Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 8534820, 8 pages
https://doi.org/10.1155/2018/8534820

Research Article

The Existence Results of Solutions for System of Fractional Differential Equations with Integral Boundary Conditions

Dongxia Zan and Run Xu

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China

Correspondence should be addressed to Run Xu; xurun_2005@163.com

Received 10 July 2018; Accepted 19 September 2018; Published 1 October 2018

Academic Editor: Douglas R. Anderson

In this paper, we investigated the system of fractional differential equations with integral boundary conditions. By using a fixed point theorem in the Banach spaces, we get the existence of solutions for the fractional differential system. By constructing iterative sequences for any given initial point in space, we can approximate this solution. As an application, an example is presented to illustrate our main results.

1. Introduction

The fractional differential equations attracted many people's attention, and they have many applications in different fields of science and biology. In recent years, the research on the qualitative properties of fractional differential equations has become a hot topic, and many results have been obtained; see [1–29]. From the literature, there are many articles investing the existence of solutions or the solutions for some nonlinear systems; see [4–10, 21].

In [6], Su studied the following system of nonlinear fractional differential equations:

$$\begin{aligned} D^{\alpha}u(t) &= f\left(t, v(t), D^{\mu}v(t)\right), \\ D^{\beta}v(t) &= g\left(t, u(t), D^{\nu}u(t)\right), \\ u(0) &= v(0) = u(1) = v(1) = 0, \end{aligned} \tag{1}$$

where $0 < t < 1$, $1 < \alpha, \beta < 2$, $\mu, \nu > 0$, $\alpha - \nu, \beta - \mu \geq 1$, f, g are given functions and D^{α}, D^{β} are Riemann-Liouville fractional derivatives of order α, β. The author gave sufficient conditions for the existence of solutions of system (1).

In [7], Wang et al. studied the existence of solutions for the following system of nonlinear fractional differential equations:

$$\begin{aligned} D^{\alpha}u(t) &= f(t, v(t)), \\ D^{\beta}v(t) &= g(t, u(t)), \\ u(0) &= v(0) = 0, \\ u(1) &= au(\xi), \\ v(1) &= bv(\xi), \end{aligned} \tag{2}$$

where $0 < t < 1$, $1 < \alpha, \beta < 2$, $0 < \xi < 1$, $0 \leq a, b \leq 1$, f, g are given functions and D^{α}, D^{β} are Riemann-Liouville fractional derivatives of order α, β.

In [8], Yang studied the existence of solutions for the following system of nonlinear fractional differential equations:

$$\begin{aligned} D^{\alpha}u(t) &= a(t) f(t, v(t)), \\ D^{\beta}v(t) &= b(t) g(t, u(t)), \\ u(0) &= v(0) = 0, \\ u(1) &= \int_0^1 \phi(t) u(t)\, dt, \\ v(1) &= \int_0^1 \psi(t) v(t)\, dt, \end{aligned} \tag{3}$$

where $0 < t < 1$, $1 < \alpha, \beta < 2$, $a, b \in C((0,1), [0, \infty))$, $\phi, \psi \in L^1[0,1]$ are nonnegative and

$f, g \in C([0,1] \times [0,+\infty), [0,+\infty))$, and D^α, D^β are also the standard Riemann-Liouville fractional derivatives of order α, β.

In [9], Alsulami et al. studied the existence of solutions for the following system of nonlinear Caputo fractional integrodifferential equations:

$$\begin{aligned} {}^cD^\alpha u(t) &= f(t, u(t), v(t)), \\ {}^cD^\beta v(t) &= g(t, u(t), v(t)), \end{aligned} \tag{4}$$

subject to the nonseparated coupled boundary conditions:

$$\begin{aligned} u(0) &= \lambda_1 v(T), \\ u'(0) &= \lambda_2 v'(T), \\ v(0) &= \mu_1 u(T), \\ v'(0) &= \mu_2 u'(T), \end{aligned} \tag{5}$$

where $t \in [0,T]$, $1 < \alpha, \beta \le 2$, ${}^cD^\alpha, {}^cD^\beta$ are Caputo fractional derivatives of order α, β, f, g are functions, and λ_i, μ_i are real constants with $\lambda_i \mu_i \ne 1 (i = 1, 2)$.

In [10], Yang et al. studied the existence of solutions for the following system of fractional differential equations with integral boundary conditions:

$$\begin{aligned} D^\alpha u(t) &= f(t, v(t)), \\ D^\beta v(t) &= g(t, u(t)), \\ u(0) &= v(0) = 0, \\ u(1) &= \int_0^1 \phi(t) u(t)\, dt, \\ v(1) &= \int_0^1 \psi(t) v(t)\, dt, \end{aligned} \tag{6}$$

where $0 < t < 1$, $1 < \alpha, \beta < 2$, $\phi, \psi \in L^1[0,1]$ are nonnegative and $f, g \in C([0,1] \times [0,+\infty), [0,+\infty))$, and D^α, D^β are also the standard Riemann-Liouville fractional derivatives of order α, β.

In this paper, we investigate the following system of fractional differential equations:

$$\begin{aligned} D^\alpha u(t) + f(t, v(t)) &= 0, \\ D^\beta v(t) + g(t, w(t)) &= 0, \\ D^\gamma w(t) + r(t, u(t)) &= 0, \\ u(0) = v(0) = w(0) &= 0, \\ u(1) &= \int_0^1 \phi(t) u(t)\, dt, \\ v(1) &= \int_0^1 \psi(t) v(t)\, dt, \\ w(1) &= \int_0^1 \eta(t) w(t)\, dt, \end{aligned} \tag{7}$$

where $0 < t < 1$, $1 < \alpha, \beta, \gamma < 2$, and $D^\alpha, D^\beta, D^\gamma$ are also the standard Riemann-Liouville fractional derivatives of order α, β, γ. The system of (7) satisfies the following conditions:

(1°) $f, g, r \in C([0,1] \times [0,+\infty), [0,+\infty))$.

(2°) $\phi, \psi, \eta : [0,1] \longrightarrow [0,\infty)$ with $\phi, \psi, \eta \in L^1[0,1]$ and $\theta_i \in (0,1) (i = 1,2,3)$, $\theta_j > 0 (j = 4,5,6)$,

where

$$\begin{aligned} \theta_1 &= \int_0^1 t^{\alpha-1} \phi(t)\, dt, \\ \theta_2 &= \int_0^1 t^{\beta-1} \psi(t)\, dt, \\ \theta_3 &= \int_0^1 t^{\gamma-1} \eta(t)\, dt; \\ \theta_4 &= \int_0^1 t^{\alpha-1} \phi(t)(1-t)\, dt, \\ \theta_5 &= \int_0^1 t^{\beta-1} \psi(t)(1-t)\, dt, \\ \theta_6 &= \int_0^1 t^{\gamma-1} \eta(t)(1-t)\, dt. \end{aligned} \tag{8}$$

We know $0 < \theta_4 \le \theta_1 < 1, 0 < \theta_5 \le \theta_2 < 1, 0 < \theta_6 \le \theta_3 < 1$.

2. Preliminaries and Basic Lemmas

In this section, we will present some definitions and lemmas, which will be used throughout this paper.

From [1], the definitions of Riemann-Liouville fractional integral and derivative are given as follows:

$$I_{a_+}^q f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)\, ds, \tag{9}$$

where $t > a, q > 0$, $f \in C[a,b]$ and Γ is the gamma function.

$$D_{a_+}^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} f(s)\, ds, \tag{10}$$

where $t > a, n-1 < q < n$ and $f : [a,\infty) \longrightarrow \mathbf{R}$.

Lemma 1 (see [2]). *If $\int_0^1 \phi(t) t^{\alpha-1} dt \ne 1$, $1 < \alpha < 2$, then, for any $\sigma \in C[0,1]$, the unique solution of the boundary value problem,*

$$\begin{aligned} D^\alpha u(t) + \sigma(t) &= 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) &= \int_0^1 \phi(t) u(t)\, dt, \end{aligned} \tag{11}$$

is given by

$$u(t) = \int_0^1 G_\alpha(t,s) \sigma(s)\, ds, \tag{12}$$

where

$$G_\alpha(t,s) = G_{1\alpha}(t,s) + G_{2\alpha}(t,s), \tag{13}$$

$$G_{1\alpha}(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{14}$$

$$G_{2\alpha}(t,s) = \frac{t^{\alpha-1}}{1-\theta_1}\int_0^1 \phi(t)\, G_{1\alpha}(t,s)\, dt. \tag{15}$$

Then $G(t,s) = (G_\alpha(t,s), G_\beta(t,s), G_\gamma(t,s))$ is Green's function of (7).

Lemma 2 (see [2]). *If $\int_0^1 \phi(t)t^{\alpha-1}dt \in [0,1)$, then $G_\alpha(t,s) \ge 0$ is continuous for all $t,s \in [0,1]$, $G_\alpha(t,s) > 0$ for all $t,s \in (0,1)$, where $G_\alpha(t,s)$ is given in (13).*

Lemma 3 (see [3]). *The function $G_{1\alpha}(t,s)$ has the following properties:*

$$\begin{aligned} &\frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-1} s \le G_{1\alpha}(t,s) \\ &\quad \le \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}, \end{aligned} \tag{16}$$

where $t,s \in [0,1]$ and $G_{1\alpha}(t,s)$ is given in (14).

Lemma 4. *Assume condition (2°) holds; then $G_\alpha(t,s)$ satisfies the following property:*

$$\begin{aligned} &\frac{(\alpha-1)\theta_4 s(1-s)^{\alpha-1} t^{\alpha-1}}{(1-\theta_1)\Gamma(\alpha)} \le G_\alpha(t,s) \\ &\quad \le \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}\left(1-t+\frac{\theta_4}{1-\theta_1}\right), \end{aligned} \tag{17}$$

where $G_\alpha(t,s)$ is given in (13) and θ_1, θ_4 are defined in (8).

Proof. From (13)–(15), we have

$$\begin{aligned} G_\alpha(t,s) &= G_{1\alpha}(t,s) + G_{2\alpha}(t,s) \\ &= G_{1\alpha}(t,s) + \frac{t^{\alpha-1}}{1-\theta_1}\int_0^1 \phi(t)\, G_{1\alpha}(t,s)\, dt \\ &\le \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2} \\ &\quad + \frac{t^{\alpha-1}}{1-\theta_1}\int_0^1 \phi(t)\frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}\, dt \\ &= \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}\left(1-t+\frac{\theta_4}{1-\theta_1}\right). \end{aligned} \tag{18}$$

From [10], we have

$$\frac{(\alpha-1)\theta_4 s(1-s)^{\alpha-1} t^{\alpha-1}}{(1-\theta_1)\Gamma(\alpha)} \le G_\alpha(t,s). \tag{19}$$

□

Throughout this paper, we list the following definitions and marks for our convenience. For details, see [11–14].

Let $(E, \|\cdot\|)$ be a real Banach space, for any $\mu, \nu \in E$.

(1) If $P \subset E$, then cone P is called partially ordered, i.e., $\mu \le \nu \Longleftrightarrow \nu - \mu \in P$.

(2) o is the zero element of E.

(3) There is a constant $M > 0$; if P is called normal, then $\|\mu\| \le M\|\nu\|$ when $o \le \mu \le \nu$.

(4) Let the mapping $A : E \longrightarrow E$, if A is an increasing operator, then $A\mu \le A\nu$ when $\mu \le \nu$.

(5) The notation $\sim$ is an equivalence relation. $\mu \sim \nu$; i.e., there exist $\lambda > 0$ and $\gamma > 0$ such that $\lambda\mu \le \nu \le \gamma\mu$.

(6) Set $P_h = \{\mu \in E \mid \mu \sim h\}$, where $h > o$.

(7) Φ is the class of function $\varphi : (0,1) \longrightarrow (0,1)$, which satisfies the condition $\varphi(t) > t$ for $t \in (0,1)$.

Lemma 5 (see [14]). *Let P be a normal cone in a real Banach space E, $h > o$. The operator T satisfies the following conditions:*

(i) T is an increasing

(ii) There is $h_0 \in P_h$ such that $Th_0 \in P_h$

(iii) For any $x \in P$ and $t \in (0,1)$, there exists $\varphi \in \Phi$ such that $T(t\mu) \ge \varphi(t)T(\mu)$

Then,

(1) T has a unique fixed point $\mu^* \in P_h$

(2) for any initial value $\mu_0 \in P_h$, constructing successively the sequence $\mu_{n+1} = T\mu_n$, $n = 0, 1, \ldots$, we have $\mu_n \longrightarrow \mu^*$ as $n \longrightarrow \infty$.

3. Main Results

Set $X = \{u(t) \mid u(t) \in C[0,1]\}$, the norm $\|u\|_X = \max_{t\in[0,1]}|u(t)|$. For $(u,v,w) \in X\times X\times X$, we define the norm $\|u,v,w\|_{X\times X\times X} = \max_{t\in[0,1]}\{\|u\|_X, \|v\|_X, \|w\|_X\}$.

We define $S = \{(u,v,w) \in X\times X\times X \mid u(t) \ge 0, v(t) \ge 0, w(t) \ge 0\}$, $P = \{u \in X \mid u(t) \ge 0\}$.

We give the definition of the partially ordered in the space $X\times X\times X$; let (u_1,v_1,w_1), $(u_2,v_2,w_2) \in S$, then $(u_1,v_1,w_1) \le (u_2,v_2,w_2) \Longleftrightarrow u_1 \le u_2, v_1 \le v_2, w_1 \le w_2$.

We also define $\lambda(h_1,h_2,h_3) = (\lambda h_1, \lambda h_2, \lambda h_3)$, where $\lambda > 0, h_1, h_2, h_3 > o$.

Lemma 6. *Define $S_h = \{(\mu,\nu,\omega)$; there exist $\lambda(\mu,\nu,\omega)$, $\gamma(\mu,\nu,\omega) > 0$ such that $\lambda(h_1,h_2,h_3) \le (\mu,\nu,\omega) \le \gamma(h_1,h_2,h_3)\}$,*

$$\begin{aligned} P_{h_1} &= \{u \in X \mid u(t) \ge 0\}, \\ P_{h_2} &= \{v \in X \mid v(t) \ge 0\}, \\ P_{h_3} &= \{w \in X \mid w(t) \ge 0\}; \end{aligned} \tag{20}$$

And then $S_h = P_{h_1} \times P_{h_2} \times P_{h_3}$, where $h = (h_1,h_2,h_3)$.

Proof. On one hand, for any $(\mu,\nu,\omega)\in S_h$, there exists $\lambda,\gamma>0$ such that $\lambda h\le(\mu,\nu,\omega)\le\gamma h$. Then,

$$\begin{aligned}(\lambda h_1,\lambda h_2,\lambda h_3)&=\lambda(h_1,h_2,h_3)\le(\mu,\nu,\omega)\\&\le\gamma(h_1,h_2,h_3)=(\gamma h_1,\gamma h_2,\gamma h_3).\end{aligned}\tag{21}$$

From the definition of the partial order, we can get $\lambda h_1\le\mu\le\gamma h_1,\lambda h_2\le\nu\le\gamma h_2,\lambda h_3\le\omega\le\gamma h_3$. So $\mu\in P_{h_1},\nu\in P_{h_2},\omega\in P_{h_3}$.

That is to say, $(\mu,\nu,\omega)\in P_{h_1}\times P_{h_2}\times P_{h_3}$.

Hence, $S_h\subset P_{h_1}\times P_{h_2}\times P_{h_3}$.

On the other hand, for any $(\mu,\nu,\omega)\in P_{h_1}\times P_{h_2}\times P_{h_3}$, we have

$$\begin{aligned}&\mu\in P_{h_1},\\&\nu\in P_{h_2},\\&\omega\in P_{h_3}.\end{aligned}\tag{22}$$

There exist $\lambda_1,\lambda_2,\lambda_3,\gamma_1,\gamma_2,\gamma_3>0$, such that

$$\begin{aligned}&\lambda_1h_1\le\mu\le\gamma_1h_1,\\&\lambda_2h_2\le\nu\le\gamma_2h_2,\\&\lambda_3h_3\le\omega\le\gamma_3h_3.\end{aligned}\tag{23}$$

Let $\lambda=\min\{\lambda_1,\lambda_2,\lambda_3\}$, $\gamma=\max\{\gamma_1,\gamma_2,\gamma_3\}$.

From the definition of the partial order, we have

$$\begin{aligned}\lambda(h_1,h_2,h_3)&=(\lambda h_1,\lambda h_2,\lambda h_3)\le(\lambda_1h_1,\lambda_2h_2,\lambda_3h_3)\\&\le(\mu,\nu,\omega)\le(\gamma_1h_1,\gamma_2h_2,\gamma_3h_3)\\&\le(\gamma h_1,\gamma h_2,\gamma h_3)=\gamma(h_1,h_2,h_3).\end{aligned}\tag{24}$$

That is, $(\mu,\nu,\omega)\in S_h$, and thus $P_{h_1}\times P_{h_2}\times P_{h_3}\subset S_h$.

Hence $S_h=P_{h_1}\times P_{h_2}\times P_{h_3}$ holds. □

Lemma 7. *Assume conditions (1°) and (2°) hold; then $(u,v,w)\in S$ is a solution of system (7) if and only if*

$$\begin{aligned}u(t)&=\int_0^1G_\alpha(t,s)f(s,v(s))\,ds,\\v(t)&=\int_0^1G_\beta(t,s)g(s,w(s))\,ds,\\w(t)&=\int_0^1G_\gamma(t,s)r(s,u(s))\,ds.\end{aligned}\tag{25}$$

From Lemma 1, we can get that (25) holds.

Theorem 8. *Assume condition (2°) and the following conditions hold:*

(1) $f(t,x),g(t,x),r(t,x)$ are continuous and $f(t,0),g(t,0),r(t,0)\not\equiv0$

(2) $f(t,x),g(t,x),r(t,x)$ are increasing functions in x, for any $t\in[0,1]$

(3) There exists $\varphi_1,\varphi_2,\varphi_3\in\Phi$ such that

$$\begin{aligned}f(t,\lambda u)&\ge\varphi_1(\lambda)f(t,u),\\g(t,\lambda u)&\ge\varphi_2(\lambda)g(t,u),\\r(t,\lambda u)&\ge\varphi_3(\lambda)r(t,u)\end{aligned}\tag{26}$$

for any $t\in[0,1],u\in[0,\infty)$ and $\lambda\in(0,1)$.

Then system (7) has a unique positive solution $(u^*,v^*,w^*)\in S_h$, where $h(t)=(t^{\alpha-1},t^{\beta-1},t^{\gamma-1}),t\in[0,1]$.

Proof. For $(u,v,w)\in S$, we define operators T_1,T_2,T_3, and T by

$$\begin{aligned}T_1u(t)&=\int_0^1G_\alpha(t,s)f(s,v(s))\,ds,\\T_2v(t)&=\int_0^1G_\beta(t,s)g(s,w(s))\,ds,\\T_3w(t)&=\int_0^1G_\gamma(t,s)r(s,u(s))\,ds,\\T(u,v,w)(t)&=\Bigg(\int_0^1G_\alpha(t,s)f(s,v(s))\,ds,\\&\quad\int_0^1G_\beta(t,s)g(s,w(s))\,ds,\\&\quad\int_0^1G_\gamma(t,s)r(s,u(s))\,ds\Bigg).\end{aligned}\tag{27}$$

If $T_1,T_2,T_3:P\longrightarrow P$ then $T:S\longrightarrow S$, that is,

$$\begin{aligned}T(u,v,w)(t)&=(T_1u(t),T_2v(t),T_3w(t))\\&=(u(t),v(t),w(t));\end{aligned}\tag{28}$$

To begin with, we prove that condition (i) of Lemma 5 holds. For any $(u_1,v_1,w_1),(u_2,v_2,w_2)\in S$ with $(u_1,v_1,w_1)\le(u_2,v_2,w_2)$, according to the definition of the partially order, we have $u_1(t)\le u_2(t),v_1(t)\le v_2(t),w_1(t)\le w_2(t)$.

From Lemma 6 and condition (2), we have

$$\begin{aligned}T_1u_1(t)&=\int_0^1G_\alpha(t,s)f(s,v_1(s))\,ds\\&\le\int_0^1G_\alpha(t,s)f(s,v_2(s))\,ds=T_1u_2(t),\\T_2v_1(t)&=\int_0^1G_\beta(t,s)g(s,w_1(s))\,ds\\&\le\int_0^1G_\beta(t,s)g(s,w_2(s))\,ds=T_2v_2(t),\\T_3w_1(t)&=\int_0^1G_\gamma(t,s)r(s,u_1(s))\,ds\\&\le\int_0^1G_\gamma(t,s)r(s,u_2(s))\,ds=T_3w_2(t).\end{aligned}\tag{29}$$

So, we can get the following result:

$$\begin{aligned} T(u_1, v_1, w_1)(t) &= (T_1 u_1(t), T_2 v_1(t), T_3 w_1(t)) \\ &\le (T_1 u_2(t), T_2 v_2(t), T_3 w_2(t)) \\ &= T(u_2, v_2, w_2)(t). \end{aligned} \tag{30}$$

Hence, T is an increasing operator.

Then, we prove that condition (ii) of Lemma 5 holds. Let $h_0(t) = h(t) = (h_1(t), h_2(t), h_3(t)) = (t^{\alpha-1}, t^{\beta-1}, t^{\gamma-1}), t \in [0,1]$.

According to Lemma 7, we only need to prove $T_1 h_1 \in P_{h_1}, T_2 h_2 \in P_{h_2}, T_3 h_3 \in P_{h_3}$.

Applying (17) and Theorem 3.1 of [10], we have

$$\begin{aligned} T_1 h_1(t) &= \int_0^1 G_\alpha(t,s) f(s, h_2(s))\, ds \\ &\ge \frac{(\alpha-1)\theta_4}{(1-\theta_1)\Gamma(\alpha)} h_1(t) \int_0^1 s(1-s)^{\alpha-1} f(s,0)\, ds, \\ T_2 h_2(t) &= \int_0^1 G_\beta(t,s) g(s, h_3(s))\, ds \\ &\ge \frac{(\beta-1)\theta_5}{(1-\theta_2)\Gamma(\beta)} h_2(t) \int_0^1 s(1-s)^{\beta-1} g(s,0)\, ds, \\ T_3 h_3(t) &= \int_0^1 G_\gamma(t,s) r(s, h_1(s))\, ds \\ &\ge \frac{(\gamma-1)\theta_6}{(1-\theta_3)\Gamma(\gamma)} h_3(t) \int_0^1 s(1-s)^{\gamma-1} r(s,0)\, ds. \end{aligned} \tag{31}$$

By computation we have

$$\begin{aligned} T_1 h_1(t) &= \int_0^1 G_\alpha(t,s) f(s, h_2(s))\, ds \\ &\le \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} \left(1 - t + \frac{\theta_4}{1-\theta_1}\right) t^{\alpha-1} f\left(s, s^{\beta-1}\right) ds \\ &\le \frac{1 - t + \theta_4/(1-\theta_1)}{\Gamma(\alpha)} h_1(t) \int_0^1 (1-s)^{\alpha-2} f(s,1)\, ds, \\ T_2 h_2(t) &= \int_0^1 G_\beta(t,s) g(s, h_3(s))\, ds \\ &\le \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta)} \left(1 - t + \frac{\theta_5}{1-\theta_2}\right) t^{\beta-1} g\left(s, s^{\gamma-1}\right) ds \\ &\le \frac{1 - t + \theta_5/(1-\theta_2)}{\Gamma(\beta)} h_2(t) \int_0^1 (1-s)^{\beta-2} g(s,1)\, ds, \\ T_3 h_3(t) &= \int_0^1 G_\gamma(t,s) r(s, h_1(s))\, ds \\ &\le \int_0^1 \frac{(1-s)^{\gamma-2}}{\Gamma(\gamma)} \left(1 - t + \frac{\theta_6}{1-\theta_3}\right) t^{\gamma-1} r\left(s, s^{\alpha-1}\right) ds \\ &\le \frac{1 - t + \theta_6/(1-\theta_3)}{\Gamma(\gamma)} h_3(t) \int_0^1 (1-s)^{\gamma-2} r(s,1)\, ds. \end{aligned} \tag{32}$$

From [10], we know $f(s,1) > f(s,0) \ge 0, s \in [0,1]$. Then we have

$$\begin{aligned} 0 &< \int_0^1 s(1-s)^{\alpha-1} f(s,0)\, ds \\ &\le \int_0^1 (1-s)^{\alpha-2} f(s,1)\, ds, \\ 0 &< \int_0^1 s(1-s)^{\beta-1} g(s,0)\, ds \\ &\le \int_0^1 (1-s)^{\beta-2} g(s,1)\, ds, \\ 0 &< \int_0^1 s(1-s)^{\gamma-1} r(s,0)\, ds \\ &\le \int_0^1 (1-s)^{\gamma-2} r(s,1)\, ds. \end{aligned} \tag{33}$$

Let

$$\begin{aligned} m_1 &= \frac{(\alpha-1)\theta_4}{(1-\theta_1)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f(s,0)\, ds > 0, \\ m_2 &= \frac{1 - t + \theta_4/(1-\theta_1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f(s,1)\, ds \\ &> 0, \\ m_3 &= \frac{(\beta-1)\theta_5}{(1-\theta_2)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} g(s,0)\, ds > 0, \\ m_4 &= \frac{1 - t + \theta_5/(1-\theta_2)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-2} g(s,1)\, ds \\ &> 0, \\ m_5 &= \frac{(\gamma-1)\theta_6}{(1-\theta_3)\Gamma(\gamma)} \int_0^1 s(1-s)^{\gamma-1} r(s,0)\, ds > 0, \\ m_6 &= \frac{1 - t + \theta_6/(1-\theta_3)}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-2} r(s,1)\, ds > 0. \end{aligned} \tag{34}$$

Since $(1-t) + \theta_4/(1-\theta_1) - (\alpha-1)\theta_4/(1-\theta_1) = ((1-t)(1-\theta_1) + (2-\alpha)\theta_4)/(1-\theta_1) > 0$,

we can obtain $m_1 \le m_2, m_3 \le m_4, m_5 \le m_6$.

Hence, $m_1 h_1(t) \le T_1 h_1(t) \le m_2 h_1(t), m_3 h_2(t) \le T_2 h_2(t) \le m_4 h_2(t), m_5 h_3(t) \le T_3 h_3(t) \le m_6 h_3(t)$.

That is, $T_1 h_1 \in P_{h_1}, T_2 h_2 \in P_{h_2}, T_3 h_3 \in P_{h_3}$.

Therefore, $T h_0 \in S_h$.

Last, we prove that condition (iii) of Lemma 5 holds. For any $\lambda \in (0,1)$ and $(u,v,w) \in S$, by condition (3), we can get

$$\begin{aligned} T_1(\lambda u)(t) &= \int_0^1 G_\alpha(t,s) f(s,\lambda v(s))\,ds \\ &\geq \varphi_1(\lambda) \int_0^1 G_\alpha(t,s) f(s,v(s))\,ds \\ &= \varphi_1(\lambda) T_1 u(t), \\ T_2(\lambda v)(t) &= \int_0^1 G_\beta(t,s) g(s,\lambda w(s))\,ds \\ &\geq \varphi_2(\lambda) \int_0^1 G_\beta(t,s) g(s,w(s))\,ds \\ &= \varphi_2(\lambda) T_2 v(t), \\ T_3(\lambda w)(t) &= \int_0^1 G_\gamma(t,s) r(s,\lambda u(s))\,ds \\ &\geq \varphi_3(\lambda) \int_0^1 G_\gamma(t,s) r(s,u(s))\,ds \\ &= \varphi_3(\lambda) T_3 w(t). \end{aligned} \tag{35}$$

Thus,

$$\begin{aligned} &T(\lambda(u,v,w))(t) \\ &\quad = \left(T_1(\lambda u)(t), T_2(\lambda v)(t), T_3(\lambda w)(t)\right) \\ &\quad \geq \left(\varphi_1(\lambda) T_1 u(t), \varphi_2(\lambda) T_2 v(t), \varphi_3(\lambda) T_3 w(t)\right). \end{aligned} \tag{36}$$

Let $\varphi(\lambda) = \min\{\varphi_1(\lambda), \varphi_2(\lambda), \varphi_3(\lambda)\}$. Hence, $T(\lambda(u,v,w))(t) \geq \varphi(\lambda) T(u,v,w)(t)$.

We have proved that the operator T satisfies all the conditions of Lemma 5. Hence T has a unique fixed point $(u^*, v^*, w^*) \in S_h$.

Moreover, we give the sequences, whose limit is the solution of system (7).

For any $(u^0, v^0, w^0) \in S_h$, we can construct the sequences as follows:

$$\begin{aligned} u_n(t) &= \int_0^1 G_\alpha(t,s) f\left(s, v_{n-1}(s)\right) ds, \\ v_n(t) &= \int_0^1 G_\beta(t,s) g\left(s, w_{n-1}(s)\right) ds, \\ w_n(t) &= \int_0^1 G_\gamma(t,s) r\left(s, u_{n-1}(s)\right) ds. \end{aligned} \tag{37}$$

The limit of $u_n(t), v_n(t), w_n(t)$ exists as u^*, v^*, w^*, respectively.

From Lemma (16), we have

$$\begin{aligned} u_n(t) &\longrightarrow u^* \ (n \longrightarrow \infty), \\ v_n(t) &\longrightarrow v^* \ (n \longrightarrow \infty), \\ w_n(t) &\longrightarrow w^* \ (n \longrightarrow \infty). \end{aligned} \tag{38}$$

Hence, (u^*, v^*, w^*) is a positive solution of system (7). □

4. An Example

Example 1. Consider the system of fractional differential equations:

$$\begin{aligned} D^{5/4} u(t) + [v(t)]^{\tau_1} + b_1(t) &= 0, \\ D^{6/5} v(t) + [w(t)]^{\tau_2} + b_2(t) &= 0, \\ D^{7/6} w(t) + [u(t)]^{\tau_3} + b_3(t) &= 0 \\ u(0) = v(0) = w(0) &= 0, \\ u(1) &= \int_0^1 t^3 u(t)\,dt, \\ v(1) &= \int_0^1 t^2 v(t)\,dt, \\ w(1) &= \int_0^1 t w(t)\,dt, \end{aligned} \tag{39}$$

where $0 < t < 1$, $\tau_1, \tau_2, \tau_3 \in (0,1)$, $b_1, b_2, b_3 : [0,1] \longrightarrow [0,\infty)$ are continuous with $b_i \not\equiv 0 (i = 1,2,3)$, then system (39) has a unique positive solution.

Proof. Compared with (7), we know $\alpha = 5/4$, $\beta = 6/5$, $\gamma = 7/6$ and

$$\begin{aligned} f(t,x) &= x^{\tau_1} + b_1(t), \\ g(t,x) &= x^{\tau_2} + b_2(t), \\ r(t,x) &= x^{\tau_3} + b_3(t), \\ \phi(t) &= t^3, \\ \psi(t) &= t^2, \\ \eta(t) &= t; \end{aligned} \tag{40}$$

After calculation, we get

$$\begin{aligned} \theta_1 &= \int_0^1 t^{\alpha-1} \phi(t)\,dt = \frac{4}{17}, \\ \theta_2 &= \int_0^1 t^{\beta-1} \psi(t)\,dt = \frac{5}{16}, \\ \theta_3 &= \int_0^1 t^{\gamma-1} \eta(t)\,dt = \frac{6}{13}, \\ \theta_4 &= \int_0^1 t^{\alpha-1} \phi(t)(1-t)\,dt = \frac{16}{357}, \\ \theta_5 &= \int_0^1 t^{\beta-1} \psi(t)(1-t)\,dt = \frac{25}{336}, \\ \theta_6 &= \int_0^1 t^{\gamma-1} \eta(t)(1-t)\,dt = \frac{36}{247}. \end{aligned} \tag{41}$$

We find $\theta_1, \theta_2, \theta_3 \in (0,1)$, $\theta_4, \theta_5, \theta_6 > 0$. The functions f, g, r are continuous and $f(t,0) = b_1(t) \not\equiv 0$, $g(t,0) = b_2(t) \not\equiv 0$,

$r(t,0) = b_3(t) \not\equiv 0$. In addition, $f(t,x), g(t,x), r(t,x)$ are increasing in x.

Let $\varphi_1(t) = t^{\tau_1}, \varphi_2(t) = t^{\tau_2}, \varphi_3(t) = t^{\tau_3}$, then $\varphi_1, \varphi_2, \varphi_3 \in \Phi$.

For any $\lambda \in (0,1)$, we have

$$\begin{aligned} f(t,\lambda x) &= \lambda^{\tau_1} x^{\tau_1} + b_1(t) \geq \lambda^{\tau_1}\left[x^{\tau_1} + b_1(t)\right] \\ &= \varphi_1(\lambda) f(t,x), \\ g(t,\lambda x) &= \lambda^{\tau_2} x^{\tau_2} + b_2(t) \geq \lambda^{\tau_2}\left[x^{\tau_2} + b_2(t)\right] \\ &= \varphi_2(\lambda) g(t,x), \\ r(t,\lambda x) &= \lambda^{\tau_3} x^{\tau_3} + b_3(t) \geq \lambda^{\tau_3}\left[x^{\tau_3} + b_3(t)\right] \\ &= \varphi_3(\lambda) r(t,x); \end{aligned} \tag{42}$$

So, system (39) satisfies all conditions of Theorem 8.

Hence, (39) has a unique positive solution $(u^*, v^*, w^*) \in S_h$,

where $h(t) = (h_1(t), h_2(t), h_3(t)) = (t^{1/4}, t^{1/5}, t^{1/6})$, and we can get this positive solution by the following way.

For any $(u^0, v^0, w^0) \in S_h$, we can construct the sequences as follows:

$$\begin{aligned} u_n(t) &= \int_0^1 G_\alpha(t,s)\left\{\left[v_{n-1}(s)\right]^{\tau_1} + b_1(s)\right\} ds, \\ v_n(t) &= \int_0^1 G_\beta(t,s)\, g\left\{\left[w_{n-1}(s)\right]^{\tau_2} + b_2(s)\right\} ds, \\ w_n(t) &= \int_0^1 G_\gamma(t,s)\, r\left\{\left[u_{n-1}(s)\right]^{\tau_3} + b_3(s)\right\} ds, \\ & \qquad n = 1, 2, \ldots, \end{aligned} \tag{43}$$

where

$$\begin{aligned} G_{1\alpha}(t,s) &= \begin{cases} \dfrac{t^{1/4}(1-s)^{1/4} - (t-s)^{1/4}}{\Gamma(5/4)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{1/4}(1-s)^{1/4}}{\Gamma(5/4)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_{1\beta}(t,s) &= \begin{cases} \dfrac{t^{1/5}(1-s)^{1/5} - (1-s)^{1/5}}{\Gamma(6/5)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{1/5}(1-s)^{1/5}}{\Gamma(6/5)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_{1\gamma}(t,s) &= \begin{cases} \dfrac{t^{1/6}(1-s)^{1/6} - (t-s)^{1/6}}{\Gamma(7/6)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{1/6}(1-s)^{1/6}}{\Gamma(7/6)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_{2\alpha}(t,s) &= \frac{t^{1/4}}{1-\theta_1}\int_0^1 t^3 G_{1\alpha}(t,s)\, ds \\ &= \frac{17t^{1/4}}{13\Gamma(5/4)}\left[\frac{4}{17}(1-s)^{1/4} - \frac{4}{5}(1-s)^{5/4} \right. \\ &\quad + \frac{48}{45}(1-s)^{9/4} - \frac{384}{585}(1-s)^{13/4} \\ &\quad \left. + \frac{1536}{9945}(1-s)^{17/4}\right], \\ G_{2\beta}(t,s) &= \frac{t^{1/5}}{1-\theta_2}\int_0^1 t^2 G_{1\beta}(t,s)\, ds \\ &= \frac{16t^{1/5}}{11\Gamma(6/5)}\left[\frac{5}{16}(1-s)^{1/5} - \frac{5}{6}(1-s)^{6/5} \right. \\ &\quad \left. + \frac{25}{33}(1-s)^{11/5} - \frac{125}{528}(1-s)^{16/5}\right], \\ G_{2\gamma}(t,s) &= \frac{t^{1/6}}{1-\theta_3}\int_0^1 t G_{1\gamma}(t,s)\, ds \\ &= \frac{13t^{1/6}}{7\Gamma(7/6)}\left[\frac{6}{13}(1-s)^{1/6} - \frac{6}{7}(1-s)^{7/6} \right. \\ &\quad \left. + \frac{36}{91}(1-s)^{13/6}\right], \\ G_\alpha(t,s) &= G_{1\alpha}(t,s) + G_{2\alpha}(t,s), \\ G_\beta(t,s) &= G_{1\beta}(t,s) + G_{2\beta}(t,s), \\ G_\gamma(t,s) &= G_{1\gamma}(t,s) + G_{2\gamma}(t,s). \end{aligned} \tag{44}$$

Then, the limit of $u_n(t), v_n(t), w_n(t)$ is the solution of system (39).

That is,

$$\begin{aligned} u_n(t) &\longrightarrow u^* \ (n \longrightarrow \infty), \\ v_n(t) &\longrightarrow v^* \ (n \longrightarrow \infty), \\ w_n(t) &\longrightarrow w^* \ (n \longrightarrow \infty). \end{aligned} \tag{45}$$

From Theorem 8, system (39) has a unique positive solution. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research is supported by National Science Foundation of China (11671227).

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