Research Article

A New Algorithm for Solving Terminal Value Problems of $q$-Difference Equations

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We propose a new algorithm for solving the terminal value problems on a $q$-difference equations. Through some transformations, the terminal value problems which contain the first- and second-order delta-derivatives have been changed into the corresponding initial value problems; then with the help of the methods developed by Liu and H. Jafari, the numerical solution has been obtained and the error estimate has also been considered for the terminal value problems. Some examples are given to illustrate the accuracy of the numerical methods we proposed. By comparing the exact solution with the numerical solution, we find that the convergence speed of this numerical method is very fast.

1. Introduction

The $q$-difference equations theory appeared already at the beginning of the 20th century. Intensive works have been made especially by Jackson [1], Carmichael [2], Mason [3], Adams [4], Trijitzinsky [5], and others. But from the 1930s to the beginning of the 1980s, only nonsignificant interest in the area appeared.

The numerical solution of the $q$-difference equations plays an important role in many fields of science and engineering, mainly in physical science, as their solutions can provide more insight into the physical aspects of the problem.

Recently, many authors focus on the study of $q$-difference equations, especially on the numerical solution problems, such as [6, 7]. In [6], by using the differential transformation method, the authors studied the following strongly nonlinear $q$-difference equation:

$$x^{\Delta^2} + (2\gamma + \epsilon y_1 x (t)) x^{\Delta} + \Omega^2 x + x^2 = 0 \text{ on } q^{N_0}$$  \hspace{1cm} (1)

with $x(0) = a$, $x^{\Delta}(0) = b$. The parameters $\gamma$, $\gamma_1$ represent the linear damping parameters, $\epsilon$ is the nonlinearity parameter, $\Omega$ is the frequency of underdamped motion, and $0 < q < 1$. They established a numerical method and obtained the numerical results of (1). Later, Jafari et al. [7] proposed a numerical algorithm to obtain the numerical solutions for the first- and second-order dynamic equations on $q^{N_0}$. As an application, they investigated the numerical solution of (1) and, through the error analysis, they proved the effectiveness of the numerical algorithm they proposed.

From the works of [6] and [7], one can easily see that when $q > 1$, the methods they proposed are no longer effective, since for the formula

$$y(q^{N_0}) = y(0) + q^{N_0} y^{\Delta}(0) + O(q^{2N_0}),$$  \hspace{1cm} (2)

when $N_0 \to \infty$, $q^{2N_0} \to \infty$, so we cannot estimate $y(q^{N_0})$ by $y(0) + q^{N_0} y^{\Delta}(0)$. In this paper, we want to give a new numerical method to gain the numerical solution of the general first- and second-order dynamic equations on $q^{N_0}$ with $q > 1$. We assume that $q > 1$ always holds in the following unless otherwise is stated.

2. Preliminary

For $q$-difference equation, the theory of time scales plays an important role, which was initiated by Hilger [8] in his Ph.D. thesis to unify both difference and differential calculus in a consistent way. Since then many authors have investigated
the dynamic equations on time scales (see [9–11], etc.).
This theory is a powerful tool for mathematical analysis in economics, population models, quantum physics, and so on.

Before giving our main result, first we list some basic properties about time scales which could be found in ([8, 12, 13]).

**Definition 1.** A time scale is an arbitrary nonempty closed subset \(T\) of the real number \(\mathbb{R}\).

**Definition 2.** For \(t \in T\) we define the forward jump operator \(\sigma: T \to T\) and the backward jump operator \(\rho: T \to T\) by
\[
\sigma(t) = \inf \{ s \in T : s > t \},
\]
\[
\rho(t) = \sup \{ s \in T : s < t \},
\]
respectively.

**Definition 3.** Let \(T\) be a time scale, for \(t \in T\); if \(\sigma(t) > t\), we say that \(t\) is right-scattered, while if \(\rho(t) < t\), we say that \(t\) is left-scattered. Also, if \(t < \sup T\) and \(\sigma(t) = t\), then \(t\) is called right-dense, and if \(t > \inf T\) and \(\rho(t) = t\), then \(t\) is called left-dense.

**Definition 4.** A function \(f: T \to \mathbb{R}\) is called rd-continuous provided it is continuous at right-dense points in \(T\) and its left-sided limits exist (finite) at left-dense points in \(T\). The set of rd-continuous functions \(f: T \to \mathbb{R}\) is denoted by
\[
C_{rd}(T) = C_{rd}(\mathbb{R}) = \mathbb{R}.
\]

**Definition 5.** Assume that \(f: T \to \mathbb{R}\) and let \(t \in T^k\), where
\[
T^k = \{ (\rho(\sup T), \sup T), \text{ if } \sup T < \infty, \mathbb{T}, \text{ if } \sup T = \infty. \}
\]

Then we define \(f^{\Delta}(t)\) to be the number (provided it exists) with property that given any \(\varepsilon > 0\), there is a neighborhood \(U\) of \(t\) such that
\[
|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|,
\]
for all \(s \in U\). We call \(f^{\Delta}(t)\) the delta (or Hilger) derivative of \(f(t)\) and it turns out that \(f^{\Delta}\) is the usual derivative if \(T = \mathbb{R}\) and is the usual forward difference operator if \(T = \mathbb{Z}\).

**Lemma 6.** The delta derivative of \(f(t)\) on right-scattered point \(t\) can be calculated by
\[
f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \hspace{1cm} (7)
\]

In the following, we give two lemmas which will be useful to prove our main result.

**Lemma 7.** Assume that \(f: q^0 \to \mathbb{R}\) is delta differentiable at \(t \in q^0\). Let \(s = 1/\sqrt{\varepsilon}\) and denote \(p(s) = f(t)\); then \(p: q^{-N} \to \mathbb{R}\) is delta differentiable at \(s/q\) \(\epsilon q^{-N}\) and
\[
p^{\Delta} \left( \frac{s}{q} \right) = -q^t f^{\Delta} (t). \hspace{1cm} (8)
\]

Proof. By the definition of delta derivative of \(f(t)\) and Lemma 6, we know that, for \(t \in q^{-N}\),
\[
f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q - 1)t}, \hspace{1cm} (9)
\]
\[
p^{\Delta} \left( \frac{s}{q} \right) = \frac{p(s) - p(s/q)}{(1 - 1/q) s}; \hspace{1cm} (10)
\]
notice that
\[
f(qt) = p \left( \frac{1}{qt} \right) = p \left( \frac{s}{q} \right); \hspace{1cm} (11)
\]
then
\[
p^{\Delta} \left( \frac{s}{q} \right) = \frac{(q^2 f^{\Delta}(t))}{(q - 1)t} = q^t f^{\Delta} (t). \hspace{1cm} (12)
\]

**Lemma 8.** Assume that \(f: q^N \to \mathbb{R}\) is twice delta differentiable at \(t \in q^N\). Let \(s = 1/\sqrt{\varepsilon}\) and denote \(p(s) = f(t)\); then \(p: q^{-N} \to \mathbb{R}\) is twice delta differentiable at \(s/q^2 \epsilon q^{-N}\) and
\[
p^{\Delta \Delta} \left( \frac{s}{q^2} \right) = q^{2t} f^{\Delta \Delta} (t) + q^{3}(q + 1) t^{3} f^{\Delta} (qt). \hspace{1cm} (13)
\]

Proof. Since
\[
f^{\Delta \Delta}(t) = \frac{f^{\Delta}(qt) - f^{\Delta}(t)}{(q - 1)t}
\]
and on the other hand
\[
p^{\Delta \Delta} \left( \frac{s}{q^2} \right) = \frac{p^{\Delta}(s/q) - p^{\Delta}(s/q^{2})}{s/q - s/q^{2}}
\]
\[
= \frac{q^{2}t \left[ p^{\Delta}(s/q) - p^{\Delta}(s/q^{2}) \right]}{(q - 1)t}
\]
\[
= \frac{q^{2}t \left[ p^{\Delta}(1/qt) - p^{\Delta}(1/q^{2}t) \right]}{(q - 1)t}, \hspace{1cm} (14)
\]
thus
\[
f^{\Delta \Delta}(t) = \frac{1}{qt^{2}} \left[ \frac{p^{\Delta}(1/qt) - p^{\Delta}(1/q^{2}t)}{(q - 1)t} \right]
\]
\[
= \frac{1}{qt^{2}} p^{\Delta \Delta} \left( \frac{s}{q^2} \right) - \frac{(q + 1) t^{3} f^{\Delta} (qt)}{(q - 1)t}, \hspace{1cm} (15)
\]
that is, (12) holds true.
3. Numerical Method

In this section we present a numerical method to solve the terminal value problems involving the first- and second-order q-difference equations.

3.1. The First-Order. Consider the following TVP on \( q^n \) with \( q > 1 \):

\[
f(t, x(t), x^\Delta(t)) = 0,
\]

\[
x(\infty) = a. \tag{16}
\]

Here \( f \in C_\text{cl}(q^n \times \mathbb{R}^1 \times \mathbb{R}^1), \) \( x(\infty) = a \) means that \( \lim_{n \to \infty} x(q^n) = 0, \) \( x^\Delta(t) \) represents delta derivative; in fact, we can easily see that when \( q > 1, \) \( x^\Delta(t) = \frac{x(qt) - x(t)}{(q-1)t}, \) where \( D_q x(t) \) is the q-difference operator. For the concept of q-difference operator, one can see [14]. We assume that (16) with (17) has an unique solution.

First, we give a necessary condition for the existence of (16) with (17).

**Lemma 9.** If \( x(t) \) is the solution of (16) and (17), then

\[
f(\infty, a, 0) = 0. \tag{18}
\]

Especially, if \( f(t, x(t), x^\Delta(t)) \equiv F(x(t), x^\Delta(t)), \) then

\[
F(a, 0) = 0. \tag{19}
\]

Now we present a numerical method to solve (16) and (17).

**Algorithm 10.** Step 1. Make the transformation \( s = \frac{1}{t} \) and let \( y(s) = x(t) \); by (8), (16) with (17) is transformed into

\[
f \left( \frac{1}{s}, y(s), -\frac{s^2}{q} y^\Delta \left( \frac{s}{q} \right) \right) = 0, \quad s \in q^{-N_0}
\]

\[
y(0) = a. \tag{20}
\]

Step 2. Motivated by the work of Liu [6], let \( y(0) = a, \) \( t_1 = 0, \) \( t_i = q^{-N_0+i-1} (i = 2, \ldots, N_0 + 1) \) and \( s_i = q t_i; \) then (20) will be changed into

\[
f \left( \frac{1}{q t_i}, y(t_{i+1}), -\frac{q^2}{q} t_i^2 y^\Delta(t_i) \right) = 0; \tag{21}
\]

notice that

\[
y^\Delta(t_i) = \frac{y(t_{i+1}) - y(t_i)}{(q-1)t_i}; \tag{22}
\]

then

\[
f \left( \frac{1}{q t_i}, y(t_{i+1}), -\frac{q}{q-1} t_i \left[ y(t_{i+1}) - y(t_i) \right] \right) = 0. \tag{23}
\]

Step 3. From the above equality, set \( y(t_1) = a \) and solve \( y(t_{i+1}), \) where obviously \( y(t_{i+1}) \) only depends on \( y(t_i), q, \) and \( t_i; \) thus we can obtain all the values of \( y \) at each point \( t_i. \)

**Example 11.** Consider the following \( q \)-difference equation \( (q > 1): \)

\[
x^\Delta(t) = -x(t) + \frac{qt^2 + t + 1}{(t + 1)(qt + 1)}, \tag{24}
\]

\[
x(\infty) = 1.
\]

It is easy to see that \( x(t) = t/(t + 1) \) is the solution of (24); we now use Algorithm 10 to obtain the numerical solution of (24).

Let \( s = 1/t \) and \( y(s) = x(t); \) by (23), (24) can be transformed into

\[
-\frac{q}{q-1} t_i \left[ y(t_{i+1}) - y(t_i) \right] = -y(t_{i+1}) + \frac{qt_i^2 + t + 1}{(t + 1)(qt_i + 1)} \tag{25}
\]

\[
y(0) = 1.
\]

Solving the above equations, we can obtain

\[
y(t_{i+1}) = \frac{(q/(q-1)) t_i y(t_i)}{1 - (q/(q-1)) t_i} + \frac{qt_i^2 + t + 1}{(t + 1)(qt_i + 1)} \tag{26}
\]

\[
+ \frac{q^2 t_i^2 + t + 1}{(t + 1)(qt_i + 1)} \left( 1 - (q/(q-1)) t_i \right).
\]

Let \( y(t_i) = 1; \) by recurrence formula (26), we can obtain all values of \( y(t_i); \) we calculate it in Table 1 for \( q = 3 \) and \( N_0 = 50. \)

From Table 1, we can see that the maximum error is \( 0.44 \times 10^{-6} \) and the average error is \( 0.44 \times 10^{-10}. \)

Figure 1 shows the comparison of exact value and approximate value of Example 11 when \( q = 1.1 \) and \( N_0 = 50. \)

3.2. The Second-Order. Consider the following TVP on \( q^n \) with \( q > 1: \)

\[
f \left( t, x(t), x^\Delta(t), x^{2\Delta}(t) \right) = 0, \tag{27}
\]

\[
\lim_{n \to \infty} x(q^n) = a, \tag{28}
\]

\[
\lim_{n \to \infty} q^{2n} x^{\Delta^n}(q^n) = b.
\]

Here \( f \in C_\text{cl}(q^n \times \mathbb{R}^2 \to \mathbb{R}), \) \( x^\Delta(t) \) represents delta derivative. Also we assume that (27) with (28) has only one solution.

First, we give a necessary condition for the existence of solution for (27) with (28).

**Lemma 12.** If \( x(t) \) is the solution of (27) and (28), then

\[
f(\infty, a, 0, 0) = 0. \tag{29}
\]

Especially, if \( f(t, x(t), x^\Delta(t), x^{2\Delta}(t)) \equiv F(x(t), x^\Delta(t), x^{2\Delta}(t)), \) then

\[
F(a, 0, 0) = 0. \tag{30}
\]
Let \( t = 1/s, \ x(t) = y(s); \) then by (8), we know that 
\[ y^{\Delta}(0) = -qb. \]
On the other hand, notice that \( \lim_{n \to \infty} q^n = 0; \) then
\[ y^{\Delta}(0) = \lim_{n \to \infty} \frac{y(q^n) - y(0)}{q^n}. \]  
(31)

Let \( N_0 \) be a nature number; then using the Taylor formula on time scales, we have
\[ y(q^{-N_0}) = y(0) + q^{-N_0} y^{\Delta}(0) + \frac{q^{-2N_0}}{2} y^{\Delta\Delta}(0) + \cdots; \]  
(32)

thus
\[ y(q^{-N_0}) y^{\Delta}(0) = a - bq^{1-N_0}; \]  
(33)

set \( t_1 = 0, t_2 = q^{-N_0}, \) and \( t_i = q^{-N_0+(i-2)}, i = 3, 4, \ldots, N_0 + 2; \) let \( y_i = y(t_i); \) then
\[ y_1 = a, \]
\[ y_2 = y(q^{-N_0}) = a - bq^{1-N_0}. \]  
(34)

Note that the delta derivative of \( y(t) \) at \( t_i \) can be calculated as
\[ y^{\Delta}(t_i) = \frac{y(qt_i) - y(t_i)}{(q-1) t_i} = \frac{y_{i+1} - y_i}{(q-1) t_i}; \]  
(35)
\[ y^{\Delta\Delta}(t_i) = \frac{y^{\Delta}(qt_i) - y^{\Delta}(t_i)}{(q-1) t_i} = \frac{(y_{i+2} - y_{i+1})/ (q-1) t_{i+1} - (y_{i+1} - y_i) / (q-1) t_i}{(q-1) t_i}. \]  
(36)

Let \( A_i, B_i, C_i \) be as follows:
\[ A_i = \frac{1}{(q-1)^2 t_i t_{i+1}}, \]
\[ B_i = \frac{t_i + t_{i+1} (q-1)^2 t_i t_{i+1}}, \]  
(37)
\[ C_i = \frac{1}{(q-1)^3 t_i^2}. \]

In the following, we give a method to solve the numerical solution of (27) with (28).
Algorithm 13. Step 1. Make the transformation $s = 1/t$ and let $y(s) = x(t)$; by (8) and (12), (27) with (28) is transformed into

$$
\begin{align*}
& f\left(\frac{1}{s}, y(s), -\frac{s^2}{q} y'\left(\frac{s}{q}\right), \frac{s^4}{q^2} y^{(\Delta)}\left(\frac{s}{q^2}\right)\right) \\
& + \left(\frac{q+1}{q^3}\right) y^\Delta\left(\frac{s}{q^3}\right) = 0, \quad s \in q^{-N_0} \quad (38)
\end{align*}
$$

$y(0) = a,$

$y^\Delta(0) = -qb.$

Step 2. Let $y(0) = a$, $y(q^{-N_0}) = a - bq^{1-N_0}$; set $t_1 = 0$, $t_2 = q^{-N_0}$, $t_i = q^{-N_0+i-2}$ ($i = 3, 4, \ldots, N_0 + 2$) and $s_i = q^5 t_i$; then (38) will be changed into

$$
\begin{align*}
& f\left(\frac{1}{q^3 t_i}, y(t_{i+2}), -\frac{q^2 t_i (y_{i+2} - y_{i+1})}{q - 1}, \frac{q^4 t_i^2 (y_{i+4} - y_{i+3})}{q - 1} \right) \\
& + q^3 (q + 1) t_i^2 y^\Delta(t_i) = 0;
\end{align*}
$$

notice that

$$
y^\Delta(t_{i+1}) = \frac{y(t_{i+2}) - y(t_{i+1})}{(q - 1) t_{i+1}}; \quad (40)
$$

then

$$
\begin{align*}
& f\left(\frac{1}{q^3 t_i}, y_{i+2}, -\frac{q^2 t_i (y_{i+2} - y_{i+1})}{q - 1}, \right) \\
& q^4 t_i^2 (A_i y_{i+2} - B_i y_{i+1} + C_i y_i) + q^3 (q + 1) \\
& \cdot t_i^2 (y_{i+1} - y_i) \right) = 0,
\end{align*}
$$

$y_1 = a,$

$y_2 = a - bq^{1-N_0}.$

Step 3. From (41), we can solve $y_{i+2}$ step by step.

Now we consider an example to illustrate the efficiency of the numerical method.

Example 14. Consider the following second-order terminal value problem on $q^{-N_0}(q > 1)$

$$
x^{(\Delta)}(t) + \frac{q + 1}{q^2 t} x^{(\Delta)}(t) + x(t) = 1 + \frac{1}{t}, \quad (42)
$$

$\lim_{n \to N_0} x\left(q^n\right) = 1,$

$\lim_{n \to N_0} q^{2n} x\left(q^n\right) = -\frac{1}{q} \quad (43)$

It is easy to see that $x(t) = 1/t + 1$ is the solution of (42) with the terminal value condition (43).
Table 2: The comparison of exact value and approximate value for $q = 2$ and $N_0 = 10$.

(a)

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![Graph](image)

Figure 2: For $q = 1.1$, $N_0 = 50$.

Additional Points

2000 AMS Subject Classification is as follows: 39A12, 39A13, 37N30.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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