Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 8509860, 7 pages
https://doi.org/10.1155/2018/8509860

Research Article

A New Algorithm for Solving Terminal Value Problems of q-Difference Equations

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Received 21 January 2018; Revised 30 March 2018; Accepted 11 April 2018; Published 14 May 2018

Academic Editor: Douglas R. Anderson

We propose a new algorithm for solving the terminal value problems on a q-difference equations. Through some transformations, the terminal value problems which contain the first- and second-order delta-derivatives have been changed into the corresponding initial value problems; then with the help of the methods developed by Liu and H. Jafari, the numerical solution has been obtained and the error estimate has also been considered for the terminal value problems. Some examples are given to illustrate the accuracy of the numerical methods we proposed. By comparing the exact solution with the numerical solution, we find that the convergence speed of this numerical method is very fast.

1. Introduction

The q-difference equations theory appeared already at the beginning of the 20th century. Intensive works have been made especially by Jackson [1], Carmichael [2], Mason [3], Adams [4], Trjitzinsky [5], and others. But from the 1930s to the beginning of the 1980s, only nonsignificant interest in the area appeared.

The numerical solution of the q-difference equations plays an important role in many fields of science and engineering, mainly in physical science, as their solutions can provide more insight into the physical aspects of the problem.

Recently, many authors focus on the study of q-difference equations, especially on the numerical solution problems, such as [6, 7]. In [6], by using the differential transformation method, the authors studied the following strongly nonlinear q-difference equation:

$$x^{\Delta\Delta} + \left(2\gamma + \varepsilon\gamma_1 x(t)\right) x^{\Delta} + \Omega^2 x + x^2 = 0 \text{ on } \overline{q^{\mathbb{N}_0}} \qquad (1)$$

with $x(0) = a$, $x^{\Delta}(0) = b$. The parameters γ, γ_1 represent the linear damping parameters, ε is the nonlinearity parameter, Ω is the frequency of underdamped motion, and $0 < q < 1$. They established a numerical method and obtained the numerical results of (1). Later, Jafari et al. [7] proposed a numerical algorithm to obtain the numerical solutions for the first- and second-order dynamic equations on $\overline{q^{N}}$. As an application, they investigated the numerical solution of (1) and, through the error analysis, they proved the effectiveness of the numerical algorithm they proposed.

From the works of [6] and [7], one can easily see that when $q > 1$, the methods they proposed are no longer effective, since for the formula

$$y\left(q^{N_0}\right) = y(0) + q^{N_0} y^{\Delta}(0) + O\left(q^{2N_0}\right), \qquad (2)$$

when $N_0 \to \infty$, $q^{2N_0} \to \infty$, so we cannot estimate $y(q^{N_0})$ by $y(0) + q^{N_0} y^{\Delta}(0)$. In this paper, we want to give a new numerical method to gain the numerical solution of the general first- and second-order dynamic equations on $q^{\mathbb{N}_0}$ with $q > 1$. We assume that $q > 1$ always hold in the following unless otherwise is stated.

2. Preliminary

For q-difference equation, the theory of time scales plays an important role, which was initiated by Hilger [8] in his Ph.D. thesis to unify both difference and differential calculus in a consistent way. Since then many authors have investigated the dynamic equations on time scales (see [9–11], etc.). This theory is a powerful tool for mathematical analysis in economics, population models, quantum physics, and so on.

Before giving our main result, first we list some basic properties about time scales which could be found in ([8, 12, 13]).

Definition 1. A time scale is an arbitrary nonempty closed subset $\mathbb{T}$ of the real number $\mathbb{R}$.

Definition 2. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\begin{aligned} \sigma(t) &:= \inf\{s \in \mathbb{T} : s > t\}, \\ \rho(t) &:= \sup\{s \in \mathbb{T} : s < t\}, \end{aligned} \tag{3}$$

respectively.

Definition 3. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$; if $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.

Definition 4. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}). \tag{4}$$

Definition 5. Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$, where

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases} \tag{5}$$

Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \le \varepsilon |\sigma(t) - s|, \tag{6}$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of $f(t)$ and it turns out that f^{Δ} is the usual derivative if $\mathbb{T} = \mathbb{R}$ and is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

Lemma 6. *The delta derivative of $f(t)$ on right-scattered point t can be calculated by*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{7}$$

In the following, we give two lemmas which will be useful to prove our main result.

Lemma 7. *Assume that $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is delta differentiable at $t \in q^{\mathbb{N}_0}$. Let $s = 1/t$ and denote $p(s) = f(t)$; then $p : q^{-\mathbb{N}_0} \to \mathbb{R}$ is delta differentiable at $(s/q) \in q^{-\mathbb{N}_0}$ and*

$$p^{\Delta}\left(\frac{s}{q}\right) = -qt^2 f^{\Delta}(t). \tag{8}$$

Proof. By the definition of delta derivative of $f(t)$ and Lemma 6, we know that, for $t \in q^{\mathbb{N}_0}$,

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(qt) - f(t)}{(q-1)t}, \\ p^{\Delta}\left(\frac{s}{q}\right) &= \frac{p(s) - p(s/q)}{(1 - 1/q)s}; \end{aligned} \tag{9}$$

notice that

$$f(qt) = p\left(\frac{1}{qt}\right) = p\left(\frac{s}{q}\right); \tag{10}$$

then

$$p^{\Delta}\left(\frac{s}{q}\right) = \frac{qt^2(f(t) - f(qt))}{(q-1)t} = -qt^2 f^{\Delta}(t). \tag{11}$$

□

Lemma 8. *Assume that $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is twice delta differentiable at $t \in q^{\mathbb{N}_0}$. Let $s = 1/t$ and denote $p(s) = f(t)$; then $p : q^{-\mathbb{N}_0} \to \mathbb{R}$ is twice delta differentiable at $s/q^2 \in q^{-\mathbb{N}_0}$ and*

$$p^{\Delta\Delta}\left(\frac{s}{q^2}\right) = q^3 t^4 f^{\Delta\Delta}(t) + q^3(q+1)t^3 f^{\Delta}(qt). \tag{12}$$

Proof. Since

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{f^{\Delta}(qt) - f^{\Delta}(t)}{(q-1)t} \\ &= \frac{-\left(1/q^3t^2\right)p^{\Delta}\left(1/q^2t\right) + \left(1/qt^2\right)p^{\Delta}(1/qt)}{(q-1)t} \end{aligned} \tag{13}$$

and on the other hand

$$\begin{aligned} p^{\Delta\Delta}\left(\frac{s}{q^2}\right) &= \frac{p^{\Delta}(s/q) - p^{\Delta}\left(s/q^2\right)}{s/q - s/q^2} \\ &= \frac{q^2t^2\left[p^{\Delta}(s/q) - p^{\Delta}\left(s/q^2\right)\right]}{(q-1)t} \\ &= \frac{q^2t^2\left[p^{\Delta}(1/qt) - p^{\Delta}\left(1/q^2t\right)\right]}{(q-1)t}, \end{aligned} \tag{14}$$

thus

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{1}{qt^2}\frac{\left[p^{\Delta}(1/qt) - p^{\Delta}\left(1/q^2t\right)\right]}{(q-1)t} \\ &\quad - \frac{\left(1/qt^2 - 1/q^3t^2\right)q^3t^2 f^{\Delta}(qt)}{(q-1)t} \\ &= \frac{1}{q^3t^4}p^{\Delta\Delta}\left(\frac{s}{q^2}\right) - \frac{(q+1)f^{\Delta}(qt)}{t}; \end{aligned} \tag{15}$$

that is, (12) holds true. □

3. Numerical Method

In this section we present a numerical method to solve the terminal value problems involving the first- and second-order q-difference equations.

3.1. The First-Order. Consider the following TVP on $q^{\mathbb{N}_0}$ with $q > 1$:

$$f\left(t, x(t), x^{\Delta}(t)\right) = 0, \tag{16}$$

$$x(\infty) = a. \tag{17}$$

Here $f \in C_{rd}(q^{N_0} \times \mathbb{R}^1 \times \mathbb{R}^1, \mathbb{R}^1)$, $x(\infty) = a$ means that $\lim_{n\to\infty} x(q^n) = 0$, $x^{\Delta}(t)$ represents delta derivative; in fact, we can easily see that when $q > 1$, $x^{\Delta}(t) = D_q x(t) = (x(qt) - x(t))/(q-1)t$, where $D_q x(t)$ is the q-difference operator. For the concept of q-difference operator, one can see [14]. We assume that (16) with (17) has an unique solution.

First, we give a necessary condition for the existence of (16) with (17).

Lemma 9. *If $x(t)$ is the solution of (16) and (17), then*

$$f(+\infty, a, 0) = 0. \tag{18}$$

Especially, if $f(t, x(t), x^{\Delta}(t)) \equiv F(x(t), x^{\Delta}(t))$, then

$$F(a, 0) = 0. \tag{19}$$

Now we present a numerical method to solve (16) and (17).

Algorithm 10. Step 1. Make the transformation $s = 1/t$ and let $y(s) = x(t)$; by (8), (16) with (17) is transformed into

$$\begin{gathered} f\left(\frac{1}{s}, y(s), -\frac{s^2}{q} y^{\Delta}\left(\frac{s}{q}\right)\right) = 0, \quad s \in q^{-\mathbb{N}_0} \\ y(0) = a. \end{gathered} \tag{20}$$

Step 2. Motivated by the work of Liu [6], let $y(0) = a$, $t_1 = 0$, $t_i = q^{-N_0+i-1}$ $(i = 2, \ldots, N_0 + 1)$ and $s_i = qt_i$; then (20) will be changed into

$$f\left(\frac{1}{qt_i}, y(t_{i+1}), -qt_i^2 y^{\Delta}(t_i)\right) = 0; \tag{21}$$

notice that

$$y^{\Delta}(t_i) = \frac{y(t_{i+1}) - y(t_i)}{(q-1)t_i}; \tag{22}$$

then

$$f\left(\frac{1}{qt_i}, y(t_{i+1}), \frac{-q}{q-1} t_i \left[y(t_{i+1}) - y(t_i)\right]\right) = 0. \tag{23}$$

Step 3. From the above equality, set $y(t_1) = a$ and solve $y(t_{i+1})$, where obviously $y(t_{i+1})$ only depends on $y(t_i)$, q, and t_i; thus we can obtain all the values of y at each point t_i.

Example 11. Consider the following q-difference equation $(q > 1)$:

$$\begin{gathered} x^{\Delta}(t) = -x(t) + \frac{qt^2 + t + 1}{(t+1)(qt+1)}, \\ x(\infty) = 1. \end{gathered} \tag{24}$$

It is easy to see that $x(t) = t/(t+1)$ is the solution of (24); we now use Algorithm 10 to obtain the numerical solution of (24).

Let $s = 1/t$ and $y(s) = x(t)$; by (23), (24) can be transformed into

$$\begin{aligned} &\frac{-q}{q-1} t_i \left[y(t_{i+1}) - y(t_i)\right] \\ &\quad = -y(t_{i+1}) + \frac{qt_i^2 + t_i + 1}{(t_i+1)(qt_i+1)}, \\ &y(0) = 1. \end{aligned} \tag{25}$$

Solving the above equations, we can obtain

$$\begin{aligned} y(t_{i+1}) = {} & -\frac{(q/(q-1))t_i}{1 - (q/(q-1))t_i} y(t_i) \\ & + \frac{qt_i^2 + t_i + 1}{(t_i+1)(qt_i+1)(1 - (q/(q-1))t_i)}. \end{aligned} \tag{26}$$

Let $y(t_1) = 1$; by recurrence formula (26), we can obtain all values of $y(t_i)$; we calculate it in Table 1 for $q = 3$ and $N_0 = 10$.

From Table 1, we can see that the maximum error is 0.44×10^{-9} and the average error is 0.44×10^{-10}.

Figure 1 shows the comparison of exact value and approximate value of Example 11 when $q = 1.1$ and $N_0 = 50$.

3.2. The Second-Order. Consider the following TVP on $q^{\mathbb{N}_0}$ with $q > 1$:

$$f\left(t, x(t), x^{\Delta}(t), x^{\Delta\Delta}(t)\right) = 0, \tag{27}$$

$$\begin{gathered} \lim_{n\to\infty} x(q^n) = a, \\ \lim_{n\to\infty} q^{2n} x^{\Delta}(q^n) = b. \end{gathered} \tag{28}$$

Here $f \in C_{\text{rd}}(q^{\mathbb{N}_0} \times \mathbb{R}^3 \to \mathbb{R})$, $x^{\Delta}(t)$ represents delta derivative. Also we assume that (27) with (28) has only one solution.

First, we give a necessary condition for the existence of solution for (27) with (28).

Lemma 12. *If $x(t)$ is the solution of (27) and (28), then*

$$f(+\infty, a, 0, 0) = 0. \tag{29}$$

Especially, if $f(t, x(t), x^{\Delta}(t), x^{\Delta\Delta}(t)) \equiv F(x(t), x^{\Delta}(t), x^{\Delta\Delta}(t))$, then

$$F(a, 0, 0) = 0. \tag{30}$$

Table 1: The comparison of exact value and approximate value.

(a)

Time domain	0.000050805263425	0.000152415790276	0.000457247370828
Exact value	0.999949197317618	0.999847607436757	0.999542961608775
Approximate value	0.999949196887419	0.999847607436790	0.999542961608775
Error (≤)	-0.43×10^{-9}	0.327×10^{-13}	-0.111×10^{-15}
Absolute error (≤)	0.44×10^{-9}	0.328×10^{-13}	0.112×10^{-15}

(b)

0.001371742112483	0.004115226337449	0.012345679012346	0.037037037037037
0.998630136986301	0.995901639344262	0.987804878048780	0.964285714285714
0.998630136986301	0.995901639344262	0.987804878048780	0.964285714285714
0	0	0	0
0	0	0	0

(c)

0.111111111111111	0.333333333333333	1.000000000000000
0.900000000000000	0.750000000000000	0.500000000000000
0.900000000000000	0.750000000000000	0.500000000000000
0	0	0
0	0	0

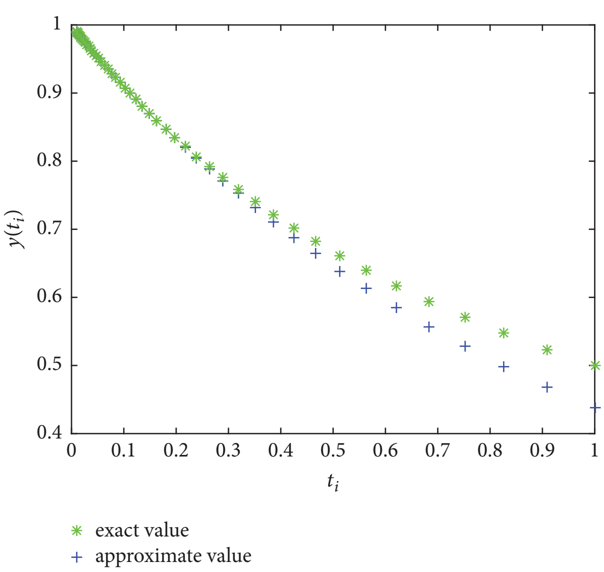

Figure 1: For $q = 1.1$, $N_0 = 50$.

Let $t = 1/s$, $x(t) = y(s)$; then by (8), we know that $y^{\Delta}(0) = -qb$. On the other hand, notice that $\lim_{n\to\infty} q^{-n} = 0$; then

$$y^{\Delta}(0) = \lim_{n\to\infty} \frac{y(q^{-n}) - y(0)}{q^{-n}}. \tag{31}$$

Let N_0 be a nature number; then using the Taylor formula on time scales, we have

$$y\left(q^{-N_0}\right) = y(0) + q^{-N_0} y^{\Delta}(0) + \frac{q^{-2N_0}}{2} y^{\Delta^2}(0) + \cdots; \tag{32}$$

thus

$$y\left(q^{-N_0}\right) \approx y(0) + q^{-N_0} y^{\Delta}(0) = a - bq^{1-N_0}; \tag{33}$$

set $t_1 = 0$, $t_2 = q^{-N_0}$, and $t_i = q^{-N_0+(i-2)}$, $i = 3, 4, \ldots, N_0 + 2$; let $y_i = y(t_i)$; then

$$\begin{aligned} y_1 &= a, \\ y_2 &= y\left(q^{-N_0}\right) \approx a - bq^{1-N_0}. \end{aligned} \tag{34}$$

Note that the delta derivative of $y(t)$ at t_i can be calculated as

$$y^{\Delta}(t_i) = \frac{y(qt_i) - y(t_i)}{(q-1)t_i} = \frac{y_{i+1} - y_i}{(q-1)t_i}, \tag{35}$$

$$\begin{aligned} y^{\Delta\Delta}(t_i) &= \frac{y^{\Delta}(qt_i) - y^{\Delta}(t_i)}{(q-1)t_i} \\ &= \frac{(y_{i+2} - y_{i+1})/(q-1)t_{i+1} - (y_{i+1} - y_i)/(q-1)t_i}{(q-1)t_i} \\ &= A_i y_{i+2} - B_i y_{i+1} + C_i y_i, \end{aligned} \tag{36}$$

where

$$\begin{aligned} A_i &= \frac{1}{(q-1)^2 t_i t_{i+1}}, \\ B_i &= \frac{t_i + t_{i+1}}{(q-1)^2 t_i^2 t_{i+1}}, \\ C_i &= \frac{1}{(q-1)^2 t_i^2}. \end{aligned} \tag{37}$$

In the following, we give a method to solve the numerical solution of (27) with (28).

Algorithm 13. Step 1. Make the transformation $s = 1/t$ and let $y(s) = x(t)$; by (8) and (12), (27) with (28) is transformed into

$$\begin{aligned} & f\left(\frac{1}{s}, y(s), -\frac{s^2}{q} y^{\Delta}\left(\frac{s}{q}\right), \frac{s^4}{q^3} y^{\Delta\Delta}\left(\frac{s}{q^2}\right) \right. \\ & \left. \quad + \frac{(q+1)s^3}{q^3} y^{\Delta}\left(\frac{s}{q^2}\right)\right) = 0, \quad s \in q^{-\mathbb{N}_0} \\ & y(0) = a, \\ & y^{\Delta}(0) = -qb. \end{aligned} \tag{38}$$

Step 2. Let $y(0) = a$, $y(q^{-N_0}) = a - bq^{1-N_0}$; set $t_1 = 0$, $t_2 = q^{-N_0}$, $t_i = q^{-N_0+i-2}$ $(i = 3, 4, \ldots, N_0 + 2)$ and $s_i = q^2 t_i$; then (38) will be changed into

$$\begin{aligned} & f\left(\frac{1}{q^2 t_i}, y(t_{i+2}), -q^3 t_i^2 y^{\Delta}(t_{i+1}), q^5 t_i^4 y^{\Delta\Delta}(t_i) \right. \\ & \left. \quad + q^3 (q+1) t_i^3 y^{\Delta}(t_i)\right) = 0; \end{aligned} \tag{39}$$

notice that

$$y^{\Delta}(t_{i+1}) = \frac{y(t_{i+2}) - y(t_{i+1})}{(q-1)t_{i+1}}; \tag{40}$$

then

$$\begin{aligned} & f\left(\frac{1}{q^2 t_i}, y_{i+2}, \frac{-q^2 t_i (y_{i+2} - y_{i+1})}{q-1}, \right. \\ & \quad q^5 t_i^4 (A_i y_{i+2} - B_i y_{i+1} + C_i y_i) + q^3 (q+1) \\ & \left. \quad \cdot t_i^2 (y_{i+1} - y_i)\right) = 0, \\ & y_1 = a, \\ & y_2 = a - bq^{1-N_0}. \end{aligned} \tag{41}$$

Step 3. From (41), we can solve y_{i+2} step by step.

Now we consider an example to illustrate the efficiency of the numerical method.

Example 14. Consider the following second-order terminal value problem on $q^{\mathbb{N}_0} (q > 1)$

$$x^{\Delta\Delta}(t) + \frac{q+1}{q^2 t} x^{\Delta}(t) + x(t) = 1 + \frac{1}{t}, \tag{42}$$

$$\begin{aligned} & \lim_{n\to\infty} x(q^n) = 1, \\ & \lim_{n\to\infty} q^{2n} x(q^n) = -\frac{1}{q}. \end{aligned} \tag{43}$$

It is easy to see that $x(t) = 1/t + 1$ is the solution of (42) with the terminal value condition (43).

Let $t = 1/s$, $y(s) = x(t)$; then (42) can be transformed into

$$\begin{aligned} & \frac{s^4}{q^3} y^{\Delta\Delta}\left(\frac{s}{q^2}\right) + \frac{(q+1)s^3}{q^3} y^{\Delta}\left(\frac{s}{q^2}\right) \\ & \quad - \frac{(q+1)s^3}{q^3} y^{\Delta}\left(\frac{s}{q}\right) + y(s) = 1 + s, \end{aligned} \tag{44}$$

$$\begin{aligned} & y(0) = 1, \\ & y^{\Delta}(0) = 1. \end{aligned} \tag{45}$$

Set $s_i = q^2 t_i$; by (41), we have

$$\begin{aligned} & q^5 t_i^4 (A_i y_{i+2} - B_i y_{i+1} + C_i y_i) + (q+1) q^3 t_i^3 \frac{y_{i+1} - y_i}{t_{i+1} - t_i} \\ & \quad - (q+1) q^3 t_i^3 \frac{y_{i+2} - y_{i+1}}{t_{i+2} - t_{i+1}} + y_{i+2} = 1 + t_{i+2}. \end{aligned} \tag{46}$$

Therefore

$$D_i y_{i+2} + E_i y_{i+1} + F_i y_i = 1 + t_{i+2}, \tag{47}$$

where

$$\begin{aligned} & D_i = 1 - \frac{(q+1) q^2 t_i^2}{q-1} + q^5 t_i^4 A_i, \\ & E_i = -q^5 t_i^4 B_i + \frac{(q+1)^2 q^2 t_i^2}{q-1}, \\ & F_i = -\frac{(q+1) q^3 t_i^2}{q-1} + q^5 t_i^4 C_i. \end{aligned} \tag{48}$$

Notice that $y_1 = 1$, $y_2 = 1 + q^{-N_0}$; then by recurrence formula (47), we can obtain all values of y_i; we calculate it in Table 2.

From Table 2, we can see that the maximum error is 1×10^{-8} and the average error is 1×10^{-9}.

Figure 2 shows the comparison of exact value and approximate value of Example 14 when $q = 1.1$ and $N_0 = 50$.

4. Discussion and Conclusion

In this paper we have presented a new algorithm for solving the terminal value problems on a q-difference equations. It makes a considering improvement: this establishing method is easy to program implementation, since the numerical calculation formula we established is a relatively simple iteration formula; on the other hand, this establishing method could be generalized to the third-order case, the fourth order case, and so on.

For problem (20) or (37)-(38), suppose that $y^*(t)$ is the exact solution and $y(t)$ is the numerical solution; from Algorithms 10 and 13, we can see that $y(0) = y^*(0)$; that is, $y(0)$ is the exact value. However, $y(q^{-N_0}) - y^*(q^{-N_0}) = O(q^{-2N_0})$; notice that $q > 1$; the error of $y(q^{-N_0})$ converges to zero exponentially. In this sense, we may say that the convergence speed of this numerical method is very fast. In

Table 2: The comparison of exact value and approximate value for $q = 2$ and $N_0 = 10$.

(a)

Time domain	0.001953125	0.00390625	0.0078125
Exact value	1.001953125	1.00390625	1.0078125
Approximate value	1.001953127793965	1.003906249999968	1.007812500000085

(b)

0.015625	0.03125	0.0625	0.125	0.25	0.5	1
1.015625	1.03125	1.0625	1.125	1.25	1.5	2
1.015625	1.03125	1.0625	1.125	1.25	1.5	2

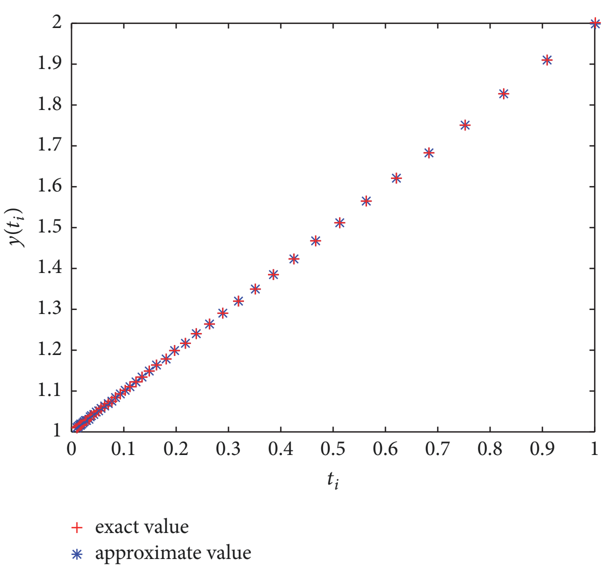

Figure 2: For $q = 1.1$, $N_0 = 50$.

fact, the calculation precision can be further improved. By (32), the computational accuracy depends on the number of terms on the right-hand side. The more a higher-order derivative of $y^*(t)$ at zero is obtained, the smaller the error would be.

As far as we know, there are very few mathematicians involved in the terminal values problems of q-difference equations. In [15], the authors examine "terminal" value problems for dynamic equations on time scales; that is, a dynamic equation whose solutions are asymptotic at infinity. They present a number of new theorems that guarantee the existence and uniqueness of solutions, as well as some comparison-type results. And in [16], the authors propose a nonpolynomial collocation method for solving a class of terminal (or boundary) value problems for differential equations of fractional order, and they study the order of convergence of the proposed algorithm. But for the numerical algorithm on the terminal values problems of q-difference equations, we have not found the relevant conclusion.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Additional Points

2000 AMS Subject Classification is as follows: 39A12, 39A13, 37N30.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by NSF of Shandong Province (ZR2015AM026), the Project of Shandong Provincial Higher Educational Science and Technology (J15LI07), and NSF of China (11201213).

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