The Existence Results of Solutions for System of Fractional Differential Equations with Integral Boundary Conditions

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In this paper, we investigated the system of fractional differential equations with integral boundary conditions. By using a fixed point theorem in the Banach spaces, we get the existence of solutions for the fractional differential system. By constructing iterative sequences for any given initial point in space, we can approximate this solution. As an application, an example is presented to illustrate our main results.

1. Introduction

The fractional differential equations attracted many people’s attention, and they have many applications in different fields of science and biology. In recent years, the research on the qualitative properties of fractional differential equations has become a hot topic, and many results have been obtained; see [1–29]. From the literature, there are many articles investing the existence of solutions or the solutions for some nonlinear systems; see [4–10, 21].

In [6], Su studied the following system of nonlinear fractional differential equations:

\[
\begin{align*}
&D^\alpha u(t) = f(t, v(t)), \\
&D^\beta v(t) = g(t, u(t)), \quad u(0) = v(0) = u(1) = v(1) = 0,
\end{align*}
\]

where \(0 < t < 1\), \(1 < \alpha, \beta < 2\), \(\mu, \nu > 0\), \(\alpha - \nu, \beta - \mu \geq 1\), \(f, g\) are given functions and \(D^\alpha, D^\beta\) are Riemann-Liouville fractional derivatives of order \(\alpha, \beta\). The author gave sufficient conditions for the existence of solutions of system (1).

In [7], Wang et al. studied the existence of solutions for the following system of nonlinear fractional differential equations:

\[
\begin{align*}
&D^\alpha u(t) = a(t) f(t, v(t)), \\
&D^\beta v(t) = b(t) g(t, u(t)), \quad u(0) = v(0) = 0, \\
&u(1) = \int_0^1 \phi(t) u(t) \, dt, \\
v(1) = \int_0^1 \psi(t) v(t) \, dt,
\end{align*}
\]

where \(0 < t < 1\), \(1 < \alpha, \beta < 2\), \(0 < \xi < 1\), \(0 \leq a, b \leq 1\), \(f, g\) are given functions and \(D^\alpha, D^\beta\) are Riemann-Liouville fractional derivatives of order \(\alpha, \beta\).

In [8], Yang studied the existence of solutions for the following system of nonlinear fractional differential equations:

\[
\begin{align*}
&D^\alpha u(t) = a(t) f(t, v(t)), \\
&D^\beta v(t) = b(t) g(t, u(t)), \quad u(0) = v(0) = 0, \\
&u(1) = \int_0^1 \phi(t) u(t) \, dt, \\
v(1) = \int_0^1 \psi(t) v(t) \, dt,
\end{align*}
\]

where \(0 < t < 1\), \(1 < \alpha, \beta < 2\), \(a, b \in C((0, 1), [0, \infty))\), \(\phi, \psi \in L^1[0, 1]\) are nonnegative and
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Let \( f, g \in C([0, 1] \times [0, +\infty), [0, +\infty)) \), and \( D^\alpha, D^\beta \) are also the standard Riemann-Liouville fractional derivatives of order \( \alpha, \beta \).

In [9], Alsulami et al. studied the existence of solutions for the following system of nonlinear Caputo fractional integrodifferential equations:

\[
\begin{align*}
\frac{d^\alpha u(t)}{dt^\alpha} &= f(t, u(t), v(t)), \\
\frac{d^\beta v(t)}{dt^\beta} &= g(t, u(t), v(t)),
\end{align*}
\]

subject to the nonseparated coupled boundary conditions:

\[
\begin{align*}
u(0) &= \lambda_1 v(T), \\
u'(0) &= \lambda_2 v'(T), \\
u(0) &= \mu_1 u(T), \\
u'(0) &= \mu_2 u'(T),
\end{align*}
\]

where \( t \in [0, T] \), \( 1 < \alpha, \beta \leq 2 \), \( \frac{d^\alpha u(t)}{dt^\alpha}, \frac{d^\beta v(t)}{dt^\beta} \) are Caputo fractional derivatives of order \( \alpha, \beta \), \( f, g \) are functions, and \( \lambda, \mu_i \) are real constants with \( \lambda, \mu_i \neq 1 (i = 1, 2) \).

In [10], Yang et al. studied the existence of solutions for the following system of fractional differential equations with integral boundary conditions:

\[
\begin{align*}
\frac{d^\alpha u(t)}{dt^\alpha} &= f(t, u(t)), \\
\frac{d^\beta v(t)}{dt^\beta} &= g(t, v(t)), \\
\frac{d^\gamma w(t)}{dt^\gamma} &= r(t, u(t)), \\
u(0) &= V(0) = W(0) = 0, \\
u'(0) &= \int_0^1 \phi(t) u(t) dt, \\
u(1) &= \int_0^1 \psi(t) v(t) dt, \\
u'(1) &= \int_0^1 \eta(t) w(t) dt,
\end{align*}
\]

where \( 0 < t < 1, 1 < \alpha, \beta, \gamma < 2 \), and \( D^\alpha, D^\beta, D^\gamma \) are also the standard Riemann-Liouville fractional derivatives of order \( \alpha, \beta, \gamma \). The system of (7) satisfies the following conditions:

\[
\begin{align*}
(1') \ f, g, r \in C([0, 1] \times [0, +\infty), [0, +\infty)) . \\
(2') \ \phi, \psi, \eta : [0, 1] \rightarrow [0, \infty) \text{ with } \phi, \psi, \eta \in L^1[0, 1] \text{ and } \theta_i \in (0, 1)(i = 1, 2, 3), \ \theta_j > 0 (j = 4, 5, 6),
\end{align*}
\]

where

\[
\begin{align*}
\theta_1 &= \int_0^1 t^{n-1} \phi(t) dt, \\
\theta_2 &= \int_0^1 t^{n-1} \psi(t) dt, \\
\theta_3 &= \int_0^1 t^{n-1} \eta(t) dt, \\
\theta_4 &= \int_0^1 t^{n-1} \phi(t)(1-t) dt, \\
\theta_5 &= \int_0^1 t^{n-1} \psi(t)(1-t) dt, \\
\theta_6 &= \int_0^1 t^{n-1} \eta(t)(1-t) dt.
\end{align*}
\]

We know \( 0 < \theta_4 \leq \theta_1 < 1, 0 < \theta_5 \leq \theta_2 < 1, 0 < \theta_6 \leq \theta_3 < 1 \).

2. Preliminaries and Basic Lemmas

In this section, we will present some definitions and lemmas, which will be used throughout this paper.

From [1], the definitions of Riemann-Liouville fractional integral and derivative are given as follows:

\[
\begin{align*}
I^q_0 f(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \\
D^q_0 f(t) &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-1} f(s) ds,
\end{align*}
\]

where \( t > a, q > 0, f \in C[a,b] \text{ and } \Gamma \text{ is the gamma function.} \)

\[
\begin{align*}
D^q_0 f(t) &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-1} f(s) ds,
\end{align*}
\]

where \( t > a, n-1 < q < n \) and \( f : [a, \infty) \rightarrow \mathbb{R} \).

Lemma 1 (see [2]). If \( \int_0^1 \phi(t)^{n-1} dt \neq 1, 1 < \alpha < 2 \), then, for any \( \sigma \in C[0,1] \), the unique solution of the boundary value problem,

\[
\begin{align*}
D^\alpha u(t) + \sigma(t) &= 0, \quad 0 < t < 1, \\
u(0) &= 0, \quad u(1) = \int_0^1 \phi(t) u(t) dt,
\end{align*}
\]

is given by

\[
\begin{align*}
u(t) &= \int_0^1 G_{\alpha}(t,s) \sigma(s) ds,
\end{align*}
\]
where
\[ G_1(t, s) = G_{1\alpha}(t, s) + G_{2\alpha}(t, s), \]
\[ G_{1\alpha}(t, s) = \begin{cases} \frac{t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1} (1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \]
\[ G_{2\alpha}(t, s) = \frac{t^{\alpha-1}}{1-\theta_4} \int_0^1 \phi(t) G_{1\alpha}(t, s) dt. \]

Then \( G(t, s) = (G_\alpha(t, s), G_\beta(t, s), G_\gamma(t, s)) \) is Green's function of (7).

**Lemma 3** (see [3]). The function \( G_{1\alpha}(t, s) \) has the following properties:
\[ \frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1} (1-t) (1-s)^{\alpha-2} \leq G_{1\alpha}(t, s) \]
\[ \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1} (1-t) (1-s)^{\alpha-2}, \]
where \( t, s \in [0, 1] \) and \( G_{1\alpha}(t, s) \) is given in (14).

**Lemma 4.** Assume condition (2') holds; then \( G_\alpha(t, s) \) satisfies the following property:
\[ \frac{(\alpha-1) \theta_4 s (1-s)^{\alpha-2}}{(1-\theta_4) \Gamma(\alpha)} \leq G_\alpha(t, s) \]
\[ \leq t^{\alpha-1} (1-s)^{\alpha-2} \frac{1}{\Gamma(\alpha)} \left(1 - t + \frac{\theta_4}{1 - \theta_4}\right), \]
where \( G_\alpha(t, s) \) is given in (13) and \( \theta_1, \theta_4 \) are defined in (8).

**Proof.** From (13)–(15), we have
\[ G_\alpha(t, s) = G_{1\alpha}(t, s) + G_{2\alpha}(t, s) \]
\[ = G_{1\alpha}(t, s) + \frac{t^{\alpha-1}}{1-\theta_4} \int_0^1 \phi(t) G_{1\alpha}(t, s) dt \]
\[ \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1} (1-t) (1-s)^{\alpha-2} \]
\[ + \frac{t^{\alpha-1}}{1-\theta_4} \int_0^1 \phi(t) \frac{1}{\Gamma(\alpha)} \left(1 - t + \frac{\theta_4}{1 - \theta_4}\right) dt \]
\[ = \frac{t^{\alpha-1} (1-s)^{\alpha-2}}{\Gamma(\alpha)} \left(1 - t + \frac{\theta_4}{1 - \theta_4}\right). \]

From [10], we have
\[ \frac{(\alpha-1) \theta_4 s (1-s)^{\alpha-2}}{(1-\theta_4) \Gamma(\alpha)} \leq G_\alpha(t, s). \]

Throughout this paper, we list the following definitions and marks for our convenience. For details, see [11–14].

Let \((E, \| \cdot \|)\) be a real Banach space, for any \( \mu, \nu \in E \).

(1) If \( P \subset E \), then cone \( P \) is called partially ordered, i.e., \( \mu \leq \nu \iff \nu - \mu \in P \).

(2) \( o \) is the zero element of \( E \).

(3) There is a constant \( M > 0 \); if \( P \) is called normal, then \( \| \mu \| \leq M \| \nu \| \) when \( o \leq \mu \leq \nu \).

(4) Let the mapping \( A : E \to E \), if \( A \) is an increasing operator, then \( \lambda \mu \leq \lambda \nu \) when \( \mu \leq \nu \).

(5) The notation \( \sim \) is an equivalence relation. \( \mu \sim \nu \); i.e., there exist \( \lambda > 0 \) and \( \gamma > 0 \) such that \( \lambda \mu \leq \gamma \nu \).

(6) Set \( P_h = \{ \mu \in E \mid \mu \sim h \} \), where \( h > 0 \).

(7) \( \Phi \) is the class of function \( \phi : (0, 1) \to (0, 1) \), which satisfies the condition \( \phi(t) > t \) for \( t \in (0, 1) \).

**Lemma 5** (see [14]). Let \( P \) be a normal cone in a real Banach space \( E \), \( h > 0 \). The operator \( T \) satisfies the following conditions:

(i) \( T \) is an increasing
(ii) There is \( h_3 \in P_h \) such that \( T h_3 \in P_h \)
(iii) For any \( x \in P \) and \( t \in (0, 1) \), there exists \( \phi \in \Phi \) such that \( T(t \mu) \geq \phi(t) T(\mu) \)

Then,

(i) \( T \) has a unique fixed point \( \mu^* \in P_h \)
(ii) for any initial value \( \mu_0 \in P_h \), constructing successively the sequence \( \mu_{n+1} = T \mu_n, n = 0, 1, \ldots \), we have \( \mu_n \to \mu^* \) as \( n \to \infty \).

**3. Main Results**

Set \( X = \{ u(t) \mid u(t) \in C[0, 1] \} \), the norm \( \| u \|_X = \max_{t \in [0,1]} \| u(t) \| \).

For \( (u, v, w) \in X \times X \times X \), we define the norm \( \| u, v, w \|_{X \times X \times X} = \max_{t \in [0,1]} \| u(t) \|_X \| v(t) \|_X \| w(t) \|_X \).

We define \( S = \{ (u, v, w) \in X \times X \times X \mid u(t) \geq 0, \nu(t) \geq 0, \omega(t) \geq 0, P = \{ u \in X \mid u(t) \geq 0 \} \}

We give the definition of the partially ordered in the space \( X \times X \times X \); let \( (u_1, v_1, w_1), (u_2, v_2, w_2) \in S \), then \( u_1, v_1, w_1 \leq u_2, v_2, w_2 \iff u_1 \leq u_2, v_1 \leq v_2, w_1 \leq w_2 \).

We also define \( \lambda(h_1, h_2, h_3) = (\lambda h_1, \lambda h_2, \lambda h_3) \), where \( \lambda > 0, h_1, h_2, h_3 > 0 \).

**Lemma 6.** Define \( S_h = \{ (\mu, v, \omega) : \ EXISTS \lambda(\mu, v, \omega) > 0 \ SUCH THAT \ \lambda(h_1, h_2, h_3) \leq (\mu, v, \omega) \leq \gamma(h_1, h_2, h_3) \}, \)
\[ P_{h_1} = \{ u \in X \mid u(t) \geq 0 \}, \]
\[ P_{h_2} = \{ v \in X \mid v(t) \geq 0 \}, \]
\[ P_{h_3} = \{ w \in X \mid w(t) \geq 0 \}; \]
And then \( S_h = P_{h_1} \times P_{h_2} \times P_{h_3}, \) where \( h = (h_1, h_2, h_3) \).
Theorem 8. Assume condition (2) and the following conditions hold:

1. \( f(t,x),g(t,x),r(t,x) \) are continuous and \( f(t,0),g(t,0),r(t,0) \neq 0 \)
2. \( f(t,x),g(t,x),r(t,x) \) are increasing functions in \( x \), for any \( t \in [0,1] \)

Then system (7) has a unique positive solution \( (u,v,w) \in S \), where \( h(t) = (t^{a-1}, t^{b-1}, t^{c-1}), t \in [0,1] \).

Proof. For \( (u,v,w) \in S \), we define operators \( T_1, T_2, T_3, \) and \( T \) by

\[
T_1 u (t) = \int_0^1 G_a (t,s) f(s,v(s)) \, ds, \\
T_2 v (t) = \int_0^1 G_b (t,s) g(s,w(s)) \, ds, \\
T_3 w (t) = \int_0^1 G_c (t,s) r(s,u(s)) \, ds, \\
T (u,v,w) (t) = (T_1 u (t), T_2 v (t), T_3 w (t))
\]

(27) To begin with, we prove that condition (i) of Lemma 5 holds. For any \( (u_1, v_1, w_1), (u_2, v_2, w_2) \in S \) with \( (u_1, v_1, w_1) \leq (u_2, v_2, w_2) \), according to the definition of the partially order, we have \( u_1 (t) \leq u_2 (t), v_1 (t) \leq v_2 (t), w_1 (t) \leq w_2 (t) \).

From Lemma 6 and condition (2), we have

\[
T_1 u_1 (t) \leq T_1 u_2 (t), \\
T_2 v_1 (t) \leq T_2 v_2 (t), \\
T_3 w_1 (t) \leq T_3 w_2 (t)
\]

(29) Hence, \( S_h = P_{h_1} \times P_{h_2} \times P_{h_3} \) holds.

Proof. For \( (u,v,w) \in S \), we define operators \( T_1, T_2, T_3, \) and \( T \) by
By computation we have
\[
T(u_1, v_1, w_1)(t) = (T_1u_1(t), T_2v_1(t), T_3w_1(t))
\]
\[
\leq (T_1u_2(t), T_2v_2(t), T_3w_2(t))
\]
\[
= T(u_2, v_2, w_2)(t).
\]
Hence, \( T \) is an increasing operator.

Applying (17) and Theorem 3.1 of [10], we have
\[
T, h_1 \in P_h, T_2h_2 \in P_h, T_3h_3 \in P_h.
\]
Applying (17) and Theorem 3.1 of [10], we have
\[
T, h_1 \in P_h, T_2h_2 \in P_h, T_3h_3 \in P_h.
\]

From [10], we know \( f(s, 1) > f(s, 0) \geq 0, s \in [0, 1] \). Then we have
\[
0 < \int_0^1 s(1-s)^{a-1} f(s, 0) ds
\]
\[
\leq \int_0^1 (1-s)^{a-2} f(s, 1) ds,
\]
\[
0 < \int_0^1 s(1-s)^{b-1} g(s, 0) ds
\]
\[
\leq \int_0^1 (1-s)^{b-2} g(s, 1) ds,
\]
(33)

Let
\[
m_1 = \frac{(\alpha - 1) \theta_4}{(1 - \theta_1) \Gamma(\alpha)} \int_0^1 s(1-s)^{a-1} f(s, 0) ds > 0,
\]
\[
m_2 = \frac{1-t+\theta_4/(1-\theta_1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{a-2} f(s, 1) ds > 0,
\]
(34)
\[
m_3 = \frac{(\beta - 1) \theta_5}{(1 - \theta_2) \Gamma(\beta)} \int_0^1 s(1-s)^{b-1} g(s, 0) ds > 0,
\]
\[
m_4 = \frac{1-t+\theta_5/(1-\theta_2)}{\Gamma(\beta)} \int_0^1 (1-s)^{b-2} g(s, 1) ds > 0,
\]
\[
m_5 = \frac{(\gamma - 1) \theta_6}{(1 - \theta_3) \Gamma(\gamma)} \int_0^1 s(1-s)^{c-1} r(s, 0) ds > 0,
\]
\[
m_6 = \frac{1-t+\theta_6/(1-\theta_3)}{\Gamma(\gamma)} \int_0^1 (1-s)^{c-2} r(s, 1) ds > 0.
\]

Since \( (1-t) + \theta_4/(1-\theta_1) - (\alpha - 1) \theta_4/(1-\theta_1) = ((1-t) - (1-\theta_1) + (2 - \alpha)\theta_4)/(1 - \theta_1) > 0 \), we can obtain \( m_1 \leq m_2, m_3 \leq m_4, m_5 \leq m_6 \).

Hence, \( m_1 h_1(t) \leq T_1h_1(t) \leq m_2h_1(t), m_3h_2(t) \leq T_2h_2(t) \leq m_4h_2(t), m_5h_3(t) \leq T_3h_3(t) \leq m_6h_3(t). \)

That is, \( T_1h_1 \in P_h, T_2h_2 \in P_h, T_3h_3 \in P_h. \)

Therefore, \( T h_0 \in S_h. \)
Last, we prove that condition (iii) of Lemma 5 holds. For any \( \lambda \in (0, 1) \) and \((u, v, w) \in S\), by condition (3), we get
\[
T_1(\lambda u)(t) = \int_0^1 G_\alpha(t, s) f(s, \lambda v(s)) \, ds
\geq \varphi_1(\lambda) \int_0^1 G_\alpha(t, s) f(s, v(s)) \, ds
= \varphi_1(\lambda) T_1 u(t),
\]
\[
T_2(\lambda v)(t) = \int_0^1 G_\beta(t, s) g(s, \lambda w(s)) \, ds
\geq \varphi_2(\lambda) \int_0^1 G_\beta(t, s) g(s, w(s)) \, ds
= \varphi_2(\lambda) T_2 v(t),
\]
\[
T_3(\lambda w)(t) = \int_0^1 G_\gamma(t, s) r(s, \lambda u(s)) \, ds
\geq \varphi_3(\lambda) \int_0^1 G_\gamma(t, s) r(s, u(s)) \, ds
= \varphi_3(\lambda) T_3 w(t).
\]

Thus,
\[
T(\lambda(u, v, w))(t) = (T_1(\lambda u)(t), T_2(\lambda v)(t), T_3(\lambda w)(t))
\geq (\varphi_1(\lambda) T_1 u(t), \varphi_2(\lambda) T_2 v(t), \varphi_3(\lambda) T_3 w(t)).
\]

Let \( \varphi(\lambda) = \min\{\varphi_1(\lambda), \varphi_2(\lambda), \varphi_3(\lambda)\} \). Hence, \( T(\lambda(u, v, w))(t) \geq \varphi(\lambda) T(u, v, w)(t) \).

We have proved that the operator \( T \) satisfies all the conditions of Lemma 5. Hence \( T \) has a unique fixed point \((u^*, v^*, w^*) \in S_h \).

Moreover, we give the sequences, whose limit is the solution of system (7).

For any \((u^0, v^0, w^0) \in S_h\), we can construct the sequences as follows:
\[
u_n(t) = \int_0^1 G_\alpha(t, s) f(s, v_{n-1}(s)) \, ds,
\]
\[
u_n(t) = \int_0^1 G_\beta(t, s) g(s, w_{n-1}(s)) \, ds,
\]
\[
u_n(t) = \int_0^1 G_\gamma(t, s) r(s, u_{n-1}(s)) \, ds.
\]

The limit of \( u_n(t), v_n(t), w_n(t) \) exists as \( u^*, v^*, w^* \), respectively. From Lemma (16), we have
\[
u_n(t) \to u^*(n \to \infty),
\]
\[
u_n(t) \to v^*(n \to \infty),
\]
\[
u_n(t) \to w^*(n \to \infty).
\]

Hence, \((u^*, v^*, w^*)\) is a positive solution of system (7).

\[\square\]

4. An Example

Example 1. Consider the system of fractional differential equations:
\[
D^{5/4}u(t) + [v(t)]^{1/2} + b_1(t) = 0,
\]
\[
D^{6/5}v(t) + [w(t)]^{1/2} + b_2(t) = 0,
\]
\[
D^{7/6}w(t) + [u(t)]^{1/2} + b_3(t) = 0
\]
\[
u(0) = w(0) = 0 = 0,
\]
\[
u(1) = \int_0^1 t^\lambda u(t) \, dt,
\]
\[
u(1) = \int_0^1 t^\lambda v(t) \, dt,
\]
\[
u(1) = \int_0^1 t^\lambda w(t) \, dt,
\]

where \( 0 < t < 1, \tau_1, \tau_2, \tau_3 \in (0, 1), b_1, b_2, b_3 : [0, 1] \to [0, \infty) \) are continuous and \( f(t, 0) \neq 0, g(t, 0) \neq 0, r(t, 0) \neq 0 \).

Proof. Compared with (7), we know \( \alpha = 5/4, \beta = 6/5, \gamma = 7/6 \) and
\[
f(t, x) = x^{5/4} + b_1(t),
\]
\[
g(t, x) = x^{6/5} + b_2(t),
\]
\[
r(t, x) = x^{7/6} + b_3(t),
\]
\[
\phi(t) = t^\lambda,
\]
\[
\psi(t) = t^\lambda,
\]
\[
\eta(t) = t;
\]

After calculation, we get
\[
\theta_1 = \int_0^1 t^\lambda \phi(t) \, dt = 4/17,
\]
\[
\theta_2 = \int_0^1 t^\lambda \psi(t) \, dt = 5/16,
\]
\[
\theta_3 = \int_0^1 t^\lambda \eta(t) \, dt = 6/13,
\]
\[
\theta_4 = \int_0^1 t^\lambda \phi(t)(1 - t) \, dt = 16/357,
\]
\[
\theta_5 = \int_0^1 t^\lambda \psi(t)(1 - t) \, dt = 25/336,
\]
\[
\theta_6 = \int_0^1 t^\lambda \eta(t)(1 - t) \, dt = 36/247.
\]

We find \( \theta_1, \theta_2, \theta_3 \in (0, 1), \theta_4, \theta_5, \theta_6 > 0 \). The functions \( f, g, r \) are continuous and \( f(t, 0) = b_1(t) \neq 0, g(t, 0) = b_2(t) \neq 0, r(t, 0) = b_3(t) \neq 0, \)

\[\square\]
So, system (39) satisfies all conditions of Theorem 8.

Therefore, (39) has a unique positive solution $(u^*, v^*, w^*) \in S_h$, where $h(t) = (h_1(t), h_2(t), h_3(t)) = (t^{1/4}, t^{1/5}, t^{1/6})$, and we can get this positive solution by the following way.

For any $(u^n, v^n, w^n) \in S_h$, we can construct the sequences as follows:

\[
\begin{align*}
    u_n(t) &= \int_0^1 G_{\alpha}(t,s) \left\{ \left[ v_{n-1}(s) \right]^{s_1} + b_1(s) \right\} ds, \\
    v_n(t) &= \int_0^1 G_{\beta}(t,s) g \left[ \left[ w_{n-1}(s) \right]^{s_2} + b_2(s) \right] ds, \\
    w_n(t) &= \int_0^1 G_{\gamma}(t,s) r \left[ \left[ u_{n-1}(s) \right]^{s_3} + b_3(s) \right] ds,
\end{align*}
\]

where

\[
\begin{align*}
    G_{\alpha}(t,s) &= \begin{cases} 
        t^{1/4} (1-s)^{1/4} - (t-s)^{1/4}, & 0 \leq s \leq t \leq 1, \\
        t^{1/4} (1-s)^{1/4}, & 0 \leq t \leq s \leq 1,
    \end{cases} \\
    G_{\beta}(t,s) &= \begin{cases} 
        t^{1/5} (1-s)^{1/5} - (1-s)^{1/5}, & 0 \leq s \leq t \leq 1, \\
        t^{1/5} (1-s)^{1/5}, & 0 \leq t \leq s \leq 1,
    \end{cases} \\
    G_{\gamma}(t,s) &= \begin{cases} 
        t^{1/6} (1-s)^{1/6} - (1-s)^{1/6}, & 0 \leq s \leq t \leq 1, \\
        t^{1/6} (1-s)^{1/6}, & 0 \leq t \leq s \leq 1,
    \end{cases}
\]

Then, the limit of $u_n(t), v_n(t), w_n(t)$ is the solution of system (39).

That is,

\[
\begin{align*}
    u_n(t) &\longrightarrow u^* (n \to \infty), \\
    v_n(t) &\longrightarrow v^* (n \to \infty), \\
    w_n(t) &\longrightarrow w^* (n \to \infty).
\end{align*}
\]

From Theorem 8, system (39) has a unique positive solution. 

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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**References**


