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Research Article

On Convergence of Infinite Matrix Products with Alternating Factors from Two Sets of Matrices

Victor Kozyakin

Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoi Karetny Lane 19, Moscow 127051, Russia

Correspondence should be addressed to Victor Kozyakin; kozyakin@gmail.com

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We consider the problem of convergence to zero of matrix products $A_nB_n\cdots A_1B_1$ with factors from two sets of matrices, $A_i \in \mathscr{A}$ and $B_i \in \mathscr{B}$, due to a suitable choice of matrices $\{B_i\}$. It is assumed that for any sequence of matrices $\{A_i\}$ there is a sequence of matrices $\{B_i\}$ such that the corresponding matrix products $A_nB_n\cdots A_1B_1$ converge to zero. We show that, in this case, the convergence of the matrix products under consideration is uniformly exponential; that is, $\|A_nB_n\cdots A_1B_1\| \le C\lambda^n$, where the constants $C > 0$ and $\lambda \in (0,1)$ do not depend on the sequence $\{A_i\}$ and the corresponding sequence $\{B_i\}$. Other problems of this kind are discussed and open questions are formulated.

1. Introduction

Denote by $\mathscr{M}(p,q)$ the space of matrices of dimension $p \times q$ with real elements and the topology of elementwise convergence. Let $\mathscr{A} \subset \mathscr{M}(N,M)$ and $\mathscr{B} \subset \mathscr{M}(M,N)$ be finite sets of matrices. We will be interested in the question of whether it is possible to ensure the convergence to zero of matrix products,

$$A_nB_n\cdots A_1B_1, \quad A_i \in \mathscr{A},\ B_i \in \mathscr{B}, \tag{1}$$

for all possible sequences of matrices $\{A_i\}$, due to a suitable choice of sequences of matrices $\{B_i\}$.

As an example of a problem in which such a question arises, let us consider one of the varieties of the stabilizability problem for discrete-time switching linear systems [1–5]. Consider a system whose dynamics is described by the equations

$$\begin{aligned} x(n) &= A_nu(n), \quad A_n \in \mathscr{A},\\ u(n) &= B_nx(n-1), \quad B_n \in \mathscr{B}, \end{aligned} \tag{2}$$

where the first of them describes the functioning of a plant, whose properties are uncontrollably affected by perturbations from class $\mathscr{A}$, while the second equation describes the behavior of a controller. Then, by choosing a suitable sequence of controls $\{B_n \in \mathscr{B}\}$, one can try to achieve the desired behavior of system (2), for example, the convergence to zero of its solutions:

$$x(n) = A_nB_n\cdots A_1B_1x(0). \tag{3}$$

As was noted, for example, in [6, 7], the question of the stabilizability of matrix products with alternating factors from two sets, due to a special choice of factors from one of these sets, can also be treated in the game-theoretic sense.

If, in considering the switching system, it is assumed that there are actually no control actions, that is, $B_n \equiv I$, then (2) take the form

$$x(n) = A_nx(n-1), \quad A_n \in \mathscr{A}. \tag{4}$$

In this case, the problem of the stabilizability of the corresponding switching system turns into the problem of its stability for all possible perturbations of the plant in class $\mathscr{A}$, that is, into the problem of convergence to zero of the solutions

$$x(n) = A_n\cdots A_1x(0) \tag{5}$$

of (4) for all possible sequences of matrices $\{A_i \in \mathscr{A}\}$. Convergence to zero of the matrix products $A_n\cdots A_1$, arising in this case, has been investigated by many authors (see, e.g., [2, 8–11], as well as the bibliography in [12]).

The presence of alternating factors in the products of matrices (1) substantially complicates the problem of convergence of the corresponding matrix products for all possible sequences of matrices $\{A_i \in \mathscr{A}\}$ due to a suitable choice of sequences of matrices $\{B_i \in \mathscr{B}\}$ in comparison with the problem of convergence of matrix products $A_n \cdots A_1$ for all possible sequences of matrices $\{A_i \in \mathscr{A}\}$. A discussion of the arising difficulties can be found, for example, in [13]. One of the applications of the results obtained in this paper for analyzing the new concept of the so-called minimax joint spectral radius is also described there.

2. Path-Dependent Stabilizability

Every product (1) is a matrix of dimension $N \times N$; that is, it is an element of the space $\mathscr{M}(N,N)$. As is known, the space $\mathscr{M}(N,N)$ with the topology of elementwise convergence is normable; therefore we assume that $\|\cdot\|$ is some norm in it. We note here that since all norms in the space $\mathscr{M}(N,N)$ are equivalent, the choice of a particular norm when considering the convergence of products (1) is inessential. Nevertheless, in what follows, it will be convenient for us to assume that the norm $\|\cdot\|$ in $\mathscr{M}(N,N)$ is submultiplicative; that is, for any two matrices X, Y, the inequality $\|XY\| \le \|X\| \cdot \|Y\|$ holds. In particular, a norm on $\mathscr{M}(N,N)$ is submultiplicative if it is generated by some vector norm on $\mathbb{R}^N$; that is, its value on matrix A is defined by the equality $\|A\| = \sup_{x\neq 0}(\|Ax\|/\|x\|)$, where $\|x\|$ and $\|Ax\|$ are the norms of the corresponding vectors in $\mathbb{R}^N$.

Definition 1. The matrix products (1) are said to be *path-dependent stabilizable* by choosing the factors $\{B_n\}$ if for any sequence of matrices $\{A_n \in \mathscr{A}\}$ there exists a sequence of matrices $\{B_n \in \mathscr{B}\}$ for which

$$\|A_n B_n \cdots A_1 B_1\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{6}$$

As an example, consider the case where sets $\mathscr{A}$ and $\mathscr{B}$ consist of square matrices of dimension $N \times N$, and $\mathscr{B} = \{I\}$, where I is the identical matrix. In this case, Definition 1 of the path-dependent stabilizability of the matrix products (1) reduces to the following condition:

$$\|A_n \cdots A_1\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{7}$$

for each sequence $\{A_n \in \mathscr{A}\}$. As is known, in this case, convergence (7) is uniformly exponential. Namely, the following statement, which was repeatedly "discovered" by many authors, is true (see, e.g., [2, 8–11]).

Theorem A (on exponential convergence). *Let the set of matrices $\mathscr{A}$ be such that for each sequence $\{A_n \in \mathscr{A}\}$ convergence (7) holds. Then there exist constants $C > 0$ and $\lambda \in (0,1)$ such that*

$$\|A_n \cdots A_1\| \le C\lambda^n, \quad n = 1, 2, \ldots, \tag{8}$$

for each sequence $\{A_n \in \mathscr{A}\}$.

Our goal is to prove that an analogue of Theorem A (on exponential convergence) is valid for the path-dependent stabilizable matrix products (1).

Theorem 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be the sets of matrices for which the matrix products (1) are path-dependent stabilizable. Then there exist constants $C > 0$ and $\lambda \in (0,1)$ such that for any sequence of matrices $\{A_n \in \mathscr{A}\}$ there is a sequence of matrices $\{B_n \in \mathscr{B}\}$ for which*

$$\|A_n B_n \cdots A_1 B_1\| \le C\lambda^n, \quad n = 1, 2, \ldots. \tag{9}$$

To prove the theorem, we need the following auxiliary assertion.

Lemma 3. *Let the conditions of Theorem 2 be satisfied. Then there exist constants $k_* > 0$ and $\mu \in (0,1)$ such that for any sequence of matrices $\{A_n \in \mathscr{A}\}$ there is a positive integer $k \le k_*$ and a set of matrices $B_1, \ldots, B_k \in \mathscr{B}$ for which $\|A_k B_k \cdots A_1 B_1\| \le \mu < 1$.*

Proof. By Definition 1 of the path-dependent stabilizability of the matrix products (1) for each matrix sequence $\{A_n \in \mathscr{A}\}$ there exists a natural k such that

$$\|A_k B_k \cdots A_1 B_1\| < 1, \tag{10}$$

for some sequence of matrices $\{B_n \in \mathscr{B}\}$.

Given a sequence $\{A_n\}$, let us denote by $k(\{A_n\})$ the smallest k under which inequality (10) holds. To prove the lemma, it suffices to show that the quantities $k(\{A_n\})$ are uniformly bounded; that is, there is a k_* such that

$$k(\{A_n\}) \le k_*, \quad \forall \{A_n \in \mathscr{A}\}. \tag{11}$$

Assuming that inequality (11) is not true, for each positive integer k, we can find a sequence $\{A_n^{(k)} \in \mathscr{A}\}$ such that $k(\{A_n^{(k)}\}) \ge k$. In this case, by the definition of the number $k(\{A_n\})$,

$$\|A_m B_m \cdots A_1 B_1\| \ge 1, \quad \forall B_1, \ldots, B_m \in \mathscr{B}, \tag{12}$$

for each positive integer $m \le k - 1 \le k(\{A_n^{(k)}\}) - 1$.

Let us denote by $\mathbf{A}_k$ the set of all sequences $\{A_n \in \mathscr{A}\}$, for each of which inequalities (12) hold. Then $\{A_n^{(k)}\} \in \mathbf{A}_k$ and, therefore, $\mathbf{A}_k \neq \varnothing$. Moreover,

$$\mathbf{A}_1 \supseteq \mathbf{A}_2 \supseteq \cdots, \tag{13}$$

and each set $\mathbf{A}_k$ is closed since inequalities (12) hold for all its elements; sequences $\{A_n\} \in \mathbf{A}_k$, for each positive integer $m \le k - 1$.

We now note that each of the sets $\mathbf{A}_k$ is a subset of the topological space $\mathscr{A}^\infty$ of all sequences $\{A_n \in \mathscr{A}\}$ with the topology of infinite direct product of the finite set of matrices $\mathscr{A}$. By the Tikhonov theorem in this case $\mathscr{A}^\infty$ is a compact. Then, each of the sets $\mathbf{A}_k$ is also a compact. In this case, it follows from (13) that $\bigcap_{k=1}^\infty \mathbf{A}_k \neq \varnothing$ and, therefore, there is a sequence $\{\overline{A}_n \in \mathscr{A}\}$ such that

$$\{\overline{A}_n\} \in \bigcap_{k=1}^{\infty} \mathbf{A}_k. \tag{14}$$

By the definition of the sets $\mathbf{A}_k$, for the sequence $\{\overline{A}_n \in \mathscr{A}\}$, the inequalities

$$\left\|\overline{A}_m B_m \cdots \overline{A}_1 B_1\right\| \geq 1 \tag{15}$$

hold for each $m \geq 1$ and any $B_1, \ldots, B_m \in \mathscr{B}$ which contradicts the assumption of the path-dependent stabilizability of the matrix products (1). This contradiction completes the proof of the existence of a number k_* for which inequalities (11) are valid.

Thus, we have proven the existence of a number k_* such that, for each sequence $\{A_n \in \mathscr{A}\}$ and some corresponding sequence $\{B_n \in \mathscr{B}\}$, strict inequalities (10) are satisfied with $k = k(\{A_n\}) \leq k_*$. Moreover, since the number of all products $A_k B_k \cdots A_1 B_1$ participating in inequalities (10) is finite, then the corresponding inequalities (10) can be strengthened: there is a $\mu \in (0, 1)$ such that for any sequence of matrices $\{A_n \in \mathscr{A}\}$ there exist a natural $k \leq k_*$ and a set of matrices $B_1, \ldots, B_k \in \mathscr{B}$ for which $\|A_k B_k \cdots A_1 B_1\| \leq \mu < 1$. □

We now proceed directly to the proof of Theorem 2.

Proof of Theorem 2. Given an arbitrary sequence $\{A_n \in \mathscr{A}\}$, by Lemma 3, there exist a number $k_1 \leq k_*$ and a set of matrices $B_1, \ldots, B_{k_1}$ such that

$$\left\|A_{k_1} B_{k_1} \cdots A_1 B_1\right\| \leq \mu < 1. \tag{16}$$

Next, consider the sequence of matrices $\{A_n \in \mathscr{A},\ n \geq k_1 + 1\}$ (the "tail" of the sequence $\{A_n \in \mathscr{A}\}$ starting with the index $k_1 + 1$). Again, by virtue of Lemma 3, there exist a $k_2 \leq k_1 + k_*$ and a set of matrices $B_{k_1+1}, \ldots, B_{k_2}$ such that

$$\left\|A_{k_2} B_{k_2} \cdots A_{k_1+1} B_{k_1+1}\right\| \leq \mu < 1. \tag{17}$$

We continue in the same way constructing for each $m = 3, 4, \ldots$ numbers

$$k_m \leq k_{m-1} + k_* \tag{18}$$

and sets of matrices $B_{k_{m-1}+1}, \ldots, B_{k_m}$ for which

$$\left\|A_{k_m} B_{k_m} \cdots A_{k_{m-1}+1} B_{k_{m-1}+1}\right\| \leq \mu < 1. \tag{19}$$

Let us show that, for the obtained sequence of matrices $\{B_n\}$ for some $C > 0$ and $\lambda \in (0, 1)$, which do not depend on the sequences $\{A_n\}$ and $\{B_n\}$, inequalities (9) are valid. Fix a positive integer n and specify for it a number $p = p(n)$ such that

$$n - k_* < k_p \leq n. \tag{20}$$

Such p exists, since the sequence $\{k_m\}$ strictly increases by construction. We now represent the product $A_n B_n \cdots A_1 B_1$ in the form

$$A_n B_n \cdots A_1 B_1 = D_* D_p \cdots D_1, \tag{21}$$

where

$$\begin{aligned} D_* &= A_n B_n \cdots A_{k_p+1} B_{k_p+1}, \\ D_i &= A_{k_i} B_{k_i} \cdots A_{k_{i-1}+1} B_{k_{i-1}+1}, \quad i = 1, 2, \ldots, p. \end{aligned} \tag{22}$$

Then

$$\left\|D_*\right\| \leq \varkappa^{n-k_p} \leq \varkappa^{k_*}, \quad \text{where } \varkappa = \max_{A \in \mathscr{A}, B \in \mathscr{B}} \{1, \|AB\|\} \tag{23}$$

(since the sets $\mathscr{A}$ and $\mathscr{B}$ are finite, $\varkappa < \infty$). Further, by the definition of the matrices D_i and inequalities (19),

$$\left\|D_i\right\| \leq \mu < 1 \quad \text{for } i = 1, 2, \ldots, p. \tag{24}$$

Taking into account the fact that, by virtue of (18), for each m, the estimate $k_m \leq k_* m$ is fulfilled, from here and from (20) we obtain for the number p a lower estimate: $p \geq n/k_* - 1$. And then from the estimates established earlier for $\|D_*\|$, $\|D_1\|, \ldots, \|D_m\|$, we deduce that

$$\begin{aligned} \left\|A_n B_n \cdots A_1 B_1\right\| &\leq \left\|D_*\right\| \cdot \left\|D_p\right\| \cdots \left\|D_1\right\| \leq \varkappa^{k_*} \mu^p \\ &\leq \varkappa^{k_*} \mu^{n/k_* - 1} \leq \frac{\varkappa^{k_*}}{\mu} \left(\mu^{1/k_*}\right)^n. \end{aligned} \tag{25}$$

Hence, putting $C = \varkappa^{k_*}/\mu$ and $\lambda = \mu^{1/k_*}$, we obtain inequalities (9). □

3. Path-Independent Stabilizability

Let us now consider another variant of the stabilizability of matrix products (1) due to a suitable choice of matrices $\{B_i\}$.

Definition 4. The matrix products (1) are said to be *path-independent periodically stabilizable* by choosing the factors $\{B_n\}$ if there exists a periodic sequence of matrices $\{\overline{B}_n \in \mathscr{B}\}$ such that

$$\left\|A_n \overline{B}_n \cdots A_1 \overline{B}_1\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{26}$$

for any sequence of matrices $\{A_n \in \mathscr{A}\}$.

It is clear that path-independent periodically stabilized products (1) are path-dependent stabilized.

Theorem 5. *Let $\mathscr{A}$ and $\mathscr{B}$ be the sets of matrices for which the matrix products (1) are path-independent periodically stabilizable by a sequence of matrices $\{\overline{B}_n \in \mathscr{B}\}$. Then there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that*

$$\left\|A_n \overline{B}_n \cdots A_1 \overline{B}_1\right\| \leq C\lambda^n, \quad n = 1, 2, \ldots, \tag{27}$$

for any sequence of matrices $\{A_n \in \mathscr{A}\}$.

Proof. Denote by p the period of the sequence $\{\overline{B}_n\}$. Consider the set of $(N \times N)$-matrices:

$$\mathscr{D} = \left\{D = A_p \overline{B}_p \cdots A_1 \overline{B}_1 : A_1 \cdots A_p \in \mathscr{A}\right\}. \tag{28}$$

Since the set of matrices $\mathscr{A}$ is finite, set $\mathscr{D}$ is also finite. Moreover, by Definition 4 of path-independent periodic stabilization,

$$\left\|A_{np} \overline{B}_{np} \cdots A_1 \overline{B}_1\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{29}$$

for each sequence $\{A_n \in \mathscr{A}\}$. Hence, for each sequence $\{D_n \in \mathscr{D}\}$, there is also

$$\|D_n \cdots D_1\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{30}$$

In this case, by Theorem A (on exponential convergence), there are $k_* > 0$ and $\mu \in (0,1)$ such that

$$\left\|D_{k_*} \cdots D_1\right\| \leq \mu < 1, \quad \forall D_1 \cdots D_{k_*} \in \mathscr{D}, \tag{31}$$

or, equivalently,

$$\left\|A_{k_* p}\overline{B}_{k_* p} \cdots A_1\overline{B}_1\right\| \leq \mu < 1, \quad \forall A_1 \cdots A_{k_* p} \in \mathscr{A}. \tag{32}$$

Further, repeating the proof of the corresponding part of Theorem 2 word for word, we derive from inequalities (32) the existence of constants $C > 0$ and $\lambda \in (0,1)$ such that for any sequence of matrices $\{A_n \in \mathscr{A}\}$ inequalities (27) hold. □

4. Remarks and Open Questions

First of all, we would like to make the following remarks.

Remark 6. In the proof of Lemma 3, in fact, we used not the condition of path-dependent stabilizability of the matrix products (1) but the weaker condition that for each matrix sequence $\{A_n \in \mathscr{A}\}$ there exist a natural $k = k(\{A_n\})$ and a collection of matrices $B_1, \ldots, B_k \in \mathscr{B}$ for which equality (10) holds. Correspondingly, the statement of Theorem 2 is valid under weaker assumptions.

Theorem 7. *Let the sets of matrices $\mathscr{A}$ and $\mathscr{B}$ be such that for each matrix sequence $\{A_n \in \mathscr{A}\}$ there are a natural k and a collection of matrices $B_1, \ldots, B_k \in \mathscr{B}$ for which*

$$\left\|A_k B_k \cdots A_1 B_1\right\| < 1. \tag{33}$$

Then there exist constants $C > 0$ and $\lambda \in (0,1)$ such that for any sequence of matrices $\{A_n \in \mathscr{A}\}$ there is a sequence of matrices $\{B_n \in \mathscr{B}\}$ for which

$$\left\|A_n B_n \cdots A_1 B_1\right\| \leq C\lambda^n, \quad n = 1, 2, \ldots. \tag{34}$$

Remark 8. All the above statements remain valid for the sets of matrices $\mathscr{A}$ and $\mathscr{B}$ with complex elements.

Remark 9. Throughout the paper, in order to avoid inessential technicalities in proofs, it was assumed that the sets of matrices $\mathscr{A}$ and $\mathscr{B}$ are finite. In fact, all the above statements remain valid in the case when the sets of matrices $\mathscr{A}$ and $\mathscr{B}$ are compacts, not necessarily finite, that is, are closed and precompact.

Comparing the notions of path-dependent stabilizability and path-independent periodic stabilizability, one can note that in the second of them the requirement of periodicity of the sequence $\{\overline{B}_n\}$ stabilizing the matrix products (1) appeared. Therefore, the following less restrictive concept of path-independent stabilizability seems rather natural.

Definition 10. The matrix products (1) are said to be *path-independent stabilizable* by choosing the factors $\{B_n\}$ if there is a sequence of matrices $\{\overline{B}_n \in \mathscr{B}\}$ such that convergence (26) holds for any sequence of matrices $\{A_n \in \mathscr{A}\}$.

It is not difficult to construct an example of the sets of square matrices in which the matrix products $A_n\overline{B}_n \cdots A_1\overline{B}_1$ converge slowly enough, slower than any geometric progression. For this, it is enough to put $\mathscr{A} = \{I\}$ and $\mathscr{B} = \{I, \lambda I\}$, where $\lambda \in (0,1)$, and define sequence $\{\overline{B}\}$ so that the matrix λI appears in it "fairly rare," at positions with numbers k^2, $k = 1, 2, \ldots$.

Question 11. Let the matrix products (1) be path-independent stabilizable by choosing a certain sequence of matrices $\{\overline{B}_n \in \mathscr{B}\}$. Is it possible in this case to specify a sequence of matrices $\{\widetilde{B}_n \in \mathscr{B}\}$ (possibly different from $\{\overline{B}_n \in \mathscr{B}\}$) and constants $C > 0$ and $\lambda \in (0,1)$ such that, for any sequence of matrices $\{A_n \in \mathscr{A}\}$ for all $n = 1, 2, \ldots$, the inequalities $\|A_n\widetilde{B}_n \cdots A_1\widetilde{B}_1\| \leq C\lambda^n$ will be valid?

Let us consider one more issue, which is adjacent to the topic under discussion. In the theory of matrix products, the following assertion is known [2, 8–11]: let $\mathscr{A}$ be a finite set such that for each sequence of matrices $\{A_n \in \mathscr{A}\}$ the sequence of norms $\{\|A_n \cdots A_1\|,\ n = 1, 2, \ldots\}$ is bounded. Then all such sequences of norms for the matrices are uniformly bounded; that is, there exists a constant $C > 0$ such that

$$\left\|A_n \cdots A_1\right\| \leq C, \quad n = 1, 2, \ldots, \tag{35}$$

for each sequence of matrices $\{A_n \in \mathscr{A}\}$.

Question 12. Let finite sets of matrices $\mathscr{A}$ and $\mathscr{B}$ be such that for each sequence of matrices $\{A_n \in \mathscr{A}\}$ there is a sequence of matrices $\{B_n \in \mathscr{A}\}$ for which the sequence of norms $\{\|A_nB_n \cdots A_1B_1\|,\ n = 1, 2, \ldots\}$ is bounded. Does there exist in this case a constant $C > 0$ such that for every matrix sequence $\{A_n \in \mathscr{A}\}$ there is a sequence of matrices $\{B_n \in \mathscr{A}\}$, for which the sequence of norms $\{\|A_nB_n \cdots A_1B_1\|,\ n = 1, 2, \ldots\}$ is uniformly bounded, that is, for all $n = 1, 2, \ldots$, the inequalities $\|A_nB_n \cdots A_1B_1\| \leq C$ hold?

Conflicts of Interest

The author declares that he has no conflicts of interest.

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