Characterizations of Certain Types of Type 2 Soft Graphs

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The vertex-neighbors correspondence has an essential role in the structure of a graph. The type 2 soft set is also based on the correspondence of initial parameters and underlying parameters. Recently, type 2 soft graphs have been introduced. Structurally, it is a very efficient model of uncertainty to deal with graph neighbors and applicable in applied intelligence, computational analysis, and decision-making. The present paper characterizes type 2 soft graphs on underlying subgraphs (regular subgraphs, irregular subgraphs, cycles, and trees) of a simple graph. We present regular type 2 soft graphs, irregular type 2 soft graphs, and type 2 soft trees. Moreover, we introduce type 2 soft cycles, type 2 soft cut-nodes, and type 2 soft bridges. Finally, we present some operations on type 2 soft trees by presenting several examples to demonstrate these new concepts.

1. Preliminaries and Introduction

A graph $\mathcal{G} = (X, \mathcal{E})$ consists of a nonempty set of objects $X$, called vertices, and a set $\mathcal{E}$ of two element subsets of $X$ called edges. Two vertices $x$ and $y$ are adjacent if $\{x, y\} \in \mathcal{E}$. A loop is an edge that connects a vertex to itself. A simple graph is an unweighted, undirected graph containing no multiple edges or graph loops. A graph $\mathcal{G}^{\text{耠}} = (X^{\text{耠}}, \mathcal{E}^{\text{耠}})$ is said to be a subgraph of $\mathcal{G} = (X, \mathcal{E})$ if $X^{\text{耙}} \subseteq X$ and $\mathcal{E}^{\text{耙}} \subseteq \mathcal{E}$. The graph neighborhoods of a vertex $x$ in a graph is the set of all the vertices adjacent to $x$ including $x$ itself. The graph neighborhoods of a vertex $x$ in a graph are the set of all the vertices adjacent to $x$ excluding $x$ itself. The eccentricity of the vertex $x$ is the maximum distance from $x$ to any vertex. The distance between two vertices $x$ and $y$ in a graph is the number of edges in a shortest path, denoted by $d(x, y)$. The radius of a graph is the minimum eccentricity of any vertex $x$. A graph $\mathcal{G} = (X, \mathcal{E})$ is called a tree if it is connected and contains no cycles. Equivalently (and sometimes more useful), a tree is a connected graph on $m$ vertices with exactly $m - 1$ edges. A forest is a disjoint union of trees. The degree of a vertex of a simple graph is the number of edges incident to the vertex.

A regular graph is a graph where each vertex has the same number of neighbors. A graph that is not a regular graph is called irregular graph. A graph is called neighborly irregular graph if no two adjacent vertices have the same degree. A complete graph is a graph in which each pair of graph vertices is connected by an edge. For basic definitions of graphs see [1–3].

Soft set theory [4], firstly initiated by Molodtsov, is a new mathematical tool for dealing with uncertainties. Some fruitful operations, soft set theory, are presented by Maji et al. [5] and Ali et al. [6]. We refer to Molodtsov’s soft sets as type 1 soft sets (briefly $T1SS$). Let $A$ be a set of parameters that can have an arbitrary nature (numbers, functions, sets of words, etc.). Let $U$ be a universe and the power set of $U$ is denoted by $P(U)$. The soft set is defined as follows.

**Definition 1** (see [5]). A pair $(S, A)$ is called a soft set over $U$, where $S$ is a mapping given by $S : A \rightarrow P(U)$.

Note that the set of all $T1SS$ over $U$ will be denoted by $\sigma(U)$. Many researchers take attention at applicability of soft sets in real and practical problems. In recent years, research
on soft set theory has been rapidly developed, and great progress has been achieved, including works of soft sets in graph theory. Ali et al. [7] introduced a representation of graphs based on neighborhoods. Akram et al. introduced the concepts of soft graphs [8, 9] and soft trees [10]. Let $\mathcal{G} = (X, \mathcal{E})$ be a simple graph, $A$ be any nonempty set.

Let $R \subseteq A \times X$ be an arbitrary relation from $A$ to $X$. A mapping $I$ from $A$ to $P(X)$ denoted as $I : A \rightarrow P(X)$ and defined as $I(x) = \{y \in X \mid xRy\}$ and a mapping $J$ from $A$ to $P(\mathcal{E})$ denoted as $J : A \rightarrow P(\mathcal{E})$ and defined as $J(x) = \{v \in \mathcal{E} \mid vu \in J(x)\}$. Then $(I, A)$ is a T1SS over $X$ and $(J, A)$ is a T1SS over $\mathcal{E}$. The notion of a soft graph is defined as follows.

**Definition 2** (see [9]). A 4-tuple $G = (\mathcal{G}, I, J, A)$ is called a soft graph if it satisfies;

(i) $\mathcal{G} = (X, \mathcal{E})$ is a simple graph.

(ii) $A$ is a non-empty set of parameters.

(iii) $[I, A]$ is a T1SS over $X$.

(iv) $[J, A]$ is a T1SS over $\mathcal{E}$.

(v) $(I(\alpha), J(\alpha))$ for all $\alpha \in A$, represents a subgraph of $\mathcal{G}$.

The soft graph $G^r = (\mathcal{G}, I, J, A)$ can also be written as $G^r = (\mathcal{G}, I, J) = \{T(\alpha) \mid \alpha \in A\}$, where $T(\alpha) = (I(\alpha), J(\alpha)) \forall \alpha \in A$.

**Definition 3** (see [9, 10]). Let $G^r$ be a soft graph of $\mathcal{G}$. Then $G^r$ is said to be a soft tree (resp., regular soft graph, irregular soft graph, neighborly irregular soft graph, soft cycle) if every $T(\alpha)$ is a tree (resp., regular graph, irregular graph, neighborly irregular graph, and cycle) for all $\alpha \in A$.

A soft graph $G^r$ is called a regular soft graph of degree $r$ if $T(\alpha)$ is a regular graph of degree $r$ for all $\alpha \in A$. In the rest of paper soft tree and soft cycle will be written as type 1 soft tree (briefly, T1ST) and type 1 soft cycle (briefly, T1SC), respectively. Some operations of T1STS are defined as follows.

**Definition 4** (see [10]). Let $G^r_1 = \langle I_1, J_1, A_1 \rangle$ and $G^r_2 = \langle I_2, J_2, A_2 \rangle$ be two T1STS of $\mathcal{G}$. Then $G^r_{12}$ is a type 1 soft subtree of $G^r$ if

(i) $A_2 \subseteq A_1$;

(ii) for each $x \in A_2$, $T_2(x)$ is a subtree of $T_1(x)$.

**Definition 5** (see [10]). Let $G^r_1 = \langle I_1, J_1, A_1 \rangle$ and $G^r_2 = \langle I_2, J_2, A_2 \rangle$ be two T1STS of $\mathcal{G}$. The extended union of $G^r_1$ and $G^r_2$, denoted by $G^r_1 \sqcup G^r_2 = G^r = \langle I, J, B \rangle$, where $B = A_1 \cup A_2$, is defined $\forall \alpha \in B$, as

$$
I(\alpha) = \begin{cases} 
I_1(\alpha), & \text{if } \alpha \in A_1 - A_2 \\
I_2(\alpha), & \text{if } \alpha \in A_2 - A_1 \\
I_1(\alpha) \cup I_2(\alpha), & \text{if } \alpha \in A_1 \cap A_2
\end{cases}
$$

$$
J(\alpha) = \begin{cases} 
J_1(\alpha), & \text{if } \alpha \in A_1 - A_2 \\
J_2(\alpha), & \text{if } \alpha \in A_2 - A_1 \\
J_1(\alpha) \cup J_2(\alpha), & \text{if } \alpha \in A_1 \cap A_2
\end{cases}
$$

It can be written as $G^r_1 \sqcup G^r_2 = \{T(\alpha) = (I(\alpha), J(\alpha)) \mid \alpha \in B\}$.

**Definition 6** (see [10]). Let $G^r_1 = \langle I_1, J_1, A_1 \rangle$ and $G^r_2 = \langle I_2, J_2, A_2 \rangle$ be two T1STS in $\mathcal{G}$. The restricted intersection of $G^r_1$ and $G^r_2$, denoted by $G^r_1 \cap G^r_2 = G^r = \langle I, J, C \rangle$, where $B = A_1 \cap A_2$, is defined $\forall \alpha \in B$, as $I(\alpha) = I_1(\alpha) \cap I_2(\alpha)$, $J(\alpha) = J_1(\alpha) \cap J_2(\alpha)$, $\forall \alpha \in A_1 \cap A_2$.

It can be written as $G^r_1 \cap G^r_2 = \{T(\alpha) = (I(\alpha), J(\alpha)) \mid \alpha \in B\}$.

**Definition 7** (see [10]). Let $G^r_1 = \langle I_1, J_1, A_1 \rangle$ and $G^r_2 = \langle I_2, J_2, A_2 \rangle$ be two T1STS in $\mathcal{G}$. The AND operation of $G^r_1$ and $G^r_2$, denoted by $G^r_1 \land G^r_2 = G^r = \langle I, J, C \rangle$, where $C = A_1 \times A_2$, is defined $\forall \alpha \beta \in A_1 \times A_2$, as $I(\alpha \beta) = I_1(\alpha) \cap I_2(\beta)$, $J(\alpha \beta) = J_1(\alpha) \cap J_2(\beta)$, $\forall \alpha \beta \in A_1 \times A_2$.

**Definition 8** (see [10]). Let $G^r_1 = \langle I_1, J_1, A_1 \rangle$ and $G^r_2 = \langle I_2, J_2, A_2 \rangle$ be two T1STS in $\mathcal{G}$. The OR operation of $G^r_1$ and $G^r_2$, denoted by $G^r_1 \lor G^r_2 = G^r = \langle I, J, C \rangle$, where $C = A_1 \times A_2$, is defined $\forall \alpha \beta \in A_1 \times A_2$, as $I(\alpha \beta) = I_1(\alpha) \cup I_2(\beta)$, $J(\alpha \beta) = J_1(\alpha) \cup J_2(\beta)$, $\forall \alpha \beta \in A_1 \times A_2$.

**Definition 9** (see [8]). Let $G^r = \langle I, J, A \rangle$ be a soft graph of $\mathcal{G}$. Then the complement of $G^r$ is denoted by $G^c$ and defined by $G^{c} = \langle I^{c}, J^{c}, A \rangle$, where $I^{c}(\alpha) = I(\alpha)$ and $J^{c}(\alpha)$ contain those edges which are not included in $I(\alpha)$.

A generalization of soft sets called type 2 soft sets is introduced by Chatterjee et al. [11]. Some results on type 2 soft sets are validated by Yang and Wang [12] and some new operation on type 2 soft sets are defined by Khizar et al. [13]. Let $X$ be universe set and $E$ be the set of parameters. The definition of type 2 soft set is as follows:

Let $\mathcal{G} = (X, \mathcal{E})$ be a simple graph. The set of all T1SS over $\mathcal{G}$ is denoted by $\Gamma(X)$ and the set of all T1SS over $\mathcal{E}$ is denoted by $\Gamma(\mathcal{E})$. The set of neighbors of an element $x \in X$ is denoted by $\mathcal{N}_X(x)$ and defined by $\mathcal{N}_X(x) = \{z \in X \mid xz \in \mathcal{E}\}$. Then $\mathcal{N}_X(\mathcal{A}) = \bigcup_{x \in A} \mathcal{N}_X(x)$, where $A \subseteq X$. Let a subset $\mathcal{R}$ of $\mathcal{N}_X(\mathcal{A}) \times X$ be an arbitrary relation from $\mathcal{N}_X(\mathcal{A})$ to $X$. Let $\mathcal{G} = (X, \mathcal{E})$ be a simple graph. Khizar et al. [14] introduced type 2 soft graphs by using T2SS over $X$ and T2SS over $\mathcal{E}$.
Consider a graph $G = (X, E)$, where $X$ is a set of vertices and $E$ is a set of edges. Let $A \subseteq X$ and $\Gamma(A)$ be the collection of all $T_1$-SSs over $X$. Let $I^*(A) = (I^*_x, \mathcal{N}^*_x)$ where $I^*_x : \mathcal{N}^*_x \rightarrow P(X)$ can be defined as $I^*_x(z) = \{y \in X | z \sim y\}$, for all $z \in \mathcal{N}^*_x \subseteq X$ and $\mathcal{N}^*_x$ is the set of all neighbors of $x \in A$. This $T_2$-SSs is also called a vertex-neighbors induced type 2 soft set (briefly, VN-type 2 soft set) over $X$.

Definition 12 (see [14]). Let $G = (X, E)$ be a simple graph, $A \subseteq X$ and $\Gamma(A)$ be the collection of all $T_1$-SSs over $\mathcal{S}$. Let $[I^*, A]$ be a VN-type 2 soft set over $X$. Then a mapping $I^* : A \rightarrow \Gamma(A)$ is called a $T_2$-SSs over $\mathcal{S}$ and it is denoted by $[I^*, A]$. In this case, corresponding to each vertex $x \in A$, there exists a $T_1$-SS, $(I^*_x, \mathcal{N}^*_x)$ such that $I^*_x(z) = \{y \in X | z \sim y\}$ for all $z \in \mathcal{N}^*_x \subseteq X$ and $\mathcal{N}^*_x$ is the set of all neighbors of $x \in A$. This $T_2$-SSs is also called a VN-type 2 soft set over $\mathcal{S}$.

Definition 13 (see [14]). A 5-tuple $G^* = (G, I^*, J^*, A, \mathcal{N}^*_A)$ is called a type 2 soft graph (briefly, $T_2$SG) if it satisfies following conditions:
(i) $G = (X, E)$ is a simple graph.
(ii) $A$ is a non-empty set of parameters.
(iii) $[I^*, A]$ is a VN-type 2 soft set over $X$.
(iv) $[J^*, A]$ is a VN-type 2 soft set over $\mathcal{S}$.
(v) $T_1$SS corresponding to $(I^*(x), J^*(x)) \forall x \in A$, represents a type 1 soft graph.

A $T_2$SG can also be represented by $G^* = (I^*, J^*, A, \mathcal{N}^*_A \setminus E)$ where $\mathcal{N}^*_A \setminus E$ is called a type 2 soft set over $\mathcal{S}$.

2. Certain Types of Type 2 Soft Graphs

In this section, we present regular type 2 soft graphs, irregular type 2 soft graphs, and type 2 soft trees. Moreover, we introduce type 2 soft cycles, type 2 soft cut-nodes, and type 2 soft bridges.

Definition 14. Let $G^* = (G, I^*, J^*, A, \mathcal{N}^*_A \setminus E)$ be a $T_2$SG of $G$. Then $G^*$ is said to be a regular $T_2$SG if every $T_1$-SS corresponding to every $T^*(y)$ is a regular $T_1$-SS for all $y \subseteq A$. A $T_2$SG $G^*$ is called a regular $T_2$SG of degree $r$ if every $T_1$-SS corresponding to $T^*(y)$ is a regular $T_1$-SS of degree $r$ for all $y \subseteq A$.

Example 15. Consider a graph $G = (X, E)$ as shown in Figure 1. Let $A = \{e_1, e_2\}$. It may be written that $\mathcal{N}^*_{e_1} = \{e_1, e_2\}$ and $\mathcal{N}^*_{e_2} = \{e_2, e_1\}$. Let $[I^*, A]$ and $[J^*, A]$ be two $T_2$SSs over $X$ and $\mathcal{S}$ respectively, such that $I^*(x) = (I^*_x, \mathcal{N}^*_x)$ and $J^*(x) = (J^*_x, \mathcal{N}^*_x)$ for all $x \in A$. Define $I^*_x(z) = \{y \in X | z \sim y\}$ and $J^*_x(z) = \{y \in X | z \sim y\}$ for all $z \in \mathcal{N}^*_x \subseteq X$, and $I^*_x(z) = \{y \in X | z \sim y\}$ and $J^*_x(z) = \{y \in X | z \sim y\}$ for all $z \in \mathcal{N}^*_x \subseteq X$. Then $T_2$SSs $[I^*, A]$ and $[J^*, A]$ are as follows:

\[
I^*(e_1) = \{e_1, e_2, e_3, e_4, e_5, e_6\}, J^*(e_2) = \{e_1, e_2, e_3, e_4, e_5, e_6\},
\]

\[
I^*(e_2) = \{e_1, e_2, e_3, e_4, e_5, e_6\}, J^*(e_1) = \{e_1, e_2, e_3, e_4, e_5, e_6\}.
\]

Note that if $G^*$ is a regular $T_2$SG of $G$ then $G^*$ need not be a complete graph.

Example 16. Consider a graph $G = (X, E)$ as shown in Figure 3. Let $A = \{e_1, e_2\}$. It may be written that $\mathcal{N}^*_{e_1} = \{e_1, e_2, e_3\}$ and $\mathcal{N}^*_{e_2} = \{e_1, e_2\}$. Let $[I^*, A]$ and $[J^*, A]$ be two $T_2$SSs over $X$ and $\mathcal{S}$ respectively, such that $I^*(x) = (I^*_x, \mathcal{N}^*_x)$ and $J^*(x) = (J^*_x, \mathcal{N}^*_x)$ for all $x \in A$. Define $I^*_x(z) = \{y \in X | z \sim y \leq d(z, y) \leq 1\}$ and $J^*_x(z) = \{\alpha \beta \in \mathcal{S} | \alpha \beta \in z \in \mathcal{N}^*_x \subseteq X, \text{and} I^*_x(z) = \{y \in X | z \sim y \leq d(z, y) \leq 1\}$, $J^*_x(z) = \{\alpha \beta \in \mathcal{S} | \alpha \beta \in z \in \mathcal{N}^*_x \subseteq X, \text{and} I^*_x(z) = \{y \in X | z \sim y \leq d(z, y) \leq 1\}$ for all $z \in \mathcal{N}^*_x \subseteq X$. Then $T_2$SSs $[I^*, A]$ and $[J^*, A]$ are as follows:

\[
I^*(e_1) = \{e_1, e_2, e_3, e_4, e_5\}, J^*(e_2) = \{e_1, e_2, e_3, e_4, e_5\},
\]

\[
I^*(e_2) = \{e_1, e_2, e_3, e_4, e_5\}, J^*(e_1) = \{e_1, e_2, e_3, e_4, e_5\}.
\]
Then

\[ T^*(e_2) = \{e_1, [e_1, e_2, e_3, e_4, e_5], e_6, e_7, e_8, e_9, e_{10}\} \]

\[ T^*(e_3) = \{e_4, [e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9] \} \]

Then \([J^*, A]\) and \([J^*, A] \) are as follows:

\[ I^*(e_2) = \{e_1, [e_1, e_2, e_3, e_4, e_5] \} \]

\[ I^*(e_3) = \{e_4, [e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9] \} \]

The complement of \( S^* = (T^*(e_2), T^*(e_3)) \) is \( S^* = (T^*(e_2), T^*(e_3)) \) as shown in Figure 4.

**Proposition 20.** If \( S^* \) is a regular T2SG of \( G \) then \( S^* \) is a regular T2SG of \( G \).

**Proof.** Let \( S^* \) be a regular T2SG of \( G \). Let \((T_\alpha, N_{\alpha})\) be a T1SG corresponding to \( T^*(\alpha) \) for all \( \alpha \in A \). Then \( T_\alpha(x) \) \( \forall x \in N_{\alpha} \) is a regular subgraph of \( G \). Since complement of a regular graph is regular, \( T^*_\alpha(x) \) \( \forall x \in N_{\alpha} \) is a regular subgraph. This implies that \( T1SG \) corresponding to \( T^*(\alpha) \) for all \( \alpha \in A \) is regular \( T1SG \). Hence \( S^* \) is a regular T2SG of \( G \).

**Proposition 21.** Let \( G \) be a regular graph. Then every \( T2SG \) of \( G \) may not be a regular \( T2SG \) of \( G \).

**Example 22.** Consider a regular graph \( G = (X, \mathcal{E}) \), where \( X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\} \) and \( \mathcal{E} = \{e_1e_2, e_2e_3, e_3e_4, e_4e_5, e_5e_6, e_6e_7, e_7e_8, e_8e_9, e_9e_{10}\} \). Let \( A = \{e_1, e_2\} \). It may be written that \( N_{\mathcal{E}_e} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\} \). Let \( [I^*, A] \) and \([J^*, A] \) be two T2SSs over \( X \) and \( \mathcal{E} \) respectively, such that \( I^*(x) = (I_e, N_{\mathcal{E}_e}) \) and \( J^*(x) = (J_e, N_{\mathcal{E}_e}) \) for all \( x \in A \). Define \( I_e(z) = \{y \in X : d(z, y) \leq 1\} \), \( J_e(z) = \{y \in X : d(z, y) \leq 2\} \) for all \( z \in \mathcal{E} \), and \( I_e(z) = \{y \in X = 1\} \), \( J_e(z) = \{y \in X \neq 1\} \) for all \( z \in \mathcal{E} \). Then \( T2SSs \) \([I^*, A] \) and \([J^*, A] \) are as follows:

\[ I^*(e_1) = \{e_2, [e_2, e_1, e_5, e_6, e_7, e_8, e_9, e_{10}] \} \]

\[ J^*(e_1) = \{e_2, [e_2, e_1, e_5, e_6, e_7, e_8, e_9, e_{10}] \} \]

Then \( S^* = (T^*(e_1), T^*(e_2)) \) is not a regular \( T2SG \) of \( G \) as shown in Figure 5.

**Definition 23.** Let \( G^* \) be a \( T2SG \) of \( G \). Then \( G^* \) is said to be irregular \( T2SG \) if \( T1SG \) corresponding to \( T^*(\gamma) \) is an irregular \( T1SG \) for all \( \gamma \in A \).

**Example 24.** Consider a graph \( G = (X, \mathcal{E}) \) as shown in Figure 6. Let \( A = \{e_3, e_4\} \). It may be written that \( N_{\mathcal{E}_e} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\} \). Let \([I^*, A] \) and \([J^*, A] \) be...
two $T2SSs$ over $X$ and $\mathcal{E}$ respectively, such that $I^*(x) = (I_x, \mathcal{N}_x)$ and $J^*(x) = (J_x, \mathcal{N}_x)$ for all $x \in A$. Define $I_x(z) = \{y \in X \mid z \mathcal{R}_y \iff d(z, y) \leq 1\}$, $J_x(z) = \{\alpha \beta \in \mathcal{E} \mid \{\alpha, \beta\} \subseteq I_x(z)\}$ for all $z \in \mathcal{N}_x \subseteq X$, and $I_{e_i}(z) = \{y \in X \mid z \mathcal{R}_y \iff d(z, y) \leq 1\}$, $J_{e_i}(z) = \{\alpha \beta \in \mathcal{E} \mid \{\alpha, \beta\} \subseteq I_{e_i}(z)\}$ for all $z \in \mathcal{N}_{e_i} \subseteq X$. Then $T2SSs$ $[I^*, A]$ and $[J^*, A]$ are as follows:

$$I^*(e_5) = \{\{e_1, e_4, e_5, e_7\}, \{e_6, e_5, e_6, e_7\}\},$$

$$J^*(e_5) = \{\{e_1, e_2, e_4, e_5, e_6, e_7\}, \{e_6, e_5, e_6, e_7\}\},$$

$$I^*(e_6) = \{\{e_5, e_6, e_5, e_7\}, \{e_7, e_5, e_6, e_7\}\},$$

$$J^*(e_6) = \{\{e_5, e_5, e_6, e_7\}, \{e_7, e_5, e_6, e_7\}\}. \quad (5)$$

Then $\mathcal{G}^* = (T^*(e_6), T^*(e_5))$ is an irregular $T2SG$ of $\mathcal{E}$ as shown in Figure 7.

**Definition 25.** Let $\mathcal{G}^*$ be a type 2 soft graph of $\mathcal{E}$. Then $\mathcal{G}^*$ is said to be neighborly irregular $T1SG$ if $T1SG$ corresponding to $T^*(y)$ is an neighborly irregular $T1SG$ for all $y \in A$. 

**Figure 4:** $\mathcal{G}^* = (T^*(e_2), T^*(e_3))$.

**Figure 5:** $\mathcal{G}^* = (T^*(e_1), T^*(e_2))$.

**Figure 6:** Simple graph.
Example 26. Consider a graph $\mathcal{G} = (X, \varepsilon)$ as shown in Figure 8. Let $A = \{e_1, e_6\}$. It may be written that $\mathcal{N}_{\mathcal{E}} e_1 = \{e_2, e_4\}$ and $\mathcal{N}_{\mathcal{E}} e_6 = \{e_5, e_3\}$. Let $[I^*, A]$ and $[J^*, A]$ be two T2SSs over $X$ and $\varepsilon$ respectively, such that $I^*(x) = \{I_x, \mathcal{N}_{\mathcal{E}} x\}$ and $J^*(x) = \{I_x, \mathcal{N}_{\mathcal{E}} x\}$ for all $x \in A$. Define $I^*(x) = \{y \in X \mid z \mathcal{R} y \iff d(z, y) \leq 1\}; I^*_E(z) = \{|x \in \mathcal{E} \mid \{x, \beta\} \subseteq I_x(z)\}$ for all $z \in \mathcal{N}_{\mathcal{E}} x \subseteq X$, and $I^*_F(z) = \{|x \in X \mid z \mathcal{R} y \iff d(z, y) \leq 1\}; I^*_E(z) = \{|x \in \mathcal{E} \mid \{x, \beta\} \subseteq I_x(z)\}$ for all $z \in \mathcal{N}_{\mathcal{E}} x \subseteq X$. Then T2SSs $I^*, A]$ and $[J^*, A]$ are as in the following: $I^*(e_1) = \{e_2, e_3, e_4, e_6\}, e_3, e_4, e_6, e_5, e_3, e_4, e_5, e_3, e_4, e_5\}, I^*(e_6) = \{e_2, e_3, e_4, e_6\}, e_3, e_4, e_6, e_5, e_3, e_4, e_5\}, I^*(e_3) = \{e_2, e_3, e_4, e_6\}, e_3, e_4, e_6, e_5, e_3, e_4, e_5\}, I^*(e_4) = \{e_2, e_3, e_4, e_6\}, e_3, e_4, e_6, e_5, e_3, e_4, e_5\}$, $I^*(e_5) = \{e_2, e_3, e_4, e_6\}, e_3, e_4, e_6, e_5, e_3, e_4, e_5\}$. Then $\mathcal{B}^* = (T^*(e_1), T^*(e_6))$ is a neighborly irregular T2SG of $\mathcal{G}$ as shown in Figure 9.

Definition 27. Let $\mathcal{G}^*$ be a T2SG of a simple graph $\mathcal{G}$. Let $(T_{\alpha}, \mathcal{N}_{\mathcal{E}} \alpha)$ be a T1SS corresponding to $T^*(\alpha)$ for all $\alpha \in A$. An edge $uv$ in $\mathcal{B}^*$ is said to be a type 2 soft bridge if its removal disconnect the subgraph $T_{\alpha}(u), V_x \in \mathcal{N}_{\mathcal{E}} \alpha$.

Definition 28. Let $\mathcal{G}^*$ be a T2SG of a simple graph $\mathcal{G}$. Let $(T_{\alpha}, \mathcal{N}_{\mathcal{E}} \alpha)$ be a T1SS corresponding to $T^*(\alpha)$ for all $\alpha \in A$. An vertex $x$ in $\mathcal{B}^*$ is said to be a type 2 soft cut-vertex if its removal disconnect the subgraph $T_{\alpha}(x), V_x \in \mathcal{N}_{\mathcal{E}} \alpha$.

Example 29. Consider $T2SG \mathcal{B}^*$ defined in Example 24. In the Figure 7, type 2 soft bridges of $\mathcal{B}^*$ are $e_1e_3, e_2e_5, e_4e_6$ in $T_{\alpha}(e_1), e_3e_4, e_5e_6$ in $T_{\alpha}(e_2), e_1e_2, e_4e_5$ in $T_{\alpha}(e_3), e_2e_4, e_3e_5$ in $T_{\alpha}(e_4)$. Moreover, type 2 cut-vertices of $\mathcal{B}^*$ are $e_1$ in $T_{\alpha}(e_1), e_5$ in $T_{\alpha}(e_3), e_3$ in $T_{\alpha}(e_4), e_4$ in $T_{\alpha}(e_5), e_2$ in $T_{\alpha}(e_6)$.

Definition 30. Let $\mathcal{B}^* = (\mathcal{G}, I^*, J^*, A, \mathcal{N}_{\mathcal{E}} A)$ be a T2SG of $\mathcal{G}$. Then $\mathcal{B}^*$ is said to be a type 2 soft tree (briefly, T2ST) if T1SS corresponding to every $T^*(\gamma)$ is a T1ST for all $\gamma \in A$.

Example 31. Consider a graph $\mathcal{G} = (X, \varepsilon)$ as shown in Figure 10. Let $A = \{c, f\} \subset X$ and $\mathcal{N}_{\mathcal{E}} c = \{b, d\}, \mathcal{N}_{\mathcal{E}} f = \{e, g\}$. Let $[I^*, A]$ and $[J^*, A]$ be two neighbor-induced T2SSs over $X$ and $\varepsilon$ respectively, such that $I^*(x) = \{I_x, \mathcal{N}_{\mathcal{E}} x\}$ and $J^*(x) = \{I_x, \mathcal{N}_{\mathcal{E}} x\}$ for all $x \in A$. We define $I^*_E(z) = \{|y \in X \mid z \mathcal{R} y \iff d(z, y) \leq 1\}; I^*_F(z) = \{|x \in \mathcal{E} \mid \{x, \beta\} \subseteq I_x(z)\}$ for all $z \in \mathcal{N}_{\mathcal{E}} x \subseteq X$ and $I^*_F(z) = \{|y \in X \mid z \mathcal{R} y \iff d(z, y) \leq 1\}; I^*_E(z) = \{|x \in \mathcal{E} \mid \{x, \beta\} \subseteq I_x(z)\}$ for all $z \in \mathcal{N}_{\mathcal{E}} x \subseteq X$. Then T2SSs $[I^*, A]$ and $[J^*, A]$ are as in the following: $I^*(c) = \{|b, g, e\}, \{d, \{a, g, f\}\}, J^*(c) = \{|b, g, e\}, \{d, \{a, g, f\}\}, I^*(f) = \{|b, d, g\}, \{e, \{a, b, c\}\}, J^*(f) = \{|b, d, g\}, \{e, \{a, b, c\}\}$. Therefore, $\mathcal{B}^* = (T^*(c), T^*(f))$ is a T2ST of $\mathcal{G}$ as shown in Figure 11. It is also called VN-type 2 soft tree.

Hence, $\mathcal{B}^* = (T^*(c), T^*(f))$ is a T2SG of $\mathcal{G}$. It is also called VN-type 2 soft graph. It may be written that $\mathcal{G} = \bigcup_{\alpha \in A} \mathcal{N}_{\mathcal{E}} \alpha = \{b, d, e, g\}$. We may symbolize $\alpha \in \mathcal{N}_{\mathcal{E}} \alpha$, as $x \mapsto \alpha$ and denote a set of associations of $A$, as $\langle A \rightarrow \mathcal{G} \rangle = \{x \mapsto \alpha \mid \alpha \in \mathcal{N}_{\mathcal{E}} \alpha\}$. Then tabular representation of T2ST is given in Table 1.

Theorem 32. Let $(T_{\gamma}, \mathcal{N}_{\mathcal{E}} \gamma)$ be a T1SS corresponding to $T^*(\gamma)$ for all $\gamma \in A$. Let $T_{\gamma}(x), V_x \in \mathcal{N}_{\mathcal{E}} \gamma$ be subgraph with $n \geq 3$ vertices of $\mathcal{B}^*$ and $\mathcal{B}^*$ a T2SG of $\mathcal{G}$. Then $\mathcal{B}^*$ is not a complete T2SG of $\mathcal{G}$.

Proof. Let $(T_{\gamma}, \mathcal{N}_{\mathcal{E}} \gamma)$ be a T1SS corresponding to $T^*(\gamma)$ for all $\gamma \in A$. Suppose the contrary that $\mathcal{G}$ is a complete T2SG, then every subgraph $T_{\gamma}(x)$, for all $x \in \mathcal{N}_{\mathcal{E}} \alpha$, is complete. Suppose $\alpha, \beta$ be arbitrary nodes of $T_{\gamma}(x)$ and they are connected by an edge $ab$. Since $T_{\gamma}(x)$ is subgraph with $n \geq 3$ vertices of $\mathcal{G}$, then we can always find at least one vertex $\eta$ which is connected to $\alpha$ by an edge $a\eta$ and to $\beta$ by an edge $b\eta$.
because $T^*_y(x)$ is a complete graph. Then there exists a cycle $\alpha \rho y$. Therefore, $T^*_y(x) \forall x \in \mathcal{N}_y$ cannot be a T1ST which contradicts the fact that $T^*_y(x)$ is a connected subgraph of T2SG. Hence, $\mathcal{G}^*$ cannot be a complete T2SG.

**Definition 33.** Let $\mathcal{G}^*$ be a T2SG and $(T_y, \mathcal{N}_y)$ be a T1ST corresponding to $T^*(y)$ for any $y \in A$. Then $\mathcal{G}^*$ is called type 2 soft forest if $T^*_y(x)$ consists of more than one disconnected tree for all $x \in \mathcal{N}_y$.

**Definition 34.** Let $\mathcal{G}^*$ be a T2SG of $\mathcal{G}$. Then $\mathcal{G}^*$ is said to be a type 2 soft cycle (briefly, T2SC) if T1SG corresponding to $T(y)$ is a type 1 soft cycle of $\mathcal{G}$ for each $y \in A$.

**Example 35.** Consider a simple graph $\mathcal{G} = (X, \mathcal{E})$, where $X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $\mathcal{E} = \{e_1e_2, e_2e_3, e_3e_1, e_3e_4, e_4e_5, e_5e_6, e_6e_7, e_7e_8, e_8e_9, e_9e_{10}, e_{10}e_1\}$. Let $A = \{e_9, e_8\} \subset X$. It may be written that $\mathcal{N}_e_{10} = \{e_{10}, e_7\}$ and $\mathcal{N}_e_8 = \{e_7, e_{10}\}$. Let $[I^*, A]$ and $[J^*, A]$ be two T2SSs over $X$ and $\mathcal{G}$, respectively, such that $T^*(x) = (I^*_x, \mathcal{N}_e_{10})$ and $T^*(x) = (J^*_x, \mathcal{N}_e_8)$ for all $x \in A$. Define $I^*_e(z) = \{y \in X | z \mathcal{R} y \iff 5 \geq d(z, y) \geq 3\}$, $I^*_e(z) = \{[\alpha \beta] \in \mathcal{E} | (\alpha, \beta) \subseteq I^*_e(z)\}$ for all $z \in \mathcal{N}_e_z \subset X$, and $I^*_e(z) = \{y \in X | z \mathcal{R} y \iff 5 \geq d(z, y) \geq 3\}$, $I^*_e(z) = \{[\alpha \beta] \in \mathcal{E} | (\alpha, \beta) \subseteq I^*_e(z)\}$ for all $z \in \mathcal{N}_e_z \subset X$. Then T2SSs $[I^*, A]$ and $[J^*, A]$ are as follows:

$$
I^*(e_9) = \{e_7e_2, e_2e_3, e_1, e_{10}, \{e_5, e_3, e_4\}\},
$$

$$
J^*(e_9) = \{e_7e_3, e_3e_4, e_4e_1, e_{10}, \{e_5e_3, e_4\}\},
$$

$$
I^*(e_8) = \{e_7e_2, e_2e_3, e_1, e_{10}, \{e_5e_3, e_4\}\},
$$

$$
J^*(e_8) = \{e_7e_3, e_3e_4, e_4e_1, e_{10}, \{e_5e_3, e_4\}\}.
$$

One can check that $\mathcal{G}^* = (T^*(e_9), T^*(e_8))$ is a T2SC of $\mathcal{G}$ as shown in Figure 12. It is also called VN-type 2 soft cycle.

**Theorem 36.** If $\mathcal{G}^*$ is a T2SC of $\mathcal{G}$ then $\mathcal{G}^*$ is not a T2ST of $\mathcal{G}$.

**Proof.** Let $\mathcal{G}^*$ be a T2SC of $\mathcal{G}$. Let $(T_y, \mathcal{N}_y)$ be a type 1 soft cycle corresponding to $T^*(y)$ for all $y \in A$. By definition, tree does not contain cycle. Then $T^*_y(x)$ is not a tree for all $x \in \mathcal{N}_y$, so that $(T^*_y, \mathcal{N}_y)$ is not a type 1 soft tree. Hence $\mathcal{G}^*$ is not a T2ST of $\mathcal{G}$.

The converse of above theorem is not true in general; i.e., if $\mathcal{G}^*$ is not a T2ST then $\mathcal{G}$ need not be a T2SC. The following example illustrates it.
Example 37. Consider a graph \( \mathcal{G} = (X, \mathcal{E}) \) as shown in Figure 13.

Let \( \{a, b\} \subseteq X \). Then \( \mathcal{N}_{\mathcal{B}_a} = \{b, c, d\}, \mathcal{N}_{\mathcal{B}_b} = \{a, c, d\} \). Let \( [I^*, A] \) and \( [J^*, A] \) be two T2SSs over \( X \) and \( \mathcal{E} \), respectively, such that \( I^*(x) = (I_x, \mathcal{N}_{\mathcal{B}_x}) \) and \( J^*(x) = (J_x, \mathcal{N}_{\mathcal{B}_x}) \) for all \( x \in A \). Let \( I_a(\alpha) = \{y \in X \mid \alpha \mathcal{E} y \iff d(\alpha, y) \leq 1\} \), \( I_a(\alpha) = \{uv \in \mathcal{E} \mid \{u, v\} \subseteq I_a(\alpha)\} \forall \alpha \in \mathcal{N}_{\mathcal{B}_a} \) and \( I_b(\beta) = \{y' \in X \mid \beta \mathcal{E} y' \iff d(\beta, y') = 1\}, I_b(\beta) = \{u, v' \in \mathcal{E} \mid \{u, v'\} \subseteq I_b(\beta)\} \forall \beta \in \mathcal{N}_{\mathcal{B}_b} \). Then

\[
\begin{align*}
I^*(a) &= \{[b, \{a, b, d, c\}], [c, \{a, b, c, e\}], [d, \{a, b, d, e\}]\}, \\
J^*(a) &= \{[b, \{ab, ad, ac, bc, bd\}], [c, \{ab, ac, ce, bc\}], \\
& \quad [d, \{ab, ad, de, db\}]\}, \tag{7}
\end{align*}
\]

Figure 14 shows the respective T1SGs corresponding to \( T^*(a) = (I^*(a), J^*(a)) \) and \( T^*(b) = (I^*(b), J^*(b)) \) respectively. One can check that \( T_a(b) = (I_a(b), J_a(b)) \), \( T_a(c) = (I_a(c), J_a(c)) \), \( T_a(d) = (I_a(d), J_a(d)) \), \( T_b(c) = (I_b(c), J_b(c)) \) and \( T_b(d) = (I_b(d), J_b(d)) \) are not trees. This implies that \( \mathcal{G}^* = (T^*(a), T^*(b)) \) is not a T2ST of \( \mathcal{G} \). But \( \mathcal{G}^* \) is not a T2SC of \( \mathcal{G} \).

Proposition 38. Every T2SC of \( \mathcal{G} \) is a regular T2SG of \( \mathcal{G} \).

Proof. Suppose that \( \mathcal{G}^* \) is a T2SC. Let \( (T_y, \mathcal{N}_{\mathcal{B}_y}) \) be a T1SC corresponding to \( T^*(y) \) for any \( y \in A \). Then \( T_x(y) \) is a cycle for all \( x \in \mathcal{N}_{\mathcal{B}_x} \). Since cycle is closed path and each vertex has degree 2, this implies that \( T_x(y) \) is a regular graph for all \( x \in \mathcal{N}_{\mathcal{B}_x} \). Therefore \( (T_y, \mathcal{N}_{\mathcal{B}_y}) \) is regular T1SG. Since \( y \) was taken to be arbitrary, thus it holds for all \( y \in A \). Hence \( \mathcal{G}^* \) is a regular T2SG of \( \mathcal{G} \).

3. Operations on Type 2 Soft Trees

In this section, we present type 2 soft subtree of T2ST, union, intersection, OR operation, and AND operation of T2ST.

Definition 39. Let \( \mathcal{G}_1^* = (I_1^*, J_1^*, A_1, \mathcal{B}) \) and \( \mathcal{G}_2^* = (I_2^*, J_2^*, A_2, \mathcal{B}) \) be two T2STs of \( \mathcal{G} \). Then \( \mathcal{G}_1^* \) is a type 2 soft subtree of \( \mathcal{G}_2^* \) if

(i) \( A_2 \subseteq A_1 \),

(ii) for each \( x \in A_2 \), \( T_1^*(x) \) corresponding to \( T_1^*(x) = (I_1^*(x), J_1^*(x)) \) is a type 1 soft subtree of \( T_2^*(x) \) corresponding to \( T_2^*(x) = (I_2^*(x), J_2^*(x)) \).

Example 40. Consider a simple graph \( \mathcal{G} = (X, \mathcal{E}) \), where \( X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \) and \( \mathcal{E} = \{e_1e_2, e_2e_3, e_3e_4, e_4e_5, e_5e_6, e_6e_7, e_7e_1\} \). Let \( A = \{e_2, e_3\} \), \( B = \{e_2, e_4, e_5\} \). It may be written that \( \mathcal{N}_{\mathcal{B}_e} = \{e_3, e_1\}, \mathcal{N}_{\mathcal{B}_e} = \{e_5, e_3\} \) and \( \mathcal{N}_{\mathcal{B}_e} = \{e_1, e_6\} \).

Let \( [I^*, A] \) and \( [J^*, A] \) be two T2SSs over \( X \) and \( \mathcal{E} \), respectively, such that \( I^*(x) = (I_x, \mathcal{N}_{\mathcal{B}_x}) \) and \( J^*(x) = (J_x, \mathcal{N}_{\mathcal{B}_x}) \) for all \( x \in A \). Define \( I_{e_1}(z) = \{y \in X \mid z \mathcal{E} y \iff \} \)
Figure 14: $E^* = (T^*(a), T^*(b))$.

Figure 15: $E^* = (T^*(e_2), T^*(e_3))$.

$$d(z, y) \leq 1, I_\varepsilon(z) = \{\alpha \beta \in \mathcal{E} \mid \{\alpha, \beta\} \subseteq I_\varepsilon(z)\} \text{ for all } z \in \mathcal{N}\mathcal{E}_\varepsilon \subseteq X, I_\varepsilon(z) = \{y \in X \mid z \mathcal{R} y \implies d(z, y) \leq 1\}, I_\varepsilon(z) = \{\alpha \beta \in \mathcal{E} \mid \{\alpha, \beta\} \subseteq I_\varepsilon(z)\} \text{ for all } z \in \mathcal{N}\mathcal{E}_\varepsilon \subseteq X.

Then $T2SSS [I^*, A]$ and $[J^*, A]$ are as follows:

$$I^*(e_2) = \{(e_1, e_1, e_2, e_3), (e_3, e_2, e_3, e_4)\},$$

$$J^*(e_2) = \{(e_1, e_1, e_2, e_1), (e_3, e_2, e_3, e_4)\},$$

$$I^*(e_4) = \{(e_3, e_4, e_2, e_1), (e_5, e_4, e_5, e_6)\},$$

$$J^*(e_4) = \{(e_3, e_4, e_2, e_1), (e_5, e_4, e_5, e_6)\}.$$  \hspace{1cm} (8)

Then $E^* = (T^*(e_2), T^*(e_4))$ is a $T2ST$ of $\mathcal{E}$ as shown in Figure 15.

Let $[I^*, B]$ and $[J^*, B]$ be two $T2SSS$ over $X$ and $\mathcal{E}$, respectively, such that $I^*(x) = (I^*_e, \mathcal{N}\mathcal{E}_\varepsilon)$ and $J^*(x) = (J^*_e, \mathcal{N}\mathcal{E}_\varepsilon)$ for all $x \in B$. Define

$$I^*_e(z) = \{y \in X \mid z \mathcal{R} y \implies d(z, y) \leq 1\},$$

$$J^*_e(z) = \{\alpha \beta \in \mathcal{E} \mid \{\alpha, \beta\} \subseteq I^*_e(z)\}$$

for all $z \in \mathcal{N}\mathcal{E}_\varepsilon \subseteq X$.

Then $T2SSS [I^*, B]$ and $[J^*, B]$ are as follows:

$$I^*(e_2) = \{(e_1, e_1, e_2, e_3), (e_3, e_2, e_3, e_4)\},$$

$$J^*(e_2) = \{(e_1, e_1, e_2, e_1), (e_3, e_2, e_3, e_4)\},$$

$$I^*(e_4) = \{(e_3, e_4, e_2, e_1), (e_5, e_4, e_5, e_6)\},$$

$$J^*(e_4) = \{(e_3, e_4, e_2, e_1), (e_5, e_4, e_5, e_6)\}.$$  \hspace{1cm} (9)

Then $E^* = (T^*(e_2), T^*(e_3))$ is a $T2ST$ of $\mathcal{E}$ as shown in Figure 16.

One can check that $A \subseteq B$ and $T^*(e_3) \mathcal{E} T^*(e_2), T^*(e_3) \mathcal{E} T^*(e_4)$. Hence $E^*$ is a type 2 subtree of $E^*$.

**Theorem 41.** Let $E^* = (I^*_1, J^*_1, A_1)$ and $E^* = (I^*_2, J^*_2, A_2)$ be two $T2STS$ of $\mathcal{E}$. Then $E^*$ is a type 2 soft subtree of $E^*$ if and only if $I^*_1(x) \subseteq I^*_2(x)$ and $J^*_2(x) \subseteq J^*_1(x)$ for all $x \in A_2$.

**Proof.** Suppose $E^*$ is a type 2 soft subtree of $E^*_1$. Then, by the definition of type 2 soft subtree,

(i) $A_2 \subseteq A_1$,

(ii) For each $x \in A_2$, $T1ST$ corresponding to $T^*_2(x) = (I^*_2(x), J^*_2(x))$ is a type 1 soft subtree of $T1ST$ corresponding to $T^*_1(x) = (I^*_1(x), J^*_1(x))$.

Since $T1ST$ corresponding to $T^*_2(x)$ is a type 1 soft subtree of $T1ST$ corresponding to $T^*_1(x)$ for all $x \in A_2$. Then $I^*_2(x) \subseteq I^*_1(x)$ and $J^*_2(x) \subseteq J^*_1(x)$ for all $x \in A_2$.

Conversely, given that $I^*_2(x) \subseteq I^*_1(x)$ and $J^*_2(x) \subseteq J^*_1(x)$ for all $x \in A_2$. As $E^*$ is a $T2ST$ of $\mathcal{E}$, $T1SS$ corresponding to $T^*_1(x)$ is a $T1ST$ of $\mathcal{E}$ for all $x \in A_1$. Also, $E^*$ is a $T2ST$ of $\mathcal{E}$, $T1SS$ corresponding to $T^*_2(x)$ is a $T1ST$ of $\mathcal{E}$ for all $x \in A_2$. This implies that $T1ST$ corresponding to $T^*_2(x)$ is a type 1 soft...
Definition 42. Let \( Y^* = \langle I^*_1, J^*_1, A_1 \rangle \) and \( Y^* = \langle I^*_2, J^*_2, A_2 \rangle \) be two T2STs of \( \mathcal{G} \). The union of \( Y^* \) and \( Y^* \), denoted by \( Y^* \cup Y^* = Y^* = \langle I^*, J^*, C \rangle \), where \( C = A_1 \cup A_2 \) is defined \( \forall \alpha \in C \) as

\[
I^*(\alpha) = \begin{cases} 
I^*_1(\alpha), & \text{if } \alpha \in A_1 - A_2 \\
I^*_2(\alpha), & \text{if } \alpha \in A_2 - A_1 \\
I^*_1(\alpha) \cup I^*_2(\alpha), & \text{if } \alpha \in A_1 \cap A_2 
\end{cases}
\]

(11)

where \( I^*_1(\alpha) \cup I^*_2(\alpha) \) for all \( \alpha \in A_1 \cap A_2 \) refers to the usual type 1 soft union between the respective T1ST corresponding to \( I^*_1(\alpha) \) and \( I^*_2(\alpha) \), respectively. And,

\[
J^*(\alpha) = \begin{cases} 
J^*_1(\alpha), & \text{if } \alpha \in A_1 - A_2 \\
J^*_2(\alpha), & \text{if } \alpha \in A_2 - A_1 \\
J^*_1(\alpha) \cup J^*_2(\alpha), & \text{if } \alpha \in A_1 \cap A_2 
\end{cases}
\]

(12)

where \( J^*_1(\alpha) \cup J^*_2(\alpha) \) for all \( \alpha \in A_2 \cap A_1 \) refers to the usual type 1 soft extended union between the respective T1ST corresponding to \( J^*_1(\alpha) \) and \( J^*_2(\alpha) \), respectively.

It can be written as \( Y^*_1 \cup Y^*_2 = \{ I^*(\alpha), J^*(\alpha) \mid \alpha \in C \} \).

Theorem 43. Let \( Y^*_1 = \langle I^*_1, J^*_1, A_1 \rangle \) and \( Y^*_2 = \langle I^*_2, J^*_2, A_2 \rangle \) be two T2STs of \( \mathcal{G} \) with \( A_1 \cap A_2 = \emptyset \). Then \( Y^*_1 \cup Y^*_2 \) is a T2ST of \( \mathcal{G} \).

Proof. The union of \( Y^*_1 \cup Y^*_2 \) is defined as \( Y^* = \langle I^*, J^*, C \rangle \) where \( C = A_1 \cup A_2 \) for all \( \alpha \in C \),

\[
I^*(\alpha) = \begin{cases} 
I^*_1(\alpha), & \text{if } \alpha \in A_1 - A_2 \\
I^*_2(\alpha), & \text{if } \alpha \in A_2 - A_1 \\
I^*_1(\alpha) \cup I^*_2(\alpha), & \text{if } \alpha \in A_1 \cap A_2 
\end{cases}
\]

(13)

where \( I^*_1(\alpha) \cup I^*_2(\alpha) \) for all \( \alpha \in A_1 \cap A_2 \) refers to the usual type 1 soft extended union between the respective T1ISS corresponding to \( I^*_1(\alpha) \) and \( I^*_2(\alpha) \), respectively.

\[
J^*(\alpha) = \begin{cases} 
J^*_1(\alpha), & \text{if } \alpha \in A_1 - A_2 \\
J^*_2(\alpha), & \text{if } \alpha \in A_2 - A_1 \\
J^*_1(\alpha) \cup J^*_2(\alpha), & \text{if } \alpha \in A_1 \cap A_2 
\end{cases}
\]

(14)

where \( J^*_1(\alpha) \cup J^*_2(\alpha) \) for all \( \alpha \in A_1 \cap A_2 \) refers to the usual type 1 soft extended union between the respective T1ISS corresponding to \( J^*_1(\alpha) \) and \( J^*_2(\alpha) \), respectively.

Let \( A_1 \cap A_2 \neq \emptyset \) then union of two T2ST may not be a T2ST as it can be seen in the following example.

Example 44. Consider a graph \( \mathcal{G} \) defined in Example 40. Let \( A = \{e_2, e_4\}, B = \{e_1, e_7\} \). It may be written that \( \mathcal{N} \mathcal{E}_1 = \{e_2, e_4\}, \mathcal{N} \mathcal{E}_2 = \{e_3, e_5\} \), and \( \mathcal{N} \mathcal{E}_3 = \{e_2, e_4\} \).

Let \( I^*(\alpha) \) and \( J^*(\alpha) \) be two T2SSs over \( X \) and \( \mathcal{G} \), respectively, such that \( I^*(\alpha) = \langle I^*_1, \mathcal{N} \mathcal{E}_1 \rangle \) and \( J^*(\alpha) = \langle I^*_2, \mathcal{N} \mathcal{E}_2 \rangle \) for all \( \alpha \in A \). Define \( I^*_e(\alpha) = \{ y \in X : z \mathcal{R} y \iff d(z, \alpha) \leq 1 \} \) and \( I^*_e(\alpha) = \{ \alpha \in \mathcal{G} : [\alpha \beta] \in I^*_e(\alpha) \} \) for all \( z \in \mathcal{N} \mathcal{E}_1 \subset X \) and \( \alpha \in \mathcal{G} \). Let \( T2SS \) \( I^*(\alpha) \) and \( J^*(\alpha) \) are as follows:

\[
\]

(15)

\[
\]

Then \( Y^* = \langle I^*, J^*, e_2 \rangle \) is a T2ST of \( \mathcal{G} \).

Let \( I^*(\alpha) \) and \( J^*(\alpha) \) be two T2SSs over \( X \) and \( \mathcal{G} \), respectively, such that \( I^*(\alpha) = \langle I^*_1, \mathcal{N} \mathcal{E}_1 \rangle \) and \( J^*(\alpha) =\)
By routine calculations, it is easy to see that 

given \( A, B \subseteq X \) and \( f \) continuous, the following holds:

\[
T^*(A \cup B) = T^*(A) \cup T^*(B)
\]

One can also prove that \( T^*(A \cap B) \neq T^*(A) \cap T^*(B) \) in general.

**Definition 46.** Let \( G^*_1 = \langle I_1^*, I_1^*, A_1 \rangle \) and \( G^*_2 = \langle I_2^*, I_2^*, A_2 \rangle \) be two T2STS of \( G \). The intersection of \( G^*_1 \) and \( G^*_2 \), denoted by \( G^*_1 \cap G^*_2 = \langle I^*, I^*, C \rangle \), where \( C = A \cap B \) is defined as \( I^*(x) = I_1^*(x) \cap I_2^*(x) \), for all \( x \in A \cap B \), where \( I_1^*(x) \cap I_2^*(x) \) for all \( x \in A \cap B \) refers to the usual type 1 soft intersection between the respective T1SS corresponding to \( I_1^*(x) \) and \( I_2^*(x) \), respectively. And \( J^*(x) = J_1^*(x) \cap J_2^*(x) \) for all \( x \in A \cap B \) where \( J_1^*(x) \cap J_2^*(x) \) for all \( x \in A \cap B \) refers to the usual type 1 soft intersection between the respective T1SS corresponding to \( J_1^*(x) \) and \( J_2^*(x) \), respectively.

It can be written as \( G^*_1 \cap G^*_2 = \langle I^*(x), J^*(x) | x \in C \rangle \).

The intersection of two T2STS may not be T2STS as it can be seen in the following example.

**Example 47.** Consider a simple graph \( G \) shown in Figure 18. Let \( A = \{a, h\} \) and \( B = \{b, h\} \). It may be written that \( N \mathcal{R} A = \{v, b\} \), \( N \mathcal{R} B = \{v, g\} \) and \( N \mathcal{R} A = \{a, c\} \).

Let \( I^*, A \) and \( J^*, B \) be two T2SSs over \( X \) and \( G \), respectively. Consider the T1SGs \( T^*(x) = (I^*_x, N \mathcal{R} x) \) and \( T^*(x) = (I^*_x, N \mathcal{R} x) \) for all \( x \in A \). Define \( I^*_x(z) = \{ y \in X | z \mathcal{R} y \iff d(z, y) \leq 1 \} \), \( I^*_x(z) = \{ y \in X | z \mathcal{R} y \iff d(z, y) \leq 1 \} \), \( I^*_x(z) = \{ y \in X | z \mathcal{R} y \iff d(z, y) \leq 1 \} \), and \( I^*_x(z) = \{ y \in X | z \mathcal{R} y \iff d(z, y) \leq 1 \} \) for all \( x \in A \cap B \) where \( I^*_x(z) \cap I^*_y(z) \) for all \( x \in A \cap B \). Then \( T^*(x) \cap T^*(y) \) is not a T2STS of \( G \).

**Lemma 45.** Let \( G^*_1 = \langle S_1^*, T_1^*, A_1 \rangle \) and \( G^*_2 = \langle S_2^*, T_2^*, A_2 \rangle \) be two T2STS of \( G \). If \( A_1 \cap A_2 = \emptyset \), then their union is a T2STS of \( G \).
\( I^* (b) = \{ \{ a, \{ e, f, g \} \}, \{ c, \{ f, g, h \} \} \}, \)

\( J^* (b) = \{ \{ a, \{ ef, fg \} \}, \{ \{ fg, gh \} \} \}, \)

\( I^* (h) = \{ \{ v, \{ e, f, g, d, a, b, c \} \}, \{ g, \{ d, e, f, h, v, a \} \} \}, \)

\( J^* (h) = \{ \{ v, \{ ef, fg, ed, vd, dc \} \}, \{ g, \{ ef, ed, vh, va, vd \} \} \}. \)

Then \( \mathcal{E}^* = (T^* (b), T^* (h)) \) is a T2ST of \( \mathcal{G} \). By the definition of intersection of T2ST,

\[
I^* (h) = I^* (h) \cap I^* (h)
\]

and

\[
J^* (h) = J^* (h) \cap J^* (h),
\]

\( \{ h \} = B \cap A. \)

By routine calculations, it is easy to see that T1SG corresponding to \( T^* (h) = T^* (h) \cap T^* (h) \) is a disconnected T1SG, as shown in Figure 19. Therefore, \( \mathcal{E}^*_1 \cap \mathcal{E}^*_2 = \mathcal{E}^* = \langle I^*, J^*, A \cap B \rangle \) is not a T2ST of \( \mathcal{G} \).

**Definition 48.** Let \( \mathcal{E}^*_1 = \langle I^*_1, J^*_1, A_1 \rangle \) and \( \mathcal{E}^*_2 = \langle I^*_2, J^*_2, A_2 \rangle \) be two T2Ss of \( \mathcal{G} \). The AND operation of \( \mathcal{E}^*_1 \) and \( \mathcal{E}^*_2 \), denoted by \( \mathcal{E}^* = \langle I^*, J^*, A \rangle \), is defined by

\[
I^* (x) = I^*_1 (x) \cap I^*_2 (x)
\]

and

\[
J^* (x) = J^*_1 (x) \cap J^*_2 (x),
\]

for all \( x \in A \).

**Example 49.** Consider a simple graph \( \mathcal{G} = (X, E) \), where

\[ X = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \} \]

and \( E = \{ e_1 e_2, e_2 e_3, e_3 e_4, e_4 e_5, e_5 e_6, e_6 e_7, e_7 e_8 \} \).

Let \( A = \{ e_1, e_8 \} \) and \( B = \{ e_8 \} \). It may be written that \( \mathcal{N} \mathcal{R}_e = \{ e_8, e_1, e_6 \} \) and \( \mathcal{N} \mathcal{R}_e = \{ e_2, e_8 \} \).

Let \( I^* (A) \) and \( J^* (A) \) be two T2Ss over \( \mathcal{G} \), respectively, such that

\[
I^* (x) = (I^*_1 (x) \cap \mathcal{N} \mathcal{R}_e) \text{ and } J^* (x) = (J^*_1 (x) \cap \mathcal{N} \mathcal{R}_e)
\]

for all \( x \in A \).

Then \( \mathcal{E}^* = (I^*, J^*, A) \) is a T2ST of \( \mathcal{G} \). The OR operation on \( \mathcal{E}^* \) and \( \mathcal{E}^* \) is defined as in the following:

\[
I^* (e_1, e_8) = I^* (e_1) \cup I^* (e_8)
\]

\[
J^* (e_1, e_8) = J^* (e_1) \cup J^* (e_8)
\]

The AND operation on \( \mathcal{E}^* \) and \( \mathcal{E}^* \) is shown in Figure 20.

**Definition 50.** Let \( \mathcal{E}^*_1 = \langle I^*_1, J^*_1, A_1 \rangle \) and \( \mathcal{E}^*_2 = \langle I^*_2, J^*_2, A_2 \rangle \) be two T2Ss over \( \mathcal{G} \), respectively, such that

\[
I^* (x) = (I^*_1 (x) \cap \mathcal{N} \mathcal{R}_e) \text{ and } J^* (x) = (J^*_1 (x) \cap \mathcal{N} \mathcal{R}_e) \text{ for all } x \in A.
\]

Define \( I^*_c (z) \) as \( \{ \alpha \beta \in \mathcal{G} | \{ \{ e_8, e_1, e_6 \} \} \} \) for all \( z \in \mathcal{N} \mathcal{R}_e \sub X \). Then \( T2SSs \) \( I^*, J^*, A \) and \( I^*, J^*, A \) are as follows:

\[
I^* (e_1) = \{ \{ e_2, e_1, e_3, e_4, e_5, e_6, e_7, e_8 \} \}
\]

\[
J^* (e_1) = \{ \{ e_2, e_1, e_3, e_4, e_5, e_6, e_7, e_8 \} \}
\]

\[
I^* (e_1) = \{ \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \} \}
\]

\[
J^* (e_1) = \{ \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \} \}
\]

\[
(21)
\]

\[
(22)
\]

\[
(23)
\]

The AND operation on \( \mathcal{E}^* \) and \( \mathcal{E}^* \) is shown in Figure 20.
A × B, where $I^*(y) ∪ I^*(\eta)$ for all $(y, \eta) \in A \times B$ refers to the usual type 1 soft OR operation between the respective T1SS corresponding to $I^*_1(y)$ and $I^*_2(\eta)$ respectively and $J^*(y) \cap J^*(\eta)$ for all $(y, \eta) \in A \times B$ refers to the usual type 1 soft OR operation between the respective T1SS corresponding to $J^*_1(y)$ and $J^*_2(\eta)$ respectively.

**Example 51.** Consider a simple graph $G = (X, \mathcal{E})$ defined in Example 49. Let $A = \{e_1, e_2\}$, $B = \{e_3\}$. It may be written that $\mathcal{N} \mathcal{E}_{e_3} = \{e_4, e_5\}$, $\mathcal{N} \mathcal{E}_{e_4} = \{e_6, e_7\}$ and $\mathcal{N} \mathcal{E}_{e_5} = \{e_8, e_9\}$.

Let $[I^*, A]$ and $[J^*, A]$ be two T2SSs over $X$ and $\mathcal{E}$ respectively, such that $I^*(x) = (I^*_x, \mathcal{N} \mathcal{E}_x)$ and $J^*(x) = (J^*_x, \mathcal{N} \mathcal{E}_x)$ for all $x \in A$. Define $I^*_c(z) = \{ y \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \leq 1 \}$, $J^*_c(z) = \{ x \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \geq 0 \}$, $I^*_e(z) = \{ x \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \geq 3 \}$, $J^*_e(z) = \{ x \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \geq 3 \}$, $I^*_c(z)$ and $J^*_c(z)$ for all $z \in \mathcal{N} \mathcal{E}_x \subseteq X$. Then T2SSs $[I^*, A]$ and $[J^*, A]$ are as follows:

$$I^*(e_3) = \{ e_2, e_1, e_2, e_3 \}, \{ e_4, e_3, e_4, e_3 \},$$

$$J^*(e_3) = \{ e_2, e_1, e_2, e_3 \}, \{ e_4, e_3, e_4, e_3 \}.$$  \hspace{1cm} (24)

Then $G^* = (T^*(e_2), T^*(e_3))$ is a T2ST of $G$.

Let $[I^*, B]$ and $[J^*, B]$ be two T2SSs over $X$ and $\mathcal{E}$, respectively, such that $I^*(x) = (I^*_x, \mathcal{N} \mathcal{E}_x)$ and $J^*(x) = (J^*_x, \mathcal{N} \mathcal{E}_x)$ for all $x \in B$. Define $I^*_c(z) = \{ y \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \leq 1 \}$, $J^*_c(z) = \{ x \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \geq 0 \}$, $I^*_e(z) = \{ x \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \geq 3 \}$, $J^*_e(z) = \{ x \in X \mid \exists \mathcal{E} y \iff \bar{d}(z, y) \geq 3 \}$, $I^*_c(z)$ and $J^*_c(z)$ for all $z \in \mathcal{N} \mathcal{E}_x \subseteq X$. Then T2SSs $[I^*, B]$ and $[J^*, B]$ are as follows:

$$I^*(e_3) = \{ e_2, e_1, e_2, e_3 \}, \{ e_4, e_3, e_4, e_3 \},$$

$$J^*(e_3) = \{ e_2, e_1, e_2, e_3 \}, \{ e_4, e_3, e_4, e_3 \}.$$  \hspace{1cm} (25)

Then $G^* = (T^*(e_3))$ is a T2ST of $G$. The OR operation on $G^*$ and $G^*$ is defined as in the following:

$$J^*(e_3, e_6) = I^*(e_3) \cap J^*(e_6)$$

$$= \{ \{e_2, e_7\}, \{e_4, e_2, e_3, e_3, e_4, e_3\}\}.$$
The OR operation on $\mathcal{G}^*$ and $\mathcal{G}^{**}$ is shown in Figure 21.

\[\begin{align*}
\{(e_4, e_5), [e_1e_2, e_2e_3, e_1e_6, e_6e_2]\}, \\
\{(e_6, e_7), [e_1e_2, e_2e_3, e_3e_4, e_4e_5]\}, \\
\{(e_6, e_5), [e_1e_2, e_2e_3, e_7e_6, e_6e_5]\}\right]
\end{align*}\]

(26)

The OR operation on $\mathcal{G}^*$ and $\mathcal{G}^{**}$ is shown in Figure 21.

4. Conclusion

In above study, we have characterized type 2 soft graphs on underlying subgraphs (regular subgraphs, irregular subgraphs, cycles, trees) of a simple graph. We have presented regular type 2 soft graphs, irregular type 2 soft graphs, and type 2 soft trees. Moreover, we have introduced type 2 soft cycles, type 2 soft cut-nodes, and type 2 soft bridges. Finally, we have presented some operations on type 2 soft trees by presenting several examples to demonstrate these new concepts. In future work, we will extend our work in following areas of research:

(i) Applications of type 2 soft graphs in computer networks and social networks

(ii) Fuzzy type 2 soft graphs and their applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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