{aligned} \quad (19)$$

This implies

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{2nk-1+i} &= A, \\ \lim_{n \rightarrow \infty} y_{2nk-k-1+i} &= B. \end{aligned} \quad (20)$$

Note that $x_{2nk+i} = \max\{A, y_{2nk-k+i}/x_{2nk-1+i}\} = y_{2nk-k+i}/x_{2nk-1+i} > A$ eventually; it follows that

$$\lim_{n \rightarrow \infty} y_{2nk-k+i} = \lim_{n \rightarrow \infty} x_{2nk+i}x_{2nk-1+i} = A_i A. \quad (21)$$

Since

$$\begin{aligned} x_{2nk+1+i} &= \max \left\{ A, \frac{x_{2(n-1)k+1+i}}{x_{2nk+i}y_{2nk-k+i}} \right\}, \\ \lim_{n \rightarrow \infty} \frac{x_{2(n-1)k+1+i}}{x_{2nk+i}y_{2nk-k+i}} &= \frac{A_{1+i}}{A_i^2 A} < A_{1+i}, \end{aligned} \quad (22)$$

we see that $x_{2nk+1+i} = A$ eventually. This with (21) implies

$$\lim_{n \rightarrow \infty} \frac{x_{2(n-1)k+1+i}}{y_{2nk-k+i}} = \frac{A}{AA_i} < B. \quad (23)$$

From (23), it follows that

$$y_{2nk-k+1+i} = \max \left\{ B, \frac{x_{2(n-1)k+1+i}}{y_{2nk-k+i}} \right\} = B \quad \text{eventually.} \quad (24)$$

By $x_{2(n-1)k+2+i}/y_{2nk-k+1+i} \geq A/B \geq 1 \geq B$ eventually and (24), it follows that

$$\begin{aligned} y_{2nk-k+2+i} &= \max \left\{ B, \frac{x_{2(n-1)k+2+i}}{y_{2nk-k+1+i}} \right\} = \frac{x_{2(n-1)k+2+i}}{B} \\ &\quad \text{eventually,} \\ x_{2nk+2+i} &= \max \left\{ A, \frac{y_{2nk-k+2+i}}{x_{2nk+1+i}} \right\} \\ &= \max \{ A, x_{2(n-1)k+2+i} \} = x_{2(n-1)k+2+i} \\ &\quad \text{eventually.} \end{aligned} \quad (25)$$

If $x_{2nk+2+i} > A$ eventually, then, in a similar fashion, we can obtain the following:

- (1) $x_{2nk+3+i} = A$ eventually and $y_{2nk-k+3+i} = B$ eventually.
- (2) $x_{2nk+4+i}$ and $y_{2nk-k+4+i}$ are constant sequences eventually.

If $x_{2nk+2+i} = A$ eventually, then $y_{2nk-k+2+i} = A/B$ eventually and

$$\begin{aligned} y_{2nk-k+3+i} &= \max \left\{ B, \frac{x_{2(n-1)k+3+i}}{y_{2nk-k+2+i}} \right\} \\ &= \max \left\{ B, \frac{x_{2(n-1)k+3+i}B}{A} \right\} = \frac{x_{2(n-1)k+3+i}B}{A} \\ &\quad \text{eventually,} \end{aligned} \quad (26)$$

$$\begin{aligned} x_{2nk+3+i} &= \max \left\{ A, \frac{y_{2nk-k+3+i}}{x_{2nk+2+i}} \right\} \\ &= \max \left\{ A, \frac{x_{2(n-1)k+3+i}B}{A^2} \right\} \quad \text{eventually.} \end{aligned} \quad (27)$$

From (27), we see that if $A = B = 1$, then by $x_{2(n-1)k+3+i} \geq A$ we have

$$x_{2nk+3+i} = x_{2(n-1)k+3+i} \quad \text{eventually,} \quad (28)$$

and if $A > B$, then

$$x_{2nk+3+i} = A \quad \text{eventually} \quad (29)$$

since

$$\lim_{n \rightarrow \infty} \frac{x_{2(n-1)k+3+i}B}{A^2} = \frac{A_{3+i}B}{A^2} < A_{3+i}. \quad (30)$$

Using arguments similar to ones developed in the above given proof, we can show that, for any $s \in \mathbb{N}$, $x_{2nk+s+i}$ and $y_{2nk-k+s+i}$ are constant sequences eventually. The proof is complete. $\square$

Theorem 5. Assume $AB = 1 > B$. Then, one of the following statements holds:

- (i) x_n and y_n are periodic sequences with period $2k$ eventually.
- (ii) $x_n = A$ eventually and y_n is a periodic sequence with period 2 eventually.

Proof. Assume that $A = 1/B > B$. If $A_i > A$ for some $i \in \mathbb{N}_0$, then by Lemma 4 we see that, for any $s \in \mathbb{N}$, $x_{2nk+s+i}$ and $y_{2nk-k+s+i}$ are constant sequences eventually, which implies that x_n and y_n are periodic sequences with period $2k$ eventually. In the following, we assume that $A_i = A$ for any $i \in \mathbb{N}_0$.

We claim that $x_n = A$ eventually. Indeed, if, for some $i \in \mathbb{N}_0$, $x_{2nk+i} > A$, then by (1) we have

$$\begin{aligned} x_{2nk+i} &= \max \left\{ A, \frac{y_{2nk-k+i}}{x_{2nk-1+i}} \right\} = \frac{y_{2nk-k+i}}{x_{2nk-1+i}} \\ &\quad \text{eventually,} \end{aligned} \quad (31)$$

$$x_{2nk+1+i} = \max \left\{ A, \frac{x_{2(n-1)k+1+i}}{x_{2nk+i}y_{2nk-k+i}} \right\}, \quad (32)$$

$$\lim_{n \rightarrow \infty} \frac{x_{2(n-1)k+1+i}}{x_{2nk+i} y_{2nk-k+i}} = \lim_{n \rightarrow \infty} \frac{x_{2(n-1)k+1+i}}{x_{2nk+i}^2 x_{2nk-1+i}} \leq \frac{A_{1+i}}{A^3} < A_{1+i}. \quad (33)$$

Then, we see that

$$x_{2nk+1+i} = A \quad \text{eventually.} \quad (34)$$

From (31), we have $y_{2nk-k+i} \geq A^2$ eventually, and with (34) and $B = 1/A$, it follows that

$$\begin{aligned} y_{2nk-k+1+i} &= \max \left\{ B, \frac{x_{2(n-1)k+1+i}}{y_{2nk-k+i}} \right\} \\ &= \max \left\{ B, \frac{A}{y_{2nk-k+i}} \right\} = B \quad \text{eventually,} \\ y_{2nk-k+2+i} &= \max \left\{ B, \frac{x_{2(n-1)k+2+i}}{y_{2nk-k+1+i}} \right\} = \frac{x_{2(n-1)k+2+i}}{B} \\ &\quad \text{eventually,} \end{aligned} \quad (35)$$

$$\begin{aligned} x_{2nk+2+i} &= \max \left\{ A, \frac{y_{2nk-k+2+i}}{x_{2nk+1+i}} \right\} \\ &= \max \left\{ A, x_{2(n-1)k+2+i} \right\} = x_{2(n-1)k+2+i} \\ &\quad \text{eventually.} \end{aligned}$$

Thus, $x_{2nk+2+i}$ and $y_{2nk-k+2+i}$ are constant sequences eventually. Using arguments similar to ones developed in the proof of Lemma 4, we can show that, for any $s \in \mathbb{N}$, $x_{2nk+s+i}$ and $y_{2nk-k+s+i}$ are constant sequences eventually. Thus, $x_{2nk+i} = A_i = A$ eventually; this is a contradiction.

By Lemma 1, we see that $x_n = A$ eventually and y_n is a periodic sequence with period 2 eventually. Theorem 5 is proven. $\square$

Theorem 6. *If $A \leq 1$ and $B \leq 1$, then (1) has a solution which is not periodic eventually.*

Proof. We claim that the equation

$$z_n = \frac{z_{n-k}}{z_{n-1}} \quad (36)$$

has a nonincreasing solution. Indeed, write

$$\begin{aligned} M &= \{(x_1, \dots, x_k) : x_1 \geq \dots \geq x_k \geq \sqrt{x_1} > 0\}, \\ E &= \left\{ (x_1, \dots, x_k) : x_1 \geq \dots \geq x_k \geq \frac{x_1}{x_{k-1}} > 0 \right\}. \end{aligned} \quad (37)$$

It is easy to see that $M \subset E$ since, for any $(x_1, \dots, x_k) \in M$, we have $x_1 \geq \dots \geq x_k \geq \sqrt{x_1}$ and $x_k \geq x_k^2/x_{k-1} \geq x_1/x_{k-1}$. Define $F : M \rightarrow E$ by

$$F(x_1, \dots, x_k) = (u_1, \dots, u_k) \equiv \left(x_2, \dots, x_k, \frac{x_1}{x_k} \right) \quad (38)$$

for all $(x_1, \dots, x_k) \in M$.

We claim that F is well defined. Indeed, from (38) and the definition of M , we have

$$u_i = x_{i+1}, \quad \text{for } i \in \{1, \dots, k-1\},$$

$$u_k = \frac{x_1}{x_k},$$

$$u_1 \geq \dots \geq u_{k-1} = x_k \geq \frac{x_1}{\sqrt{x_1}} \geq \frac{x_1}{x_k} = u_k \geq \frac{u_1}{u_{k-1}}. \quad (39)$$

Thus, $(u_1, \dots, u_k) \in E$.

We also claim that F is a bijection from M to E . Indeed, let $z = (z_1, \dots, z_k)$, $y = (y_1, \dots, y_k) \in M$ with $z \neq y$; we have $F(y) \neq F(z)$. On the other hand, for any $u = (u_1, \dots, u_k) \in E$, we have

$$u_1 \geq \dots \geq u_k \geq \frac{u_1}{u_{k-1}}. \quad (40)$$

Choose

$$x = (x_1, \dots, x_k) \equiv (u_k u_{k-1}, u_1, \dots, u_{k-1}). \quad (41)$$

It follows from (40) and (41) that

$$\begin{aligned} x_1 &= u_k u_{k-1} \geq u_1 = x_2 \geq \dots \geq u_{k-1} = x_k = \frac{u_{k-1} u_k}{u_k} \\ &\geq \frac{u_{k-1} u_k}{\sqrt{u_{k-1} u_k}} = \sqrt{x_1}, \end{aligned} \quad (42)$$

which implies $x \in M$, and by (38) we obtain $F(x) = u$.

Furthermore, since $F^{-1}(u_1, \dots, u_k) = (u_k u_{k-1}, u_1, \dots, u_{k-1})$ is continuous, F is a homeomorphism from M to E .

Since $M \subset E$ and F is a homeomorphism from M onto E , it follows that $F^{-1}(M) \subset F^{-1}(E) = M$. By induction, we have

$$p = (1, 1, \dots, 1) \in F^{-n}(M) \subset F^{-n+1}(M) \quad (43)$$

for every $n \in \mathbb{N}$. Because M is an unbounded connected closed set, we know that $F^{-n}(M)$ is an unbounded connected closed set for every $n \in \mathbb{N}$. Let

$$S = \bigcap_{i=0}^{\infty} F^{-i}(M). \quad (44)$$

Then, $S \neq \emptyset$ and S is also an unbounded connected set.

Now, let $\{z_n\}_{n=-k}^{\infty}$ be a solution of (36) with the initial conditions $(z_{-k}, \dots, z_{-1}) \in S - \{p\}$; we can show that, for all $n \in \mathbb{N}$,

$$F^n(z_{-k}, \dots, z_{-1}) = (z_{n-k}, \dots, z_{n-1}) \in M - \{p\}. \quad (45)$$

Thus, $z_n \geq z_{n+1} > 1$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} z_n = 1$. It is easy to see that $\{(z_n, z_n)\}_{n=-k}^{\infty}$ is also a solution of (1) which is not periodic eventually. Theorem 6 is proven. $\square$

Theorem 7. *If $AB < 1 < A$, then (1) has an unbounded solution.*

Proof. Let $\{(x_n, y_n)\}_{n=-k}^{\infty}$ be a solution of (1) with the initial values $x_i = A$ and $y_i = B$ for $i \in \{-k, -k+1, \dots, -1\}$.

If $k = 2m$ for some $m \in \mathbb{N}$, then it is easy to see that, for $i \in \{0, 1, \dots, m-1\}$, we have $(x_{2i}, y_{2i}) = (A, A/B)$, $(x_{2i+1}, y_{2i+1}) = (A, B)$. Thus,

$$\begin{aligned}
 (x_k, y_k) &= \left(\max \left\{ A, \frac{y_0}{x_{k-1}} \right\}, \max \left\{ B, \frac{x_0}{y_{k-1}} \right\} \right) \\
 &= \left(\max \left\{ A, \frac{1}{B} \right\}, \max \left\{ B, \frac{A}{B} \right\} \right) = \left(\frac{1}{B}, \frac{A}{B} \right), \\
 (x_{k+1}, y_{k+1}) &= \left(\max \left\{ A, \frac{y_1}{x_k} \right\}, \max \left\{ B, \frac{x_1}{y_k} \right\} \right) \\
 &= \left(\max \left\{ A, B^2 \right\}, \max \left\{ B, B \right\} \right) = (A, B), \\
 (x_{k+2}, y_{k+2}) &= \left(\max \left\{ A, \frac{y_2}{x_{k+1}} \right\}, \max \left\{ B, \frac{x_2}{y_{k+1}} \right\} \right) \\
 &= \left(\max \left\{ A, \frac{1}{B} \right\}, \max \left\{ B, \frac{A}{B} \right\} \right) = \left(\frac{1}{B}, \frac{A}{B} \right), \\
 (x_{k+3}, y_{k+3}) &= \left(\max \left\{ A, \frac{y_3}{x_{k+2}} \right\}, \max \left\{ B, \frac{x_3}{y_{k+2}} \right\} \right) \\
 &= \left(\max \left\{ A, B^2 \right\}, \max \left\{ B, B \right\} \right) = (A, B), \\
 &\vdots \\
 (x_{2k-2}, y_{2k-2}) &= \left(\max \left\{ A, \frac{y_{k-2}}{x_{2k-3}} \right\}, \max \left\{ B, \frac{x_{k-2}}{y_{k-3}} \right\} \right) \\
 &= \left(\max \left\{ A, \frac{1}{B} \right\}, \max \left\{ B, \frac{A}{B} \right\} \right) = \left(\frac{1}{B}, \frac{A}{B} \right), \\
 (x_{2k-1}, y_{2k-1}) &= \left(\max \left\{ A, \frac{y_{k-1}}{x_{2k-2}} \right\}, \max \left\{ B, \frac{x_{k-1}}{y_{k-2}} \right\} \right) \\
 &= \left(\max \left\{ A, B^2 \right\}, \max \left\{ B, B \right\} \right) = (A, B), \\
 (x_{2k}, y_{2k}) &= \left(\max \left\{ A, \frac{y_k}{x_{2k-1}} \right\}, \max \left\{ B, \frac{x_k}{y_{2k-1}} \right\} \right) \\
 &= \left(\max \left\{ A, \frac{1}{B} \right\}, \max \left\{ B, \frac{1}{B^2} \right\} \right) = \left(\frac{1}{B}, \frac{1}{B^2} \right), \\
 (x_{2k+1}, y_{2k+1}) &= \left(\max \left\{ A, \frac{y_{k+1}}{x_{2k}} \right\}, \max \left\{ B, \frac{x_{k+1}}{y_{2k}} \right\} \right) \\
 &= \left(\max \left\{ A, B^2 \right\}, \max \left\{ B, B^2 A \right\} \right) = (A, B), \\
 (x_{2k+2}, y_{2k+2}) &= \left(\max \left\{ A, \frac{y_{k+2}}{x_{2k+1}} \right\}, \max \left\{ B, \frac{x_{k+2}}{y_{2k+1}} \right\} \right) \\
 &= \left(\max \left\{ A, \frac{1}{B} \right\}, \max \left\{ B, \frac{1}{B^2} \right\} \right) = \left(\frac{1}{B}, \frac{1}{B^2} \right), \\
 (x_{2k+3}, y_{2k+3}) &= \left(\max \left\{ A, \frac{y_{k+3}}{x_{2k+2}} \right\}, \max \left\{ B, \frac{x_{k+3}}{y_{2k+2}} \right\} \right) \\
 &= \left(\max \left\{ A, B^2 \right\}, \max \left\{ B, B^2 A \right\} \right) = (A, B),
 \end{aligned}$$

$$\begin{aligned}
 &\vdots \\
 (x_{3k-2}, y_{3k-2}) &= \left(\max \left\{ A, \frac{y_{2k-2}}{x_{3k-3}} \right\}, \max \left\{ B, \frac{x_{2k-2}}{y_{3k-3}} \right\} \right) \\
 &= \left(\max \left\{ A, \frac{1}{B} \right\}, \max \left\{ B, \frac{1}{B^2} \right\} \right) = \left(\frac{1}{B}, \frac{1}{B^2} \right), \\
 (x_{3k-1}, y_{3k-1}) &= \left(\max \left\{ A, \frac{y_{2k-1}}{x_{3k-2}} \right\}, \max \left\{ B, \frac{x_{2k-1}}{y_{3k-2}} \right\} \right) \\
 &= \left(\max \left\{ A, B^2 \right\}, \max \left\{ B, B^2 A \right\} \right) = (A, B).
 \end{aligned} \tag{46}$$

By induction, we can obtain that, for any $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, m-1\}$,

$$\begin{aligned}
 (x_{(2n-1)k+2i}, y_{(2n-1)k+2i}) &= \left(\frac{1}{A^{n-1}B^n}, \frac{1}{A^{n-2}B^n} \right), \\
 (x_{2nk+2i}, y_{2nk+2i}) &= \left(\frac{1}{A^{n-1}B^n}, \frac{1}{A^{n-1}B^{n+1}} \right), \\
 (x_{nk+2i+1}, y_{nk+2i+1}) &= (A, B).
 \end{aligned} \tag{47}$$

If $k = 2m + 1$ for some $m \in \mathbb{N}$, then we also can obtain that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
 (x_{(2n-1)k+2i}, y_{(2n-1)k+2i}) &= \left(\frac{1}{A^{n-1}B^n}, B \right) \\
 &\quad (i \in \{0, 1, \dots, m\}), \\
 (x_{(2n-1)k+2i+1}, y_{(2n-1)k+2i+1}) &= \left(A, \frac{1}{A^{n-2}B^n} \right) \\
 &\quad (i \in \{0, 1, \dots, m-1\}), \\
 (x_{2nk+2i}, y_{2nk+2i}) &= \left(A, \frac{1}{A^{n-1}B^{n+1}} \right) \\
 &\quad (i \in \{0, 1, \dots, m\}), \\
 (x_{2nk+2i+1}, y_{2nk+2i+1}) &= \left(\frac{1}{A^{n-1}B^n}, B \right) \\
 &\quad (i \in \{0, 1, \dots, m-1\}).
 \end{aligned} \tag{48}$$

It is easy to see that $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is an unbounded solution of (1). Theorem 7 is proven. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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