Research Article

Dynamics and Solutions of a Fifth-Order Nonlinear Difference Equation

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The main objective of this paper is to study the behavior of the rational difference equation of the fifth-order $y_{n+1} = \alpha y_n + \beta y_n y_{n-3} / (A y_{n-4} + B y_{n-3})$, $n = 0, 1, \dots$, where α, β, A , and B are real numbers and the initial conditions $y_{-4}, y_{-3}, y_{-2}, y_{-1}$ and y_0 are positive real numbers such that $A y_{-4} + B y_{-3} \neq 0$. Also, we obtain the solution of some special cases of this equation.

1. Introduction

In recent years, there has been a great interest in studying the rational difference equations. These equations describe real life situations in stochastic time series, combinatorial analysis, electrical network, number theory, biology, genetics, probability theory, physics, ecology, statistical problems, and economics, for example [1–5]. It is so important to investigate the asymptotic behavior of solutions of a nonlinear difference equations and to discuss the boundedness, periodicity, and stability (local and global) of their equilibrium points; see [6–36] and references therein.

Kalabušić et al. [6] investigated the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the difference equation

$$x_{n+1} = \frac{\beta x_{n-l} + \delta x_{n-k}}{B x_{n-l} + D x_{n-k}}. \quad (1)$$

In [7], Elabbasy et al. got the solution and the periodicity character of the recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}. \quad (2)$$

Cinar [8] found the solution of the difference equation

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}. \quad (3)$$

In [9] Obaid et al. studied the global stability, boundedness, and the periodicity of solutions of the rational difference equation

$$x_{n+1} = a x_n + \frac{b x_{n-1} + c x_{n-2} + d x_{n-3}}{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}. \quad (4)$$

Elsayed et al. [10] studied the dynamical analysis of rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-3}}{A + B x_{n-1} x_{n-3}}. \quad (5)$$

Elabbasy et al. [11] obtained the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_{n-l} x_{n-k}}{b x_{n-p} + c x_{n-q}}. \quad (6)$$

Aloqeili [12] investigated the dynamics of the difference equation

$$x_{n+1} = \frac{x_{n-k}}{A + x_n x_{n-k}}. \quad (7)$$

The aim of this paper is to study some qualitative behavior of the positive solutions of the difference equation

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \quad n = 0, 1, \dots, \quad (8)$$

where α, β, A , and B are real numbers and the initial conditions $y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 are positive real numbers such that $Ay_{-4} + By_{-3} \neq 0$.

2. Preliminaries

Let I be some interval of real numbers and let

$$G : I^{l+1} \longrightarrow I, \quad (9)$$

be a continuously differentiable function. Then for every set of initial conditions $y_{-l}, y_{-l+1}, \dots, y_0 \in I$, the difference equation

$$y_{n+1} = G(y_n, y_{n-1}, \dots, y_{n-l}), \quad n = 0, 1, \dots \quad (10)$$

has a unique solution $\{y_n\}_{n=-l}^{\infty}$.

Definition 1 (equilibrium point). A point $\bar{y} \in I$ is called an equilibrium point of the difference equation (10) if

$$\bar{y} = G(\bar{y}, \bar{y}, \dots, \bar{y}). \quad (11)$$

That is, $y_n = \bar{y}$ for $n \geq 0$ is a solution of the difference equation (10) or, equivalently, $\bar{y}$ is a fixed point of G .

Definition 2 (stability). Let $\bar{y} \in (0, \infty)$ be an equilibrium point of the difference equation (10). Then, we have the following:

(i) The equilibrium point $\bar{y}$ of the difference equation (10) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_{-l}, y_{-l+1}, \dots, y_0 \in I$ with

$$|y_{-l} - \bar{y}| + \dots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \delta, \quad (12)$$

we have

$$|y_n - \bar{y}| < \epsilon \quad \text{for all } n \geq -l. \quad (13)$$

(ii) The equilibrium point $\bar{y}$ of the difference equation (10) is called locally asymptotically stable if $\bar{y}$ is locally stable solution of (10) and there exists $\gamma > 0$, such that, for all $y_{-l}, \dots, y_{-1}, y_0 \in I$ with

$$|y_{-l} - \bar{y}| + \dots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \gamma, \quad (14)$$

we have

$$\lim_{n \rightarrow \infty} y_n = \bar{y}. \quad (15)$$

(iii) The equilibrium point $\bar{y}$ of the difference equation (10) is called global attractor if for all $y_{-l}, \dots, y_{-1}, y_0 \in I$, we have

$$\lim_{n \rightarrow \infty} y_n = \bar{y}. \quad (16)$$

(iv) The equilibrium point $\bar{y}$ of the difference equation (10) is called globally asymptotically stable if $\bar{y}$ is locally stable, and $\bar{y}$ is also a global attractor of the difference equation (10).

(v) The equilibrium point $\bar{y}$ of the difference equation (10) is called unstable if $\bar{y}$ is not locally stable.

Definition 3 (periodicity). A sequence $\{y_n\}_{n=-l}^{\infty}$ is said to be periodic with period p if $y_{n+p} = y_n$ for all $n \geq -l$. A sequence $\{y_n\}_{n=-l}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4 (Fibonacci sequence). The sequence $\{F_i\}_{i=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$, i.e., $F_i = F_{i-1} + F_{i-2}$, $F_{-2} = 1$, $F_{-1} = 1$, is called Fibonacci sequence.

Definition 5. The linearized equation of the difference equation (10) about the equilibrium $\bar{y}$ is the linear difference equation

$$z_{n+1} = \sum_{i=0}^l \frac{\partial H(\bar{y}, \bar{y}, \dots, \bar{y})}{\partial y_{n-i}} z_{n-i}. \quad (17)$$

Now, assume that the characteristic equation associated with (17) is

$$p(\lambda) = p_0 \lambda^l + p_1 \lambda^{l-1} + \dots + p_{l-1} \lambda + p_l = 0, \quad (18)$$

where

$$p_i = \frac{\partial H(\bar{y}, \bar{y}, \dots, \bar{y})}{\partial y_{n-i}}. \quad (19)$$

Theorem 6 (see [1]). Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \dots, l$, and l is nonnegative integer. Then

$$\sum_{i=1}^l |p_i| < 1 \quad (20)$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+l} + p_1 y_{n+l-1} + \dots + p_l y_n = 0, \quad n = 0, 1, \dots, \quad (21)$$

Theorem 7 (see [2]). Let $g : [\eta, \xi] \times [\eta, \xi] \times [\eta, \xi] \longrightarrow [\eta, \xi]$ be a continuous function and $[\eta, \xi]$ be an interval of real numbers. Consider the difference equation

$$y_{n+1} = g(y_n, y_{n-3}, y_{n-4}). \quad (22)$$

Assume that g satisfies the following conditions:

(i) $g(x, y, z)$ is nondecreasing in x and y in $[\eta, \xi]$ for all $z \in [\eta, \xi]$ and nonincreasing in $z \in [\eta, \xi]$ for all x and y in $[\eta, \xi]$.

(2) If $(m, M) \in [\eta, \xi] \times [\eta, \xi]$ is a solution of the system

$$M = g(M, M, m) \quad (23)$$

$$\text{and } m = g(m, m, M),$$

then

$$m = M. \quad (24)$$

Then (22) has a unique equilibrium point $\bar{y} \in [\eta, \xi]$ and every solution of (22) converges to $\bar{y}$.

3. Dynamics of (8)

In this section, we study the local stability, global stability of the solutions, and the boundedness of

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \quad (25)$$

where α, β, A , and B are positive real numbers.

3.1. Local Stability of the Equilibrium Point. In this subsection, we study the local stability of the equilibrium point of (8).

Equation (8) has a unique equilibrium point and is given by

$$\bar{y} = \alpha \bar{y} + \frac{\beta \bar{y}^2}{A \bar{y} + B \bar{y}}, \quad (26)$$

or

$$(1 - \alpha)(A + B) \bar{y}^2 = \beta \bar{y}^2. \quad (27)$$

If $(1 - \alpha)(A + B) \neq \beta$, then the only equilibrium point is $\bar{y} = 0$.

Theorem 8. *Let*

$$(A + B)^2 (1 - \alpha) > \beta (3A + B). \quad (28)$$

Then the equilibrium point $\bar{y}$ of (8) is locally asymptotically stable.

Proof. Let $G : (0, \infty)^3 \rightarrow (0, \infty)$ be a continuous function defined by

$$G(u_1, u_2, u_3) = \alpha u_1 + \frac{\beta u_1 u_2}{A u_3 + B u_2}. \quad (29)$$

Therefore, it follows that

$$\begin{aligned} \frac{\partial G(u_1, u_2, u_3)}{\partial u_1} &= \alpha + \frac{\beta u_2}{A u_3 + B u_2}, \\ \frac{\partial G(u_1, u_2, u_3)}{\partial u_2} &= \frac{\beta A u_1 u_3}{(A u_3 + B u_2)^2}, \\ \frac{\partial G(u_1, u_2, u_3)}{\partial u_3} &= -\frac{\beta A u_1 u_2}{(A u_3 + B u_2)^2}. \end{aligned} \quad (30)$$

Then

$$\begin{aligned} \frac{\partial G(\bar{y}, \bar{y}, \bar{y})}{\partial u_1} &= \alpha + \frac{\beta}{A + B}, \\ \frac{\partial G(\bar{y}, \bar{y}, \bar{y})}{\partial u_2} &= \frac{\beta A}{(A + B)^2}, \\ \frac{\partial G(\bar{y}, \bar{y}, \bar{y})}{\partial u_3} &= -\frac{\beta A}{(A + B)^2}, \end{aligned} \quad (31)$$

FIGURE 1: Plot the behavior of the zero solution of (8).

and the linearized equation of (8) about $\bar{y} = 0$ is

$$\begin{aligned} z_{n+1} - \left(\alpha + \frac{\beta}{A + B} \right) z_n - \frac{\beta A}{(A + B)^2} z_{n-3} \\ + \frac{\beta A}{(A + B)^2} z_{n-4} = 0, \end{aligned} \quad (32)$$

It follows by Theorem 6 that (8) is asymptotically stable if and only if

$$\left| \alpha + \frac{\beta}{A + B} \right| + \left| \frac{\beta A}{(A + B)^2} \right| + \left| \frac{\beta A}{(A + B)^2} \right| < 1, \quad (33)$$

and so

$$\alpha + \frac{\beta}{A + B} + \frac{2\beta A}{(A + B)^2} < 1, \quad (34)$$

and, thus,

$$\beta (3A + B) < (1 - \alpha)(A + B)^2. \quad (35)$$

The proof is complete. $\square$

Example 9. Figure 1 shows that the zero solution of the difference Equation (8) is local stability if $\alpha = 0.4$, $\beta = 0.03$, $A = 0.1$, and $B = 2$ and the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$.

3.2. Global Stability of the Equilibrium Point. In this subsection we study the global stability of the positive solutions of (8).

Theorem 10. *The equilibrium point $\bar{y}$ of (8) is global stability if*

$$B(1 - \alpha) \neq \beta. \quad (36)$$

Proof. Let η and ξ be nonnegative real numbers and assume that $g : [\eta, \xi]^3 \rightarrow [\eta, \xi]$ is a function defined by

$$g(v_1, v_2, v_3) = \alpha v_1 + \frac{\beta v_1 v_2}{A v_3 + B v_2}. \quad (37)$$

Then we can see that the function $g(v_1, v_2, v_3)$ is increasing in v_1 and v_2 and decreasing in v_3 .

FIGURE 2: Plot the behavior of the zero solution of (8) is global stable.

Assume that (m, M) is a solution of the system

$$M = g(M, M, m) \quad (38)$$

$$\text{and } m = g(m, m, M),$$

Then from (8), we see that

$$M = \alpha M + \frac{\beta M^2}{A m + B M} \quad (39)$$

$$\text{and } m = \alpha m + \frac{\beta m^2}{A M + B m},$$

and then

$$B(1-\alpha)M^2 + A(1-\alpha)Mm = \beta M^2, \quad (40)$$

$$B(1-\alpha)m^2 + A(1-\alpha)Mm = \beta m^2.$$

Subtracting these two equations, we obtain

$$[B(1-\alpha) - \beta](M^2 - m^2) = 0, \quad (41)$$

and if $B(1-\alpha) - \beta \neq 0$, then we see that $M = m$.

According to Theorem 7 the equilibrium point $\bar{y}$ is a global attractor of (8). The proof is complete. $\square$

Example 11. The zero solution of the difference Equation (8) is global stability if $\alpha = 0.01$, $\beta = 0.001$, $A = 0.5$, and $B = 0.2$ and the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$ (See Figure 2).

3.3. Existence of Boundedness and Unboundedness Solutions. Here we look at the boundedness and unboundedness solutions of (8).

Theorem 12. Every solution of (8) is bounded if

$$\alpha + \frac{\beta}{B} < 1. \quad (42)$$

Proof. Let $\{y_n\}_{n=-4}^{\infty}$ be a solution of (8). It follows from (8) that

$$\begin{aligned} y_{n+1} &= \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \\ &\leq \alpha y_n + \frac{\beta y_n y_{n-3}}{B y_{n-3}} = \left(\alpha + \frac{\beta}{B} \right) y_n. \end{aligned} \quad (43)$$

FIGURE 3: Sketch the behavior of the solution of (8) is bounded.

FIGURE 4: Sketch the behavior of the solution of (8) is unbounded.

Then

$$y_{n+1} \leq y_n \quad \text{for all } n \geq 0. \quad (44)$$

Then the sequence $\{y_n\}_{n=0}^{\infty}$ is decreasing and so is bounded from the above by $M = \max\{y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0\}$. $\square$

Theorem 13. Every solution of (8) is unbounded if $\alpha > 1$.

Proof. Let $\{y_n\}_{n=-4}^{\infty}$ be a solution of (8). It follows from (8) that

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}} > \alpha y_n, \quad \text{for all } n \geq 1. \quad (45)$$

We see that the right hand side can be written as follows:

$$\begin{aligned} x_{n+1} &= a x_n \implies \\ x_n &= a^n x_0, \end{aligned} \quad (46)$$

and this equation is unstable because $a > 1$ and $\lim_{n \rightarrow \infty} x_n = \infty$. Then by using ratio test $\{y_n\}_{n=-4}^{\infty}$ is unbounded from above. $\square$

Example 14. Figure 3 shows that (8) is bounded when $\alpha = 0.15$, $\beta = 0.1$, $A = 0.1$, and $B = 0.2$ and the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$.

Example 15. Figure 4 shows that (8) is unbounded when $\alpha = 1.2$, $\beta = 0.1$, $A = 0.1$, and $B = 0.2$ and the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$.

4. Special Cases of (8)

In this section we investigate the following special case:

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \quad (47)$$

where α, β, A , and B are integrals numbers.

4.1. First Case When $\alpha = \beta = A = B = 1$.

Theorem 16. Suppose that $\{y_n\}_{n=-4}^{\infty}$ is a solution of rational difference equation

$$y_{n+1} = y_n + \frac{y_n y_{n-3}}{y_{n-4} + y_{n-3}}. \quad (48)$$

Then, for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned} y_{4n-3} &= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-2} &= c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-1} &= b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n} &= a \prod_{i=0}^{n-1} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \end{aligned} \quad (49)$$

where $y_{-4} = e, y_{-3} = d, y_{-2} = c, y_{-1} = b, y_0 = a, \{F_i\}_{i=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \dots\}, F_{-1} = 1, F_0 = 0$, and $\prod_{i=0}^{-1} A_i = 1$.

Proof. For $n = 0$ the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned} y_{4n-8} &= a \prod_{i=0}^{n-3} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-7} &= d \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-6} &= c \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-5} &= b \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-4} &= a \prod_{i=0}^{n-2} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right). \end{aligned} \quad (50)$$

From (48), we see that

$$\begin{aligned} y_{4n-3} &= y_{4n-4} + \frac{y_{4n-4} y_{4n-7}}{y_{4n-8} + y_{4n-7}} = a \prod_{i=0}^{n-2} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\ &\quad + \left(a \prod_{i=0}^{n-2} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\ &\quad \cdot d \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\ &\quad \cdot \left(a \prod_{i=0}^{n-3} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \end{aligned}$$

$$\begin{aligned}
y_{4n-2} &= y_{4n-3} + \frac{y_{4n-3}y_{4n-6}}{y_{4n-7} + y_{4n-6}} = d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&+ \left(d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\
&\cdot c \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\cdot \left(d \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\
&+ c \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \Big)^{-1} \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&+ \left(d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\
&\cdot c \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\cdot \left(\left[\prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right] \left[d \prod_{i=0}^{n-2} \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) + c \prod_{i=0}^{n-2} \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \right] \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&+ \left(d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right. \\
&\cdot c \prod_{i=0}^{n-2} \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \Bigg) \\
&\cdot \left(d \prod_{i=0}^{n-2} \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) + c \prod_{i=0}^{n-2} \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \right)^{-1} \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\cdot \left[1 + \left(c \prod_{i=0}^{n-2} \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \right) \left(d \prod_{i=0}^{n-2} \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) + c \prod_{i=0}^{n-2} \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \right)^{-1} \right] \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\cdot \left[1 + \left(c \left(\frac{2c + d}{c + d} \right) \cdots \left(\frac{cF_{2n-3} + dF_{2n-4}}{cF_{2n-4} + dF_{2n-5}} \right) \left(\frac{cF_{2n-1} + dF_{2n-2}}{cF_{2n-2} + dF_{2n-3}} \right) \right) \right. \\
&\cdot \left. \left(d \left(\frac{c}{d} \right) \left(\frac{2c + d}{c + d} \right) \cdots \left(\frac{cF_{2n-3} + dF_{2n-4}}{cF_{2n-4} + dF_{2n-5}} \right) + c \left(\frac{2c + d}{c + d} \right) \cdots \left(\frac{cF_{2n-3} + dF_{2n-4}}{cF_{2n-4} + dF_{2n-5}} \right) \left(\frac{cF_{2n-1} + dF_{2n-2}}{cF_{2n-2} + dF_{2n-3}} \right) \right)^{-1} \right] \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\cdot \left[1 + \frac{((cF_{2n-1} + dF_{2n-2}) / (cF_{2n-2} + dF_{2n-3}))}{1 + ((cF_{2n-1} + dF_{2n-2}) / (cF_{2n-2} + dF_{2n-3}))} \right] \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\cdot \left[1 + \frac{cF_{2n-1} + dF_{2n-2}}{cF_{2n-2} + dF_{2n-3} + cF_{2n-1} + dF_{2n-2}} \right] \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[1 + \frac{cF_{2n-1} + dF_{2n-2}}{cF_{2n} + dF_{2n-1}} \right] \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[\frac{cF_{2n} + dF_{2n-1} + cF_{2n-1} + dF_{2n-2}}{cF_{2n} + dF_{2n-1}} \right] \\
&= d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[\frac{cF_{2n+1} + dF_{2n}}{cF_{2n} + dF_{2n-1}} \right] \\
&= c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right).
\end{aligned}$$

[illegible]

$$\begin{aligned}
& + b \left(\frac{2b+c}{b+c} \right) \cdots \left(\frac{bF_{2n-3} + cF_{2n-4}}{bF_{2n-4} + cF_{2n-5}} \right) \left(\frac{bF_{2n-1} + cF_{2n-2}}{bF_{2n-2} + cF_{2n-3}} \right)^{-1} \Bigg] \\
& = c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
& \cdot \left[1 + \frac{((bF_{2n-1} + cF_{2n-2}) / (bF_{2n-2} + cF_{2n-3}))}{1 + ((bF_{2n-1} + cF_{2n-2}) / (bF_{2n-2} + cF_{2n-3}))} \right] \\
& = c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
& \cdot \left[1 + \frac{bF_{2n-1} + cF_{2n-2}}{bF_{2n-2} + cF_{2n-3} + bF_{2n-1} + cF_{2n-2}} \right] \\
& = c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[1 + \frac{bF_{2n-1} + cF_{2n-2}}{bF_{2n} + cF_{2n-1}} \right] \\
& = c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
& \cdot \left[\frac{bF_{2n} + cF_{2n-1} + bF_{2n-1} + cF_{2n-2}}{bF_{2n} + cF_{2n-1}} \right] \\
& = c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[\frac{bF_{2n+1} + cF_{2n}}{bF_{2n} + cF_{2n-1}} \right] \\
& = b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right).
\end{aligned}$$

(53)

Also,

$$\begin{aligned}
y_{4n} &= y_{4n-1} + \frac{y_{4n-1}y_{4n-4}}{y_{4n-5} + y_{4n-4}} = b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
& + \left(b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\
& \cdot \left(a \prod_{i=0}^{n-2} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\
& \cdot \left(b \prod_{i=0}^{n-2} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \right) \\
& + a \prod_{i=0}^{n-2} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \Bigg)^{-1} \\
& = b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right)
\end{aligned}$$

[illegible]

$$\begin{aligned}
&= b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[1 + \frac{aF_{2n-1} + bF_{2n-2}}{aF_{2n} + bF_{2n-1}} \right] \\
&= b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \\
&\quad \cdot \left[\frac{aF_{2n} + bF_{2n-1} + aF_{2n-1} + bF_{2n-2}}{aF_{2n} + bF_{2n-1}} \right] \\
&= b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right) \left[\frac{aF_{2n+1} + bF_{2n}}{aF_{2n} + bF_{2n-1}} \right] \\
&= a \prod_{i=0}^{n-1} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right).
\end{aligned} \tag{54}$$

This completes the proof. $\square$

Theorem 17. Let $\{y_n\}_{n=-4}^{\infty}$ be a solution of rational difference equation

$$y_{n+1} = y_n - \frac{y_n y_{n-3}}{y_{n-4} + y_{n-3}}. \tag{55}$$

4.2. Second Case When $\alpha = A = B = 1$ and $\beta = -1$

Then, for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned}
y_{4n} &= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})}, \\
y_{4n+1} &= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})}, \\
y_{4n+2} &= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})}, \\
y_{4n+3} &= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_{n+1} + cF_{n+2})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})},
\end{aligned} \tag{56}$$

where $y_{-4} = e, y_{-3} = d, y_{-2} = c, y_{-1} = b, y_0 = a$ are arbitrary nonzero real numbers and $\{F_i\}_{i=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$, $F_0 = 0$.

Proof. For $n = 0$ the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
&y_{4n-5} \\
&= \frac{a b c d e}{(aF_{n-2} + bF_{n-1})(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)}, \\
&y_{4n-4} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)},
\end{aligned}$$

$$\begin{aligned}
&y_{4n-3} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_n + eF_{n+1})}, \\
&y_{4n-2} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_n + dF_{n+1})(dF_n + eF_{n+1})}, \\
&y_{4n-1} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})}.
\end{aligned} \tag{57}$$

Now, it follows from (55) that

$$\begin{aligned}
y_{4n} &= y_{4n-1} - \frac{y_{4n-1}y_{4n-4}}{y_{4n-5} + y_{4n-4}} = \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)} \right) \right) \\
&\quad \cdot \left(\frac{a b c d e}{(aF_{n-2} + bF_{n-1})(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)} + \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)} \right)^{-1} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{1}{aF_{n-1} + bF_n} \right) \right) \left(\left(\frac{1}{aF_{n-2} + bF_{n-1}} \right) + \left(\frac{1}{aF_{n-1} + bF_n} \right) \right)^{-1} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{1}{aF_{n-1} + bF_n} \right) \right) \left(\frac{(aF_{n-1} + bF_n) + (aF_{n-2} + bF_{n-1})}{(aF_{n-2} + bF_{n-1})(aF_{n-1} + bF_n)} \right)^{-1} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{aF_n + bF_{n+1}}{aF_{n-2} + bF_{n-1}} \right)^{-1} \\
&= \frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \frac{a b c d e (aF_{n-2} + bF_{n-1})}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})(aF_n + bF_{n+1})} \\
&= \frac{a b c d e (aF_n + bF_{n+1} - aF_{n-2} - bF_{n-1})}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})(aF_n + bF_{n+1})} \\
&= \frac{a b c d e (aF_{n-1} + bF_n)}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})(aF_n + bF_{n+1})} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})}.
\end{aligned} \tag{58}$$

Also, we see that

$$\begin{aligned}
y_{4n+1} &= y_{4n} - \frac{y_{4n}y_{4n-3}}{y_{4n-4} + y_{4n-3}} = \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)} \right) \right) \\
&\quad \cdot \left(\left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)} \right) + \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_{n-1} + eF_n)} \right) \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{1}{dF_n + eF_{n+1}} \right) \right) \left(\left(\frac{1}{dF_{n-1} + eF_n} \right) + \left(\frac{1}{dF_n + eF_{n+1}} \right) \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&- \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{1}{dF_n + eF_{n+1}} \right) \right) \left(\frac{dF_n + eF_{n+1} + dF_{n-1} + eF_n}{(dF_{n-1} + eF_n)(dF_n + eF_{n+1})} \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&- \left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \left(\frac{dF_{n+1} + eF_{n+2}}{dF_{n-1} + eF_n} \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \\
&- \frac{a b c d e (dF_{n-1} + eF_n)}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&= \frac{a b c d e (dF_{n+1} + eF_{n+2} - dF_{n-1} - eF_n)}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&= \frac{a b c d e (dF_n + eF_{n+1})}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})}.
\end{aligned} \tag{59}$$

Also,

$$\begin{aligned}
y_{4n+2} &= y_{4n+1} - \frac{y_{4n+1}y_{4n-2}}{y_{4n-3} + y_{4n-2}} = \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&- \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \right) \\
&\cdot \left(\left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_{n-1} + dF_n)(dF_n + eF_{n+1})} \right) + \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&- \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{1}{cF_n + dF_{n+1}} \right) \right) \left(\left(\frac{1}{cF_{n-1} + dF_n} \right) + \left(\frac{1}{cF_n + dF_{n+1}} \right) \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&- \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{1}{cF_n + dF_{n+1}} \right) \right) \left(\frac{cF_n + dF_{n+1} + cF_{n-1} + dF_n}{(cF_{n-1} + dF_n)(cF_n + dF_{n+1})} \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&- \left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{cF_{n+1} + dF_{n+2}}{cF_{n-1} + dF_n} \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})} \\
&- \frac{a b c d e (cF_{n-1} + dF_n)}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})(cF_{n+1} + dF_{n+2})}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a b c d e (cF_{n+1} + dF_{n+2} - cF_{n-1} - dF_n)}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})(cF_{n+1} + dF_{n+2})} \\
&= \frac{a b c d e (cF_n + dF_{n+1})}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_{n+1} + eF_{n+2})(cF_{n+1} + dF_{n+2})} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})},
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
y_{4n+3} &= y_{4n+2} - \frac{y_{4n+2}y_{4n-1}}{y_{4n-2} + y_{4n-1}} = \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \right) \\
&\quad \cdot \left(\left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_{n-1} + cF_n)(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) + \left(\frac{a b c d e}{(aF_{n-1} + bF_n)(bF_n + cF_{n+1})(cF_n + dF_{n+1})(dF_n + eF_{n+1})} \right) \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{1}{bF_n + cF_{n+1}} \right) \right) \left(\left(\frac{1}{bF_{n-1} + cF_n} \right) + \left(\frac{1}{bF_n + cF_{n+1}} \right) \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \\
&\quad - \left(\left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{1}{bF_n + cF_{n+1}} \right) \right) \left(\frac{bF_n + cF_{n+1} + bF_{n-1} + cF_n}{(bF_{n-1} + cF_n)(bF_n + cF_{n+1})} \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \\
&\quad - \left(\frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \right) \left(\frac{bF_{n+1} + cF_{n+2}}{bF_{n-1} + cF_n} \right)^{-1} \\
&= \frac{a b c d e}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})} \\
&\quad - \frac{a b c d e (bF_{n-1} + cF_n)}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})(bF_{n+1} + cF_{n+2})} \\
&= \frac{a b c d e (bF_{n+1} + cF_{n+2} - bF_{n-1} - cF_n)}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})(bF_{n+1} + cF_{n+2})} \\
&= \frac{a b c d e (bF_n + cF_{n+1})}{(aF_n + bF_{n+1})(bF_n + cF_{n+1})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})(bF_{n+1} + cF_{n+2})} \\
&= \frac{a b c d e (bF_n + cF_{n+1})}{(aF_n + bF_{n+1})(bF_{n+1} + cF_{n+2})(cF_{n+1} + dF_{n+2})(dF_{n+1} + eF_{n+2})}.
\end{aligned} \tag{61}$$

This completes the proof. $\square$

Example 18. Figure 5 illustrates the solutions of (55) when the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$.

4.3. Third Case When $\alpha = A = 1$, $\beta = B = -1$

Theorem 19. Assume that $\{y_n\}_{n=-4}^{\infty}$ is solutions of rational difference equation

$$y_{n+1} = y_n - \frac{y_n y_{n-3}}{y_{n-4} - y_{n-3}}, \tag{62}$$

FIGURE 5: Sketch the behavior of the solution of (55).

and then for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned}
 y_{4n} &= a \prod_{i=0}^{n-1} \left(\frac{aF_{i+3} - bF_{i+1}}{aF_{i+1} - bF_{i-1}} \right) \left(\frac{bF_{i+3} - cF_{i+1}}{bF_{i+1} - cF_{i-1}} \right) \\
 &\quad \cdot \left(\frac{cF_{i+3} - dF_{i+1}}{cF_{i+1} - dF_{i-1}} \right) \left(\frac{dF_{i+3} - eF_{i+1}}{dF_{i+1} - eF_{i-1}} \right), \\
 y_{4n+1} &= a \left(\frac{2d - e}{d - e} \right) \prod_{i=0}^{n-1} \left(\frac{aF_{i+3} - bF_{i+1}}{aF_{i+1} - bF_{i-1}} \right) \\
 &\quad \cdot \left(\frac{bF_{i+3} - cF_{i+1}}{bF_{i+1} - cF_{i-1}} \right) \left(\frac{cF_{i+3} - dF_{i+1}}{cF_{i+1} - dF_{i-1}} \right) \\
 &\quad \cdot \left(\frac{dF_{i+4} - eF_{i+2}}{dF_{i+2} - eF_i} \right), \\
 y_{4n+2} &= a \left(\frac{2d - e}{d - e} \right) \left(\frac{2c - d}{c - d} \right) \prod_{i=0}^{n-1} \left(\frac{aF_{i+3} - bF_{i+1}}{aF_{i+1} - bF_{i-1}} \right) \\
 &\quad \cdot \left(\frac{bF_{i+3} - cF_{i+1}}{bF_{i+1} - cF_{i-1}} \right) \left(\frac{cF_{i+4} - dF_{i+2}}{cF_{i+2} - dF_i} \right) \\
 &\quad \cdot \left(\frac{dF_{i+4} - eF_{i+2}}{dF_{i+2} - eF_i} \right), \\
 y_{4n+3} &= a \left(\frac{2d - e}{d - e} \right) \left(\frac{2c - d}{c - d} \right) \left(\frac{2b - c}{b - c} \right) \\
 &\quad \cdot \prod_{i=0}^{n-1} \left\{ \left(\frac{aF_{i+3} - bF_{i+1}}{aF_{i+1} - bF_{i-1}} \right) \left(\frac{bF_{i+4} - cF_{i+2}}{bF_{i+2} - cF_i} \right) \right. \\
 &\quad \cdot \left. \left(\frac{cF_{i+4} - dF_{i+2}}{cF_{i+2} - dF_i} \right) \left(\frac{dF_{i+4} - eF_{i+2}}{dF_{i+2} - eF_i} \right) \right\},
 \end{aligned} \tag{63}$$

where $y_{-4} = e$, $y_{-3} = d$, $y_{-2} = c$, $y_{-1} = b$, and $y_0 = a$ are arbitrary nonzero real numbers with $y_{-4} \neq y_{-3} \neq y_{-2} \neq y_{-1} \neq y_0$, $\{F_i\}_{i=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$, $F_0 = 0$, and $\prod_{i=0}^{-1} A_i = 1$.

Proof. The proof of Theorem 16 will be omitted. $\square$

4.4. Fourth Case When $\alpha = \beta = A = 1$ and $B = -1$

Theorem 20. Suppose that $\{y_n\}_{n=-4}^{\infty}$ is solutions of rational difference equation

$$y_{n+1} = y_n + \frac{y_n y_{n-3}}{y_{n-4} - y_{n-3}}, \tag{64}$$

and then for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned}
 y_{12n-4} &= e, \\
 y_{12n-3} &= d, \\
 y_{12n-2} &= c, \\
 y_{12n-1} &= b, \\
 y_{12n} &= a, \\
 y_{12n+1} &= -\frac{ae}{d - e}, \\
 y_{12n+2} &= \frac{ade}{(c - d)(d - e)}, \\
 y_{12n+3} &= -\frac{acde}{(b - c)(c - d)(d - e)}, \\
 y_{12n+4} &= \frac{abcde}{(a - b)(b - c)(c - d)(d - e)}, \\
 y_{12n+5} &= \frac{abce}{(a - b)(b - c)(c - d)}, \\
 y_{12n+6} &= \frac{abe}{(a - b)(b - c)}, \\
 y_{12n+7} &= \frac{ae}{a - b},
 \end{aligned} \tag{65}$$

where $y_{-4} = e$, $y_{-3} = d$, $y_{-2} = c$, $y_{-1} = b$, and $y_0 = a$ are arbitrary nonzero real numbers with $y_{-4} \neq y_{-3} \neq y_{-2} \neq y_{-1} \neq y_0$.

Proof. For $n = 0$ the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
 y_{12n-12} &= a, \\
 y_{12n-11} &= -\frac{ae}{d - e}, \\
 y_{12n-10} &= \frac{ade}{(c - d)(d - e)}, \\
 y_{12n-9} &= -\frac{acde}{(b - c)(c - d)(d - e)}, \\
 y_{12n-8} &= \frac{abcde}{(a - b)(b - c)(c - d)(d - e)}, \\
 y_{12n-7} &= \frac{abce}{(a - b)(b - c)(c - d)},
 \end{aligned}$$

$$\begin{aligned}
y_{12n-6} &= \frac{abe}{(a-b)(b-c)}, & y_{12n-3} &= d, \\
y_{12n-5} &= \frac{ae}{a-b}, & y_{12n-2} &= c, \\
y_{12n-4} &= e, & y_{12n-1} &= b.
\end{aligned} \tag{66}$$

Now, it follows from (64) that

$$\begin{aligned}
y_{12n-4} &= y_{12n-5} + \frac{y_{12n-5}y_{12n-8}}{y_{12n-9} - y_{12n-8}} = \frac{ae}{a-b} + \frac{(ae/(a-b))(abcde/(a-b)(b-c)(c-d)(d-e))}{(-acde/(b-c)(c-d)(d-e)) - (abcde/(a-b)(b-c)(c-d)(d-e))} \\
&= \frac{ae}{a-b} + \frac{(ae/(a-b))(abcde/(a-b)(b-c)(c-d)(d-e))}{((-a^2cde + abcde - abcde)/(a-b)(b-c)(c-d)(d-e))} = \frac{ae}{a-b} - \frac{eb}{a-b} = \frac{e(a-b)}{a-b} = e, \\
y_{12n-3} &= y_{12n-4} + \frac{y_{12n-4}y_{12n-7}}{y_{12n-8} - y_{12n-7}} = e + \frac{e(abce/(a-b)(b-c)(c-d))}{(abcde/(a-b)(b-c)(c-d)(d-e)) - (abce/(a-b)(b-c)(c-d))} \\
&= e + \frac{e(abce/(a-b)(b-c)(c-d))}{((abcde - abce + abce^2)/(a-b)(b-c)(c-d)(d-e))} = e + d - e = d, \\
y_{12n-2} &= y_{12n-3} + \frac{y_{12n-3}y_{12n-6}}{y_{12n-7} - y_{12n-6}} = d + \frac{d(abe/(a-b)(b-c))}{(abce/(a-b)(b-c)(c-d)) - (abe/(a-b)(b-c))} \\
&= d + \frac{d(abe/(a-b)(b-c))}{((abce - abce + abcd)/(a-b)(b-c)(c-d))} = d + \frac{d(abe/(a-b)(b-c))}{(abcd/(a-b)(b-c)(c-d))} = d + c - d = c, \\
y_{12n-1} &= y_{12n-2} + \frac{y_{12n-2}y_{12n-5}}{y_{12n-6} - y_{12n-5}} = c + \frac{c(ae/(a-b))}{(abe/(a-b)(b-c)) - (ae/(a-b))} = c + \frac{c(ae/(a-b))}{((abe - abe + ace)/(a-b)(b-c))} \\
&= c + b - c = b, \\
y_{12n} &= y_{12n-1} + \frac{y_{12n-1}y_{12n-4}}{y_{12n-5} - y_{12n-4}} = b + \frac{be}{(ae/(a-b)) - e} = b + \frac{be}{((ae - ae + eb)/(a-b))} = b + \frac{be(a-b)}{eb} = b + a - b \\
&= a, \\
y_{12n+1} &= y_{12n} + \frac{y_{12n}y_{12n-3}}{y_{12n-4} - y_{12n-3}} = a + \frac{ad}{e-d} = \frac{ae - ad + ad}{e-d} = \frac{ae}{e-d} = -\frac{ae}{d-e}, \\
y_{12n+2} &= y_{12n+1} + \frac{y_{12n+1}y_{12n-2}}{y_{12n-3} - y_{12n-2}} = -\frac{ae}{d-e} + \frac{(-ae/(d-e))c}{d-c} = -\frac{ae}{d-e} - \frac{ace}{(d-e)(d-c)} = -\frac{ae}{d-e} \left(1 + \frac{c}{d-c}\right) \\
&= -\frac{ae}{d-e} \left(\frac{d-c+c}{d-c}\right) = -\frac{aed}{(d-e)(d-c)} = \frac{aed}{(c-d)(d-e)}, \\
y_{12n+3} &= y_{12n+2} + \frac{y_{12n+2}y_{12n-1}}{y_{12n-2} - y_{12n-1}} = \frac{aed}{(c-d)(d-e)} + \frac{(aed/(c-d)(d-e))b}{c-b} = \frac{aed}{(c-d)(d-e)} + \frac{abed}{(c-d)(d-e)(c-b)} \\
&= \frac{aed}{(c-d)(d-e)} \left(1 + \frac{b}{c-b}\right) = \frac{aed}{(c-d)(d-e)} \left(\frac{c-b+b}{c-b}\right) = \frac{aced}{(c-d)(d-e)(c-b)} = -\frac{aced}{(b-c)(c-d)(d-e)}, \\
y_{12n+4} &= y_{12n+3} + \frac{y_{12n+3}y_{12n}}{y_{12n-1} - y_{12n}} = -\frac{aced}{(b-c)(c-d)(d-e)} + \frac{(-aced/(b-c)(c-d)(d-e))a}{b-a} \\
&= -\frac{aced}{(b-c)(c-d)(d-e)} - \frac{a^2ced}{(b-c)(c-d)(d-e)(b-a)} = -\frac{aced}{(b-c)(c-d)(d-e)} \left(1 + \frac{a}{b-a}\right) \\
&= -\frac{aced}{(b-c)(c-d)(d-e)} \left(\frac{b-a+a}{b-a}\right) = -\frac{abcd}{(b-c)(c-d)(d-e)(b-a)} = \frac{abcd}{(a-b)(b-c)(c-d)(d-e)},
\end{aligned}$$

$$\begin{aligned}
y_{12n+5} &= y_{12n+4} + \frac{y_{12n+4}y_{12n+1}}{y_{12n} - y_{12n+1}} = \frac{abcd}{(a-b)(b-c)(c-d)(d-e)} + \frac{(abcd/(a-b)(b-c)(c-d)(d-e))(-ae/(d-e))}{a - (-ae/(d-e))} \\
&= \frac{abcd}{(a-b)(b-c)(c-d)(d-e)} + \frac{(abcd/(a-b)(b-c)(c-d)(d-e))(-ae/(d-e))}{((ad - ae + ae)/(d-e))} \\
&= \frac{abcd}{(a-b)(b-c)(c-d)(d-e)} - \frac{abce^2}{(a-b)(b-c)(c-d)(d-e)} = \frac{abce}{(a-b)(b-c)(c-d)(d-e)}(d-e) \\
&= \frac{abce}{(a-b)(b-c)(c-d)}, \\
y_{12n+6} &= y_{12n+5} + \frac{y_{12n+5}y_{12n+2}}{y_{12n+1} - y_{12n+2}} = \frac{abce}{(a-b)(b-c)(c-d)} + \frac{(abce/(a-b)(b-c)(c-d))(ade/(c-d)(d-e))}{(-ae/(d-e)) - (ade/(c-d)(d-e))} \\
&= \frac{abce}{(a-b)(b-c)(c-d)} + \frac{(abce/(a-b)(b-c)(c-d))(ade/(c-d)(d-e))}{((-ace + ade - ade)/(c-d)(d-e))} \\
&= \frac{abce}{(a-b)(b-c)(c-d)} - \frac{abed}{(a-b)(b-c)(c-d)} = \frac{abe}{(a-b)(b-c)(c-d)}(c-d) = \frac{abe}{(a-b)(b-c)}, \\
y_{12n+7} &= y_{12n+6} + \frac{y_{12n+6}y_{12n+3}}{y_{12n+2} - y_{12n+3}} = \frac{abe}{(a-b)(b-c)} + \frac{(abe/(a-b)(b-c))(-acde/(b-c)(c-d)(d-e))}{(ade/(c-d)(d-e)) - (-acde/(b-c)(c-d)(d-e))} \\
&= \frac{abe}{(a-b)(b-c)} + \frac{(abe/(a-b)(b-c))(-acde/(b-c)(c-d)(d-e))}{((abde - acde + acde)/(b-c)(c-d)(d-e))} = \frac{abe}{(a-b)(b-c)} - \frac{ace}{(a-b)(b-c)} \\
&= \frac{ae}{(a-b)(b-c)}(b-c) = \frac{ae}{a-b},
\end{aligned}$$

(67)

This completes the proof. $\square$

Theorem 21. Every solution $\{y_n\}_{n=-4}^{\infty}$ of rational difference equation (64) is periodic with period 12.

Proof. The proof follows from the previous theorem and will be omitted. $\square$

Example 22. Figure 6 shows that the solution of (64) has a periodic solution with period 12 when the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$.

4.5. Fifth Case When $\beta = A = B = 1$ and $\alpha = -1$

Theorem 23. Suppose that $\{y_n\}_{n=-4}^{\infty}$ is solutions of rational difference equation

$$y_{n+1} = -y_n + \frac{y_n y_{n-3}}{y_{n-4} + y_{n-3}}. \quad (68)$$

Then every solution of (68) is periodic with period 12. Moreover $\{y_n\}_{n=-4}^{\infty}$ takes the form

$$\begin{aligned}
y_{12n-4} &= e, \\
y_{12n-3} &= d, \\
y_{12n-2} &= c,
\end{aligned}$$

$$y_{12n-1} = b,$$

$$y_{12n} = a,$$

$$y_{12n+1} = -\frac{ae}{d+e},$$

$$y_{12n+2} = \frac{ade}{(c+d)(d+e)},$$

$$y_{12n+3} = -\frac{acde}{(b+c)(c+d)(d+e)}$$

$$y_{12n+4} = \frac{abcde}{(a+b)(b+c)(c+d)(d+e)},$$

$$y_{12n+5} = \frac{abce}{(a+b)(b+c)(c+d)},$$

$$y_{12n+6} = \frac{abe}{(a+b)(b+c)},$$

$$y_{12n+7} = \frac{ae}{a+b},$$

(69)

where $y_{-4} = e$, $y_{-3} = d$, $y_{-2} = c$, $y_{-1} = b$, and $y_0 = a$ are arbitrary nonzero real numbers with $y_{-4} \neq -y_{-3}$, $y_{-3} \neq -y_{-2}$, $y_{-2} \neq -y_{-1}$, and $y_{-1} \neq -y_0$.

FIGURE 6: Plot the periodicity of the solution of (64).

FIGURE 7: Plot the periodicity of the solution of (68).

Proof. The proof is left to the reader. $\square$

Example 24. Figure 7 shows that the solution of (68) has a periodic solution with period 12 when the initial conditions $y_{-4} = 0.57$, $y_{-3} = 1.1$, $y_{-2} = 0.3$, $y_{-1} = 0.5$, and $y_0 = 0.1$.

4.6. Sixth Case When $\beta = A = 1$ and $\alpha = B = -1$

Theorem 25. Suppose that $\{y_n\}_{n=-4}^{\infty}$ is solutions of rational difference equation

$$y_{n+1} = -y_n + \frac{y_n y_{n-3}}{y_{n-4} - y_{n-3}}, \quad (70)$$

and then for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned} y_{4n-3} &= -d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} - bF_{2i}}{aF_{2i} - bF_{2i-1}} \right) \left(\frac{bF_{2i+1} - cF_{2i}}{bF_{2i} - cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+1} - dF_{2i}}{cF_{2i} - dF_{2i-1}} \right) \left(\frac{dF_{2i+3} - eF_{2i+2}}{dF_{2i+2} - eF_{2i+1}} \right), \\ y_{4n-2} &= c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} - bF_{2i}}{aF_{2i} - bF_{2i-1}} \right) \left(\frac{bF_{2i+1} - cF_{2i}}{bF_{2i} - cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} - dF_{2i+2}}{cF_{2i+2} - dF_{2i+1}} \right) \left(\frac{dF_{2i+3} - eF_{2i+2}}{dF_{2i+2} - eF_{2i+1}} \right), \\ y_{4n-1} &= -b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} - bF_{2i}}{aF_{2i} - bF_{2i-1}} \right) \left(\frac{bF_{2i+3} - cF_{2i+2}}{bF_{2i+2} - cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} - dF_{2i+2}}{cF_{2i+2} - dF_{2i+1}} \right) \left(\frac{dF_{2i+3} - eF_{2i+2}}{dF_{2i+2} - eF_{2i+1}} \right), \\ y_{4n} &= a \prod_{i=0}^{n-1} \left(\frac{aF_{2i+3} - bF_{2i+2}}{aF_{2i+2} - bF_{2i+1}} \right) \left(\frac{bF_{2i+3} - cF_{2i+2}}{bF_{2i+2} - cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} - dF_{2i+2}}{cF_{2i+2} - dF_{2i+1}} \right) \left(\frac{dF_{2i+3} - eF_{2i+2}}{dF_{2i+2} - eF_{2i+1}} \right), \end{aligned} \quad (71)$$

where $y_{-4} = e$, $y_{-3} = d$, $y_{-2} = c$, $y_{-1} = b$, $y_0 = a$, $y_{-4} \neq y_{-3} \neq y_{-2} \neq y_{-1} \neq y_0$, $\{F_i\}_{i=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$, $F_{-1} = 1$, $F_0 = 0$, and $\prod_{i=0}^{-1} A_i = 1$.

4.7. Seventh Case When $\beta = B = 1$ and $\alpha = A = -1$

Theorem 26. Suppose that $\{y_n\}_{n=-4}^{\infty}$ is solutions of rational difference equation

$$y_{n+1} = -y_n + \frac{y_n y_{n-3}}{y_{n-3} - y_{n-4}}, \quad (72)$$

and then for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned} y_{4n} &= \frac{a b c d e}{(aF_n - bF_{n+1})(bF_n - cF_{n+1})(cF_n - dF_{n+1})(dF_n - eF_{n+1})}, \\ y_{4n+1} &= \frac{a b c d e}{(aF_n - bF_{n+1})(bF_n - cF_{n+1})(cF_n - dF_{n+1})(dF_{n+1} - eF_{n+2})}, \\ y_{4n+2} &= \frac{a b c d e}{(aF_n - bF_{n+1})(bF_n - cF_{n+1})(cF_{n+1} - dF_{n+2})(dF_{n+1} - eF_{n+2})}, \end{aligned}$$

$$y_{4n+3} = \frac{a b c d e}{(aF_n - bF_{n+1})(bF_{n+1} - cF_{n+2})(cF_{n+1} - dF_{n+2})(dF_{n+1} - eF_{n+2})}, \quad (73)$$

where $y_{-4} = e, y_{-3} = d, y_{-2} = c, y_{-1} = b$, and $y_0 = a$ are arbitrary nonzero real numbers $y_{-4} \neq y_{-3} \neq y_{-2} \neq y_{-1} \neq y_0$ and $\{F_i\}_{i=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$, $F_0 = 0$.

4.8. Eighth Case When $A = B = 1$ and $\alpha = \beta = -1$

Theorem 27. Suppose that $\{y_n\}_{n=-4}^{\infty}$ is solutions of rational difference equation

$$y_{n+1} = -y_n - \frac{y_n y_{n-3}}{y_{n-4} + y_{n-3}}, \quad (74)$$

and then, for $n = 0, 1, 2, \dots$, we see that

$$\begin{aligned} y_{4n-3} &= -d \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+1} + dF_{2i}}{cF_{2i} + dF_{2i-1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-2} &= c \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+1} + cF_{2i}}{bF_{2i} + cF_{2i-1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n-1} &= -b \prod_{i=0}^{n-1} \left(\frac{aF_{2i+1} + bF_{2i}}{aF_{2i} + bF_{2i-1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \\ y_{4n} &= a \prod_{i=0}^{n-1} \left(\frac{aF_{2i+3} + bF_{2i+2}}{aF_{2i+2} + bF_{2i+1}} \right) \left(\frac{bF_{2i+3} + cF_{2i+2}}{bF_{2i+2} + cF_{2i+1}} \right) \\ &\quad \cdot \left(\frac{cF_{2i+3} + dF_{2i+2}}{cF_{2i+2} + dF_{2i+1}} \right) \left(\frac{dF_{2i+3} + eF_{2i+2}}{dF_{2i+2} + eF_{2i+1}} \right), \end{aligned} \quad (75)$$

where $y_{-4} = e, y_{-3} = d, y_{-2} = c, y_{-1} = b, y_0 = a, \{F_i\}_{i=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$, $F_{-1} = 1, F_0 = 0$, and $\prod_{i=0}^{-1} A_i = 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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