Research Article

Uniqueness and Novel Finite-Time Stability of Solutions for a Class of Nonlinear Fractional Delay Difference Systems

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This paper focuses on the uniqueness and novel finite-time stability of solutions for a kind of fractional-order nonlinear difference equations with time-varying delays. Under some new criteria and by applying the generalized Gronwall inequality, the new constructive results have been established in the literature. As an application, two typical examples are delineated to demonstrate the effectiveness of our theoretical results.

1. Introduction

The time difference of fractional order was firstly studied by Kuttner [1] in 1957; since then, various kinds of definitions of fractional difference were studied by many authors. We know fractional difference equations play an important role in promoting modern mathematics development and have been widely applied, especially in physics, dynamic mechanics, medicines, and communications. There is a growing tendency nowadays that many experts show their great enthusiasm for fractional difference equations, and in the past few years, a lot of achievements have been done. For an extensive collection of such results, we recommend the readers to monograph [2] and papers [3–16].

The study about uniqueness of discrete solutions for fractional difference equations is one of the most interesting and valuable topics. Recently, Abdeljawad et al. [3] studied the following nonlinear fractional difference system

$$\begin{aligned}({}^{CFR}_a\nabla^\alpha y)(t) &= f(t, y(t)), \quad t \in \mathbb{N}_{a,b}, \\ y(a) &= c.\end{aligned}\quad (1)$$

Let $f(t, y)$ satisfy Lipschitz condition: there exists a constant $A > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq A|y_1 - y_2|, \quad (2)$$

and $f : \mathbb{N}_{a,b} \times \mathbb{R} \rightarrow \mathbb{R}$ and $y : \mathbb{N}_{a,b} \rightarrow \mathbb{R}$. By Banach Contraction Principle, authors obtained that system (1) had a unique solution $x \in X$, $X = \{x : \max_{t \in \mathbb{N}_{a,b}} |x(t)| < \infty\}$, if

$$\frac{A}{B(\alpha-1)} \left((2-\alpha)(b-a) + \frac{(\alpha-1)(b-a)^2}{2} \right) < 1, \quad (3)$$

where $B(\alpha)$ is a normalization positive constant depending on α satisfying $B(0) = B(1) = 1$, and ${}^{CFR}_a\nabla^\alpha$ is a fractional operator defined as in Definition 4 of paper [3].

In [4], Abdeljawad and M. Al-Mdallal considered the fractional difference system

$$\begin{aligned}({}^{ABC}_a\nabla^\alpha y)(t) &= f(t, y(t)), \quad t \in \mathbb{N}_{a,b}, \\ y(a) &= c,\end{aligned}\quad (4)$$

such that $b \equiv a \pmod{1}$, $f(a, y(a)) = 0$. Let f admit the Lipschitz condition, A be a Lipschitz constant, and $y : \mathbb{N}_{a,b} \rightarrow \mathbb{R}$. Then system (4) had a unique solution provided that

$$A \left(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\overline{\alpha}}}{\Gamma(\alpha)B(\alpha)} \right) < 1, \quad (5)$$

where ${}^{ABC}_a\nabla^\alpha$ is a fractional operator, and we can see it in Definition 4 of paper [4].

What will happen if the nonlinear equations (1) and (4) subjecting the initial value condition extend into fractional delay difference equation? We are particularly interested in fractional difference equation involving time-varying delays.

On the other hand, stability analysis is also one of the most crucial themes for fractional nonlinear systems, such as [17, 18] researching stability in nondelay fractional systems and [14, 19–22] in delay fractional difference systems. Specifically, in [17, 18], using Lyapunov's direct method, the stability of discrete nonautonomous systems with the nabla Caputo fractional difference was studied. In [19], the authors studied a class of linear fractional difference equations with impulse effects, and they provided the generalized Mittag-Leffler stability by numerical illustration. In papers [20, 22], asymptotic stability of a fractional discrete system was discussed and the theorem for a discrete fractional Lyapunov direct method was proved. In [21], the researchers investigated the stability of the equilibrium solution of a linear fractional difference equation with the initial condition, and to achieve this target, the well-known unilateral z -transform was successfully employed. In paper [14], the researchers considered the following linear fractional difference equations with a constant delay

$${}^C\Delta_a^\nu x(t) = A_0 x(t + \nu) + A_1 x(t + \nu - k), \quad 0 < \nu < 1, \quad (6)$$

where $x(t) \in \mathbb{R}^l$, k is a fixed positive integer, and ${}^C\Delta_a^\nu x(t)$ denotes the Caputo delta fractional difference of $x(t)$ on the discrete time. The finite-time stable conclusions are presented in the addressed paper.

Deeply inspired by [3, 4, 14] and other mentioned papers, in this paper, we are concerned with the uniqueness and finite-time stability of solutions for the following fractional discrete equation with time-varying delays

$$\begin{aligned} {}^C\Delta_a^\nu x(t) &= Ax(t) + Bx(t - h(t)) + Dw(t) \\ &\quad + f(t, x(t), x(t - h(t)), w(t)), \end{aligned} \quad (7)$$

$$t \in J_1,$$

$$x(t) = \phi(t), \quad t \in J_2,$$

where ${}^C\Delta_a^\nu$ denotes the Caputo fractional difference operator with $\nu \in (0, 1]$, and $J_1 = \{t \in \mathbb{Z} \mid a + 1 - \nu \leq t \leq a + M\}$, and $J_2 = \{t \in \mathbb{Z} \mid a + 1 - \nu - h \leq t \leq a + 1 - \nu\}$, $a \in \mathbb{R}$, M is a positive integer; $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the disturbance vector, $h(t)$ is a function satisfying $0 \leq h(t) \leq h$, and $\phi(t) \in \mathbb{R}^n$ is the given function; $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times m}$ are constant matrices, and $f: J_1 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$.

Compared with some recent results in the literatures, such as [14, 17, 19–22], the chief contributions of this study contain at least the following three:

- (1) In [14, 17, 19–22], the literatures investigated the stability of fractional difference equations with constant delays, but the delay term we studied in system (7) is a bounded function with respect to the variable t . This is a significant breakthrough in dealing with fractional difference system with time-varying delays.
- (2) The model we are concerned with is more generalized, some ones in the articles are the special cases of it.

In [19, 22], the coefficients of the fractional discrete system researched by authors are one-dimensional real numbers which are too simple to describe the mathematical model well, and we adopt constant matrices as coefficients in (7). Therefore, the generalized models are originally discussed in the present paper. Furthermore, our conclusions can also be applied to the equations with function matrices, and you can see it by the following Corollaries 10 and 13.

- (3) An innovative method based on the generalized Gronwall inequality is exploited to discuss the uniqueness and finite-time stability of the solutions for the fractional-order difference equation with time-varying delays. The results established are essentially new.

The following article is organized as follows: in Section 2, we will recall some known results for our considerations. Some lemmas and definitions are useful to our work. Section 3 is devoted to researching the uniqueness of solutions for the fractional-order difference equation with time delay. Subsequently, we investigate the finite-time stability of the addressed equation, and then we will come up with the main theorem. To explain the results clearly, we finally provide two examples in Section 4.

2. Preliminaries

In this section, we plan to introduce some basic definitions and lemmas which are used throughout this paper.

Definition 1 ([23, 24]). We define

$$t^\nu := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, \quad (8)$$

as for any t and ν for which the right-hand side is defined. Here and in what follows Γ denotes the gamma function. We also appeal to the common convention that if $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^\nu = 0$.

Definition 2 ([24]). The ν -th fractional sum of a function f , for $\nu > 0$, is defined to be

$$\Delta_a^{-\nu} f(t) = \Delta_a^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s), \quad (9)$$

where $t \in \{a + \nu, a + \nu + 1, \dots\} =: \mathbb{N}_{a+\nu}$. We also define the ν -th fractional difference, where $\nu > 0$ and $0 \leq N-1 < \nu \leq N$ with $N \in \mathbb{N}$, to be $\Delta^\nu f(t) := \Delta^N \Delta^{N-\nu} f(t)$, where $t \in \mathbb{N}_{a+\nu}$.

Lemma 3 ([15]). Assume that $\mu > 0$ and f is defined on $\mathbb{N}_a$. Then

$$\begin{aligned} \Delta^{-\mu} \Delta^\mu f(t) &= f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) \\ &= f(t) + c_0 + c_1 t + \dots + c_{n-1} t^{(n-1)}, \end{aligned} \quad (10)$$

where n is the smallest integer greater than or equal to μ , $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n-1$.

Lemma 4 ([10]). Let $v \in \mathbb{R}$ and $t, s \in \mathbb{R}$ such that $(t-s)^\gamma$ is well defined, then $\Delta_s(t-s)^\gamma = -v(t-s-1)^{\gamma-1}$.

Definition 5 ([14]). Given positive numbers c_1, c_2 satisfying $c_1 < c_2$, system (7) is finite-time stable if and only if

$$\|\phi\| \leq c_1 \implies \|x(t)\| \leq c_2, \quad \forall t \in J_1 \cup J_2, \quad (11)$$

$$x(t) = \begin{cases} x(a) + \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-s-1)^{\gamma-1} [Ax(s) + Bx(s-h(s)) + Dw(s) + f(s, x(s), x(s-h(s)), w(s))], & t \in J_1, \\ \phi(t), & t \in J_2. \end{cases} \quad (13)$$

Lemma 7 (generalized Gronwall inequality). Let $\alpha > 0$, and $u(t), v(t)$ be nonnegative functions and $w(t)$ be nonnegative, nondecreasing function for $t \in \mathbb{N}_a$ such that $w(t) \leq M$, where M is a constant. If

$$u(t) \leq v(t) + w(t) \Gamma(\alpha) \Delta_a^{-\alpha} u(t), \quad (14)$$

then

$$u(t) \leq v(t) + \sum_{k=1}^{\infty} (w(t) \Gamma(\alpha))^k \Delta_a^{-k\alpha} v(t). \quad (15)$$

Proof. Define operator

$$B\phi(t) = w(t) \sum_{s=a}^{t-\alpha} (t-s-1)^{\alpha-1} \phi(s), \quad (16)$$

then from (14), we know

$$u(t) \leq v(t) + Bu(t), \quad (17)$$

which implies that $u(t) \leq \sum_{k=0}^{n-1} B^k v(t) + B^n u(t)$. The following proof process is similar to the relevant conclusion, and we can refer to Theorem 3.2 in [14]. $\square$

3. Main Results

Let $\lambda(A)$ be the set of all eigenvalues of A and $\lambda_{\max}(A) = \max\{\operatorname{Re}(\lambda) : \lambda \in \lambda(A)\}$. Assume that $\|A\|$ denotes the spectral norm defined by $\sqrt{\lambda_{\max}(A^T A)}$, and let $\|x\|$ be the norm of $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ defined by $\|x\| = \max_{t \in J_1 \cup J_2} (\sum_{i=1}^n x_i^2)^{1/2}$. Suppose that $\mathcal{B}^+(J_1)$ denotes the set of all nonnegative bounded functions on J_1 . Assume that the nonlinear function $f : J_1 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies the condition (H_1) : there exists a positive constant $l(t) \in \mathcal{B}^+(J_1)$ such that

$$\begin{aligned} & \|f(t, x_1, y_1, w_1) - f(t, x_2, y_2, w_2)\| \\ & \leq l(t) (\|x_1 - x_2\| + \|y_1 - y_2\| + \|w_1 - w_2\|), \end{aligned} \quad (18)$$

where $f(t, 0, 0, 0) = 0$. In this section, we always assume that

$$\|A\| = \bar{a},$$

$$\|B\| = \bar{b},$$

for all disturbances $w(t)$ satisfying the following condition:

$$\exists \eta > 0 : w^T(t) w(t) \leq \eta^2. \quad (12)$$

Definition 6. A function $x(t)$ is called the solution of (7) if $x(t)$ satisfies

$$\begin{aligned} \|D\| &= \bar{d}, \\ \sup_{s \in [a, t-\gamma]} l(s) &= \bar{L}. \end{aligned} \quad (19)$$

Theorem 8. Assume that $0 \leq (\bar{a} + \bar{b} + 2\bar{L})/\Gamma(\gamma) < 1$, then system (7) has a unique solution on $J_1 \cup J_2$ if the condition (H_1) holds.

Proof. Let $x(t)$ and $\tilde{x}(t)$ be any two different solutions to system (7), then $x(t)$ and $\tilde{x}(t)$ both satisfy (13). Let $z(t) = x(t) - \tilde{x}(t)$.

We can easily obtain $z(t) = 0$ for $t \in J_2$. That is to say, system (7) has a unique solution as $t \in J_2$.

When $t \in J_1 = J_{11} \cup J_{12}$, and $J_{11} = \{t \in \mathbb{Z} \mid a+1-\gamma \leq t \leq a+\gamma\}$, $J_{12} = \{t \in \mathbb{Z} \mid a+\gamma \leq t \leq a+M\}$, we get

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-s-1)^{\gamma-1} [Az(s) + Bz(s-h(s)) \\ &+ f(s, x(s), x(s-h(s)), w(s)) \\ &- f(s, \tilde{x}(s), \tilde{x}(s-h(s)), w(s))]. \end{aligned} \quad (20)$$

If $t \in J_{11}$, then

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-s-1)^{\gamma-1} [Az(s) \\ &+ f(s, x(s), x(s-h(s)), w(s)) \\ &- f(s, \tilde{x}(s), \tilde{x}(s-h(s)), w(s))]. \end{aligned} \quad (21)$$

Now applying the norm $\|\cdot\|$ on both sides of (21), we get

$$\begin{aligned} \|z(t)\| &\leq \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-s-1)^{\gamma-1} [\|A\| \|z(s)\| \\ &+ \|f(s, x(s), x(s-h(s)), w(s)) \\ &- f(s, \tilde{x}(s), \tilde{x}(s-h(s)), w(s))\|] \leq \frac{1}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-s \\ &- 1)^{\gamma-1} [(\|A\| + l(s)) \|z(s)\|] \leq \frac{\bar{a} + \bar{L}}{\Gamma(\gamma)} \sum_{s=a}^{t-\gamma} (t-s \\ &- 1)^{\gamma-1} \|z(s)\|. \end{aligned} \quad (22)$$

Applying the Generalized Gronwall Inequality in Lemma 7, we have

$$\|z(t)\| \leq 0. \quad (23)$$

Namely, $x(t) = \tilde{x}(t)$ for $t \in J_{11}$.

If $t \in J_{12}$, then applying the norm $\|\cdot\|$ on both sides of (20), it follows that

$$\begin{aligned} \|z(t)\| &\leq \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} \\ &\cdot [(\bar{a} + l(s)) \|z(s)\| + (\bar{b} + l(s)) \|z(s-h(s))\|] \\ &\leq \frac{\bar{a} + \bar{L}}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} \|z(s)\| + \frac{\bar{b} + \bar{L}}{\Gamma(\nu)} \\ &\cdot \sum_{s=a+\nu+1}^{t-\nu} (t-s-1)^{\nu-1} \times \|z(s-h(s))\|. \end{aligned} \quad (24)$$

Let $z^*(t) = \sup_{\theta \in [-h, 0]} \|z(t+\theta)\|$, $t \in J_{12}$, then we have

$$z^*(t) \leq \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} z^*(s). \quad (25)$$

Similarly, applying the generalized Gronwall inequality in Lemma 7, it follows that

$$\|z(t)\| \leq z^*(t) < 0. \quad (26)$$

Hence we can obtain $x(t) = \tilde{x}(t)$. This completes the proof. $\square$

Remark 9. When we demonstrate the uniqueness of solutions for the fractional discrete system with time delay, we find that the conditions we needed have nothing to do with the disturbance vector $w(t)$. That is to say, disturbance vector does not affect the uniqueness of solution for the system.

One can extend the constant matrices in (7) to the function form as follows:

$$\begin{aligned} {}^C\Delta_a^\nu x(t) &= A(t)x(t) + B(t)x(t-h(t)) \\ &+ D(t)w(t) \\ &+ f(t, x(t), x(t-h(t)), w(t)), \end{aligned} \quad (27)$$

$$t \in J_1,$$

$$x(t) = \phi(t), \quad t \in J_2.$$

Our conclusion (Theorem 8) can also be applied to (27) if $f(\cdot)$ satisfies the condition (H_1) . In the case, let $\|A(t)\| = \bar{a}$, $\|B(t)\| = \bar{b}$, $\|D(t)\| = \bar{d}$ in the same proof.

Corollary 10. Assume that $0 \leq (\bar{a} + \bar{b} + 2\bar{L})/\Gamma(\nu) < 1$, then system (27) has a unique solution on $J_1 \cup J_2$ if the condition (H_1) holds.

Theorem 11. Suppose that (H_1) holds, and $0 \leq (\bar{a} + \bar{b} + 2\bar{L})/\Gamma(\nu) < 1$, and there exist positive numbers $c_1 < c_2$, and $\|\phi\| \leq c_1$, then system (7) is finite-time stable on $J_1 \cup J_2$ if

$$\begin{aligned} &\left(c_1 + \frac{\eta(\bar{d} + \bar{L})M^\nu}{\Gamma(\nu+1)} \right) \\ &\cdot \left(1 + \sum_{k=1}^{\infty} (\bar{a} + \bar{b} + 2\bar{L})^k \cdot \frac{M^{k\nu}}{\Gamma(k\nu+1)} \right) \leq c_2. \end{aligned} \quad (28)$$

Proof. Let $u(t) = \sup_{\theta \in [a+1-\nu-h, t]} \|x(\theta)\|$, for $t \in J_1$. We have $\|x(s)\| \leq u(s)$, and $\|x(s-h(s))\| \leq u(s)$, $\forall s \in [0, t]$. According to (12) and (13), one can obtain

$$\begin{aligned} \|x(t)\| &\leq \|x(a)\| + \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} [\|A\| \|x(s)\| \\ &+ \|B\| \|x(s-h(s))\| + \|D\| \|w(s)\| \\ &+ \|f(s, x(s), x(s-h(s)), w(s))\|] \leq \|\phi\| \\ &+ \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} u(s) + \frac{\eta(\bar{d} + \bar{L})}{\Gamma(\nu)} \\ &\cdot \frac{(t-a)^\nu}{\nu} \leq \|\phi\| + \frac{\eta(\bar{d} + \bar{L})M^\nu}{\Gamma(\nu+1)} + \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \\ &\cdot \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} u(s). \end{aligned} \quad (29)$$

As for all $\theta \in [a+1-\nu, t]$, we have

$$\begin{aligned} \|x(\theta)\| &\leq \|\phi\| + \frac{\eta(\bar{d} + \bar{L})M^\nu}{\Gamma(\nu+1)} \\ &+ \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \sum_{s=a}^{\theta-\nu} (\theta-s-1)^{\nu-1} u(s) \\ &\leq \|\phi\| + \frac{\eta(\bar{d} + \bar{L})M^\nu}{\Gamma(\nu+1)} \\ &+ \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} u(s). \end{aligned} \quad (30)$$

Therefore, we have

$$\begin{aligned} u(t) &= \sup_{\theta \in [a+1-\nu-h, t]} \|x(\theta)\| \\ &\leq \max \left\{ \sup_{\theta \in [a+1-\nu-h, a+1-\nu]} \|x(\theta)\|, \sup_{\theta \in [a+1-\nu, t]} \|x(\theta)\| \right\} \\ &\leq \max \left\{ \|\phi\|, \|\phi\| + \frac{\eta(\bar{d} + \bar{L})M^\nu}{\Gamma(\nu+1)} \right. \\ &\quad \left. + \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} u(s) \right\} = \|\phi\| \\ &\quad + \frac{\eta(\bar{d} + \bar{L})M^\nu}{\Gamma(\nu+1)} + \frac{\bar{a} + \bar{b} + 2\bar{L}}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} u(s). \end{aligned} \quad (31)$$

According to Lemma 7, we can get

$$\begin{aligned}
 u(t) &\leq \|\phi\| + \frac{\eta(\bar{a} + \bar{L})M^\nu}{\Gamma(\nu + 1)} + \sum_{k=1}^{\infty} (\bar{a} + \bar{b} + 2\bar{L})^k \\
 &\quad \cdot \frac{1}{\Gamma(k\nu)} \sum_{s=a}^{t-k\nu} (t-s-1)^{k\nu-1} \\
 &\quad \times \left(\|\phi\| + \frac{\eta(\bar{a} + \bar{L})M^\nu}{\Gamma(\nu + 1)} \right) \\
 &\leq \left(\|\phi\| + \frac{\eta(\bar{a} + \bar{L})M^\nu}{\Gamma(\nu + 1)} \right) \\
 &\quad \cdot \left(1 + \sum_{k=1}^{\infty} (\bar{a} + \bar{b} + 2\bar{L})^k \cdot \frac{M^{k\nu}}{\Gamma(k\nu + 1)} \right).
 \end{aligned} \tag{32}$$

Hence,

$$\begin{aligned}
 \|x(t)\| \leq u(t) &\leq \left(c_1 + \frac{\eta(\bar{a} + \bar{L})M^\nu}{\Gamma(\nu + 1)} \right) \\
 &\quad \cdot \left(1 + \sum_{k=1}^{\infty} (\bar{a} + \bar{b} + 2\bar{L})^k \cdot \frac{M^{k\nu}}{\Gamma(k\nu + 1)} \right) \leq c_2.
 \end{aligned} \tag{33}$$

As for $t \in J_2$, obviously, system (7) has finite-time stability. According to Definition 5, we can obtain that system (7) is finite-time stable. This completes the proof of Theorem 11. $\square$

Remark 12. Different from the research in uniqueness of solutions, the disturbance vector $w(t)$ plays a key role in studying the finite-time stability. It can be easily analyzed by (33).

When A, B, D are function matrices instead of constant ones, our conclusion about the finite-time stability (Theorem 11) also can be applied to system (27). In this case, we assume that $\|A(t)\| = \bar{a}$, $\|B(t)\| = \bar{b}$, $\|D(t)\| = \bar{d}$ in the same proof.

Corollary 13. Given positive numbers c_1, c_2, M , such that $c_1 < c_2$, then system (27) is finite-time stable on $J_1 \cup J_2$ if all the conditions in Theorem 11 hold.

4. Example

In this section, we will present the following two examples to illustrate our main results.

Example 14. Suppose that $\nu = 1/20$, and $f(\cdot) = 0.1 \sin x(t) + 0.1 \cos x(t - h(t))$ with $x = (x_1(t), x_2(t))^T$. Consider system (7), where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{34}$$

$$B = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}. \tag{35}$$

We have $\|A\| = \bar{a} = 4$, $\|B\| = \bar{b} = 13$, and $\bar{L} = 0.1$, $\Gamma(\nu) \approx 19.47$, which imply that the condition $0 < (\bar{a} + \bar{b} + 2\bar{L})/\Gamma(\nu) < 1$ referred to in Theorem 8 holds. Therefore, the specific system (7) has a unique solution.

Remark 15. Since there are few papers researching the uniqueness of solutions for the nonlinear fractional-order difference equation with time-varying delay, one can see that all the results in [23, 25–28] can not directly be applicable to Example 14 to obtain the uniqueness of the solution. This implies that the results in this paper are essentially new.

Example 16. Suppose that

$$f(\cdot) = 0.1 \begin{pmatrix} \sqrt{x_1^2(t) + x_1^2(t-h)} + \sin x_1 \\ \sqrt{x_2^2(t) + x_2^2(t-h)} + \sin x_2 \end{pmatrix}, \tag{36}$$

and let

$$A = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}, \tag{37}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0.4 & 0.2 \end{pmatrix}, \tag{38}$$

$$D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{39}$$

Let $M = 10$, $w(t) = \sqrt{2}$, $h = 1$, $\nu = 0.03$, $\Gamma(\nu) \approx 32.79$, $\Gamma(\nu + 1) \approx 0.98$, $a = 1$, $\eta = 1.42$, $\phi(t) = (0.8, 0.8)^T$. We have $\bar{a} = 0.3$, $\bar{b} = 0.45$, $\bar{d} = 1$, $\bar{L} = 0.1$. One can assume that $c_1 = 1.14$, $c_2 \geq 463.26$, $0 < (\bar{a} + \bar{b} + 2\bar{L})/\Gamma(\nu) \approx 0.97 < 1$. By Mathematica software, we can get

$$\begin{aligned}
 &\left(c_1 + \frac{\eta(\bar{a} + \bar{L})M^\nu}{\Gamma(\nu + 1)} \right) \\
 &\quad \cdot \left(1 + \sum_{k=1}^{\infty} (\bar{a} + \bar{b} + 2\bar{L})^k \cdot \frac{M^{k\nu}}{\Gamma(k\nu + 1)} \right) \approx 463.26 \\
 &\leq c_2,
 \end{aligned} \tag{40}$$

which implies that inequality (28) holds, and we conclude that all conditions in Theorem 11 are satisfied. Therefore, the specific system (7) is finite-time stable.

5. Conclusion

In this paper, we are concerned with a nonlinear fractional-order difference equation. The addressed equation has time delay terms, which are quite different from the related references discussed in the literature [18, 23, 24, 28, 29]. The nonlinear fractional-order difference system studied in the present paper is more generalized and more practical. By applying the generalized Gronwall inequality and the definition of the finite-time stability, we employ a novel argument and the easily verifiable sufficient conditions have

been provided to determine the uniqueness and finite-time stability of the solutions for the considered equation. Finally, two typical examples have been presented at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion. Consequently, this paper shows theoretically and numerically that some related references known in the literature can be enriched and complemented.

Data Availability

The data in this study were mainly collected via discussion during our class. Readers wishing to access these data can do so by contacting the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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