The Uniqueness Theorem of the Solution for a Class of Differential Systems with Coupled Integral Boundary Conditions

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We discuss the uniqueness of the solution to a class of differential systems with coupled integral boundary conditions under a Lipschitz condition. Our main method is the linear operator theory and the solvability for a system of inequalities. Finally, an example is given to demonstrate the validity of our main results.

1. Introduction

In this paper, we study the following differential system with coupled integral boundary conditions:

\[-u''(t) = f(t, u(t), v(t)), \quad t \in (0,1),\]
\[-v''(t) = g(t, u(t), v(t)), \quad t \in (0,1),\]

\[u(0) = v(0) = 0,\]

\[u(1) = \alpha[u], \quad v(1) = \beta[v],\]

where \(\alpha[u], \beta[v]\) are bounded linear functionals on \(C[0,1]\) given by

\[\alpha[u] = \int_0^1 u(t) \, dA(t),\]

\[\beta[v] = \int_0^1 v(t) \, dB(t),\]

involving Riemann-Stieltjes integrals defined via positive Stieltjes measures of \(A, B\).

Differential systems with coupled boundary conditions have some applications in various fields of sciences and engineering, for example, the heat equation [1], reaction-diffusion phenomena [2], and interaction problems [3]. The existence of solutions for differential system with coupled boundary conditions has received a growing attention in the literature; for details, see [4–21]. For example, Asif and Khan in [4] obtained the existence of positive solution for singular sublinear system with coupled four-point boundary value conditions by using the Guo-Krasnosel’skii fixed point theorem. In [5], Cui and Sun discuss the existence of positive solutions of singular superlinear coupled integral boundary value problems by constructing a special cone and using fixed point index theory. In [7], Cui and Zou proved the existence of extremal solutions of coupled integral boundary value problems by monotone iterative method. In [10], Infante, Minhós, and Pietramala presented a general theory for existence of positive solutions of singular superlinear coupled integral boundary value problems by constructing a special cone and using fixed point index theory. In [7], Cui and Zou proved the existence of extremal solutions of coupled integral boundary value problems by monotone iterative method. In [10], Infante, Minhós, and Pietramala presented a general theory for existence of positive solutions for coupled systems by use of fixed point index theory.

The question of existence and uniqueness of solution of differential equations and differential systems is an age-old problem and it has a great importance, as much in theory as in applications. This problem has been investigated by use of a variety of nonlinear analyses such as fixed point theorem for mixed monotone operator [7, 15, 22–25], maximal principle [6], Banach's contraction mapping principle [26–29], and the linear operator theory [27, 30, 31].
For example, the authors [31] introduced a Banach space using the positive eigenfunction of linear operator related to differential system (1). They established the uniqueness results for differential system (1) under a Lipschitz condition. It should be noted that the Lipschitz constant is related to the spectral radius corresponding to the related linear operators. The obtained results are optimal from the viewpoint of theory. However, it is very difficult to determine the spectral radius for differential system (1) with general functions $A(t), B(t)$.

Motivated by the above works, we investigate the uniqueness of solutions for differential system (1) by using a system of inequalities and the linear operator theory. The main features of this paper are as follows: (1) The main results are mostly implemented to the uniqueness result for coupled boundary value problems. (2) An easy criterion to determine the uniqueness result is obtained by using a system of inequalities. (3) An example shows that the main result provides the same results with weaker conditions.

Throughout the paper, we assume that the following condition holds:

\[(H_1)\quad \alpha[t] = \int_0^t t dA(t) > 0, \quad \beta[t] = \int_0^t t dB(t) > 0, \quad \kappa = 1 - \alpha[t] \beta[1] \geq 1, \quad \frac{\alpha[t]}{\kappa} \beta[1] + 1, \quad \frac{\beta[t]}{\kappa} \alpha[1] \geq 1, \quad \frac{\alpha[t]}{\kappa} \beta[1] \geq 1, \quad \frac{\beta[t]}{\kappa} \alpha[1] \geq 1.\]

\[(H_2)\quad f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \text{ are continuous.}\]

2. Preliminaries

Let $C[0, 1]$ be the Banach space with the maximal norm given by $\|x\| = \max_{t \in [0,1]} |x(t)|$. Let $E = C[0, 1] \times C[0, 1], \| (x, y) \|_E = \max\{\|x\|, \|y\|\}$. Then $(E, \| (\cdot, \cdot) \|_E)$ is a Banach space.

**Lemma 1 (see [5]).** Let $u, v \in C[0, 1]$, then the system of BVPs

\[
\begin{align*}
-u''(t) &= x(t), \\
-v''(t) &= y(t),
\end{align*}
\]

\[t \in [0, 1], \quad u(0) = v(0) = 0, \quad u(1) = \alpha[v], \quad v(1) = \beta[u] \tag{3}
\]

has integral representation

\[
\begin{align*}
u(t) &= \int_0^1 G_1(t, s) x(s) \, ds + \int_0^1 H_1(t, s) y(s) \, ds, \\
v(t) &= \int_0^1 G_2(t, s) x(s) \, ds + \int_0^1 H_2(t, s) y(s) \, ds.
\end{align*}
\tag{4}
\]

where

\[
\begin{align*}
G_1(t, s) &= \frac{\alpha[t]}{\kappa} \int_0^1 k(s, \tau) \, dB(\tau) + k(t, s), \\
H_1(t, s) &= \frac{t}{\kappa} \int_0^1 k(s, \tau) \, dA(\tau),
\end{align*}
\]

\[
\begin{align*}
G_2(t, s) &= \frac{\beta[t]}{\kappa} \int_0^1 k(s, \tau) \, dA(\tau) + k(t, s), \\
H_2(t, s) &= \frac{t}{\kappa} \int_0^1 k(s, \tau) \, dB(\tau),
\end{align*}
\]

\[
k(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\
s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}
\tag{5}
\]

**Lemma 2 (see [5]).** The functions $k(t, s), G_i(t, s), H_i(t, s)$ ($i = 1, 2$) satisfy the following properties:

\[
\begin{align*}
G_i(t, s) &\leq \rho t, \\
H_i(t, s) &\leq \rho t,
\end{align*}
\tag{6}
\]

\[
0 < k(t, s) \leq s(1-s), \quad \forall t, s \in (0, 1),
\tag{7}
\]

where

\[
\rho = \max\left\{ \frac{\alpha[t]}{\kappa} \beta[1] + 1, \frac{\beta[t]}{\kappa} \alpha[1] \right\} + 1, \frac{1}{\kappa} \beta[1], \frac{1}{\kappa} \alpha[1].
\tag{8}
\]

With the help of Lemma 1, BVP (1) can be viewed as a fixed point in $E$ for the completely continuous operator

\[
S(u, v) = (S_1(u, v), S_2(u, v)), \quad (u, v) \in E, \tag{9}
\]

where $S_1, S_2 : E \to C[0, 1]$ are defined by

\[
\begin{align*}
S_1(u, v)(t) &= \int_0^1 G_1(t, s) f(s, u(s), v(s)) \, ds + \int_0^1 H_1(t, s) g(s, u(s), v(s)) \, ds, \\
S_2(u, v)(t) &= \int_0^1 G_2(t, s) f(s, u(s), v(s)) \, ds + \int_0^1 H_2(t, s) g(s, u(s), v(s)) \, ds.
\end{align*}
\tag{10}
\]

In order to prove our main result, the following criterion for solving system of inequalities is needed.

**Lemma 3.** Let $a, b, c, d \in [0, +\infty)$ with $a < 1, d < 1$. Then the inequality system

\[
a + b \mu \leq \lambda, \\
c + d \mu \leq \lambda \mu
\]

has a solution $(\lambda, \mu)$ with $\lambda \in (0, 1), \mu > 0$ if and only if $a, b, c, d$ satisfy

\[
(1 - d)(1 - a) > bc. \tag{11}
\]
Proof.

Necessity. The proof is obviously true for the case: $bc = 0$. So we consider the remaining case $bc \neq 0$. From the first inequality in (11), we get

$$\mu \leq \frac{\lambda - a}{b}. \quad (13)$$

Substituting it into the second inequality in (11), we have

$$c \leq (\lambda - d) \mu \leq (\lambda - d) \frac{\lambda - a}{b}. \quad (14)$$

Thus,

$$(1-d)(1-a) > (\lambda - d)(\lambda - a) \geq bc. \quad (15)$$

Sufficiency. For the case $bc = 0$, we can take $\lambda = \max\{(d + 1)/2, (d + 1)/2\}$. So we consider the last case $bc \neq 0$. Let

$$\varphi(x) = (x - d)(x - a) - bc, \quad x \in \mathbb{R}. \quad (16)$$

From the derivative of $\varphi(x)$, we conclude that $\varphi(x)$ is increasing on $[(a + d)/2, 1]$. This together with the locally sign-preserving property of $\varphi(x)$ implies that there exists $\lambda \in [(a + d)/2, 1)$ such that

$$(\lambda - d)(\lambda - a) \geq bc \quad (17)$$

The above inequality can be rewritten as

$$\frac{c}{\lambda - d} \leq \frac{\lambda - a}{b}. \quad (18)$$

Hence (11) holds for $\mu \in [c/(\lambda - d), (\lambda - a)/b]$. □

3. Main Result

For notational convenience, let

$$a_{11} = \frac{\alpha [\tau] \beta [\phi]}{\kappa} + \frac{1}{6},$$

$$a_{12} = \frac{\alpha [\phi]}{\kappa},$$

$$a_{21} = \frac{\alpha [\phi] \beta [\tau]}{\kappa} + \frac{1}{6},$$

$$a_{22} = \frac{\beta [\phi]}{\kappa},$$

where

$$\phi(t) = \frac{t(1-t)(1+t)}{6}. \quad (20)$$

Take $\varphi(t) = t$. By (7), we get

$$\int_0^1 G_1(t, s) \varphi(s) ds = \frac{\alpha [\tau] t}{\kappa} \int_0^1 \int_0^1 k(s, \tau) dB(\tau) s ds + \int_0^1 k(t, s) ds$$

$$= \frac{\alpha [\tau] t}{\kappa} \int_0^1 \int_0^1 k(s, \tau) s ds dB(\tau) + \frac{t(1-t)(1+t)}{6} \quad (21)$$

$$= \frac{\alpha [\tau] t}{\kappa} \int_0^1 \tau (1-\tau)(1-\tau) dB(\tau)$$

$$+ \frac{t(1-t)(1+t)}{6} \leq \left( \frac{\alpha [\tau] \beta [\phi]}{\kappa} + \frac{1}{6} \right) \cdot t = a_{11} t,$$

$$(22)$$

$$\int_0^1 H_1(t, s) \varphi(s) ds = \frac{\alpha [\phi] t}{\kappa} \int_0^1 \int_0^1 k(s, \tau) dA(\tau) s ds$$

$$= \frac{\alpha [\phi] t}{\kappa} \int_0^1 \int_0^1 k(s, \tau) s ds dA(\tau)$$

$$+ k(t, s) ds$$

$$= \frac{\alpha [\phi] t}{\kappa} \int_0^1 \tau (1-\tau)(1-\tau) dA(\tau)$$

$$+ \frac{t(1-t)(1+t)}{6} \leq \left( \frac{\alpha [\phi] \beta [\tau]}{\kappa} + \frac{1}{6} \right) \cdot t = a_{12} t,$$

$$\int_0^1 G_2(t, s) \varphi(s) ds$$

$$= \frac{\beta [\phi] \alpha [\phi] t}{\kappa} \int_0^1 \int_0^1 k(s, \tau) dA(\tau) s ds$$

$$+ k(t, s) ds$$

$$= \frac{\beta [\phi] t}{\kappa} \int_0^1 \int_0^1 k(s, \tau) s ds dA(\tau)$$

$$+ \frac{t(1-t)(1+t)}{6} \leq \left( \frac{\alpha [\phi] \beta [\tau]}{\kappa} + \frac{1}{6} \right) \cdot t = a_{21} t,$$

$$(23)$$

$$\int_0^1 H_2(t, s) \varphi(s) ds = \frac{\alpha [\phi] \beta [\tau]}{\kappa} \int_0^1 \int_0^1 k(s, \tau) dB(\tau) s ds$$

$$= \frac{\alpha [\phi] \beta [\tau]}{\kappa} \int_0^1 \int_0^1 k(s, \tau) s ds dB(\tau)$$

$$+ \frac{t(1-t)(1+t)}{6} \leq \left( \frac{\alpha [\phi] \beta [\tau]}{\kappa} + \frac{1}{6} \right) \cdot t = a_{22} t.$$

By use of (21), (22), (23), and (24), we present the main result of this paper.

Theorem 4. Suppose that there exist four nonnegative constants $a_1, b_1, c_1, d_1$ such that the following conditions hold:

$$|f(t, u_1, v_1) - f(t, u_2, v_2)|$$

$$\leq a_1 |u_1 - u_2| + b_1 |v_1 - v_2|, \quad t \in [0, 1], \quad u_1, u_2, v_1, v_2 \in \mathbb{R},$$

$$= a_2 t,$$

$$= a_3 t,$$

$$= a_4 t.$$
\[ |g(t, u_1, v_1) - g(t, u_2, v_2)| \leq c_1 |u_1 - u_2| + d_1 |v_1 - v_2|, \]
\[ t \in [0, 1], \quad u_1, u_2, v_1, v_2 \in \mathbb{R}, \]
\[ (1 - a_{11}a_1 - a_{12}c_1)(1 - a_{21}d_1 - a_{22}b_1) \]
\[ > (a_1 b_1 + a_{12}d_1)(a_{21}c_1 + a_{22}a_1), \]  
\begin{align*}
& a_{11}a_1 + a_{12}c_1 < 1, \\
& a_{21}d_1 + a_{22}b_1 < 1. \tag{25}\end{align*}

Then differential system (1) has a unique solution in \( E \).

**Proof.** We divide the proof into several main steps to show that the operator \( S \) has a unique point in \( E \) under the conditions of Theorem 4.

**Step 1.** It follows from (25), (26), and Lemma 3 that there exist \( \lambda \in (0, 1), \mu > 0 \) such that
\begin{align*}
(a_{11}a_1 + a_{12}c_1) + (a_{11}b_1 + a_{12}d_1) \mu & \leq \lambda, \\
(a_{21}c_1 + a_{22}a_1) + (a_{21}d_1 + a_{22}b_1) \mu & \leq \lambda \mu. \tag{27}
\end{align*}

Let us introduce a linear operator \( T \) on \( E \) as
\[ T(u, v) = (T_1(u, v), T_2(u, v)), \tag{28} \]
where \( T_1, T_2 : E \to C[0, 1] \) is given by
\begin{align*}
T_1(u, v)(t) & = \int_0^1 G_1(t, s)(a_1 u(s) + b_1 v(s)) \, ds \\
& \quad + \int_0^1 H_1(t, s)(c_1 u(s) + d_1 v(s)) \, ds, \tag{29}
\end{align*}
\begin{align*}
T_2(u, v)(t) & = \int_0^1 G_2(t, s)(c_1 u(s) + d_1 v(s)) \, ds \\
& \quad + \int_0^1 H_2(t, s)(a_1 u(s) + b_1 v(s)) \, ds.
\end{align*}

Take \( \psi(t) = \mu t \). Now, (21)-(24) and (27) show that
\begin{align*}
T_1(\varphi, \psi)(t) & = \int_0^1 G_1(t, s)(a_1 \varphi(s) + b_1 \psi(s)) \, ds \\
& \quad + \int_0^1 H_1(t, s)(c_1 \varphi(s) + d_1 \psi(s)) \, ds \\
& \leq (a_{11}a_1 + a_{12}c_1) \varphi(t) \\
& \quad + (a_1 b_1 + a_{12}d_1) \psi(t) \leq \lambda \varphi(t), \tag{30}
\end{align*}
\begin{align*}
T_2(\varphi, \psi)(t) & = \int_0^1 G_2(t, s)(c_1 \varphi(s) + d_1 \psi(s)) \, ds \\
& \quad + \int_0^1 H_2(t, s)(a_1 \varphi(s) + b_1 \psi(s)) \, ds \\
& \leq (a_{21}c_1 + a_{22}a_1) \varphi(t) \\
& \quad + (a_{21}d_1 + a_{22}b_1) \psi(t) \leq \lambda \psi(t),
\end{align*}
\( \mu \),
\[ \leq (c_1 a_{21} + a_{22}c_1) \varphi(t) + (a_{21}d_1 + a_{22}b_1) \psi(t) \leq \lambda \psi(t), \]
i.e.,
\[ T(\varphi, \psi)(t) \leq \lambda (\varphi(t), \psi(t)). \tag{32} \]

Then for \( p \in \mathbb{N} \), by induction, we obtain
\[ T^p(\varphi, \psi)(t) \leq \lambda^p (\varphi(t), \psi(t)). \tag{33} \]

**Step 2.** For all \((u, v) \in E \) with \( u(t) \geq 0 \) and \( v(t) \geq 0 \), there exists \( M = M(u, v) \in (0, +\infty) \) such that
\[ T(u, v)(t) \leq M \cdot (\varphi(t), \psi(t)), \quad t \in [0, 1]. \tag{34} \]

Indeed, by Lemma 2, we have
\[ T_1(u, v)(t) \leq \rho \left( (a_1 + c_1) \| u \| + (b_1 + d_1) \| v \| \right) t \\
= \rho \left( (a_1 + c_1) \| u \| + (b_1 + d_1) \| v \| \right) \psi(t), \tag{35} \]
\[ T_2(u, v)(t) \leq \rho \left( (a_1 + c_1) \| u \| + (b_1 + d_1) \| v \| \right) \psi(t) \]
\[ \leq \rho \left( (a_1 + c_1) \| u \| + (b_1 + d_1) \| v \| \right) \psi(t) \]
\[ \leq \rho \left( (a_1 + c_1) \| u \| + (b_1 + d_1) \| v \| \right) \psi(t). \]
So, we can take \( M = \rho \max\{1, 1/\mu\}((a_1 + c_1) \| u \| + (b_1 + d_1) \| v \|) \)
such that (34) holds.

**Step 3.** For any given \((u_0, v_0) \in E \), \( n = 1, 2, \ldots \), let \( (u_n, v_n) = S(u_{n-1}, v_{n-1}) \). By Step 2, there exists \( M > 0 \) such that
\[ T([u_1(t) - u_0(t)], [v_1(t) - v_0(t)]) \]
\[ \leq M \cdot (\varphi(t), \psi(t)), \quad t \in [0, 1]. \tag{36} \]

Notice for \( p \in \mathbb{N} \) that
\[ |u_{n+p+1}(t) - u_{n+p}(t)| = \left| S_1 \left( u_{n+p}, v_{n+p} \right) (t) \right| \\
- \left| S_1 \left( u_{n+p-1}, v_{n+p-1} \right) (t) \right| \leq \int_0^1 G_1(t, s) \\
\cdot \left| f \left( s, u_{n+p}(s), v_{n+p}(s) \right) \right| ds \\
- \left| f \left( s, u_{n+p-1}(s), v_{n+p-1}(s) \right) \right| ds \\
+ \int_0^1 H_1(t, s) \left| g \left( s, u_{n+p}(s), v_{n+p}(s) \right) \right| ds \\
- \left| g \left( s, u_{n+p-1}(s), v_{n+p-1}(s) \right) \right| ds \\
\leq \int_0^1 G_1(t, s) \left( a_1 \left| u_{n+p}(t) - u_{n+p-1}(t) \right| \\
+ b_1 \left| v_{n+p}(t) - v_{n+p-1}(t) \right| \right) ds + \int_0^1 H_1(t, s) \\
\cdot \left( c_1 \left| u_{n+p}(t) - u_{n+p-1}(t) \right| \right), \]
Thus, by (33) and (36), we obtain that
\[
\begin{align*}
\left(\left[u_{n+p+1}(t) - u_{n+p}(t)\right], \left|v_{n+p+1}(t) - v_{n+p}(t)\right|\right) &\leq T^{n+p}(\left[u_1 - u_0, \left|v_1 - v_0\right|\right)(t), \\
\left|v_{n+p+1}(t) - v_{n+p}(t)\right| &\leq T_2^{n+p}(\left[u_1 - u_0, \left|v_1 - v_0\right|\right)(t).
\end{align*}
\]
(37)

Thus, by (33) and (36), we obtain that
\[
\begin{align*}
\left(\left[u_{n+m+1}(t) - u_{n+m}(t)\right], \left|v_{n+m+1}(t) - v_{n+m}(t)\right|\right) &\leq T^{n+m}(\left[u_1 - u_0, \left|v_1 - v_0\right|\right)(t), \\
\left|v_{n+m+1}(t) - v_{n+m}(t)\right| &\leq M T^{n+m-1}(\left(\varphi, \psi\right)(t) \leq M \lambda^{n+m-1}(\varphi(t), \psi(t)).
\end{align*}
\]
(38)

Thus, for \(n, m \in \mathbb{N}\), we conclude that
\[
\begin{align*}
|u_{n+m}(t) - u_n(t)| &\leq |u_{n+m}(t) - u_{n+m-1}(t)| + \cdots + |u_{n+1}(t) - u_n(t)| \\
&\leq M \left(\lambda^{n+m-2} + \cdots + \lambda^{n-1}\right) \varphi(t) \\
&= M \frac{\lambda^{n-1} - \lambda^{n+m-1}}{1 - \lambda} \varphi(t), \\
|v_{n+m}(t) - v_n(t)| &\leq M \frac{\lambda^{n-1} - \lambda^{n+m-1}}{1 - \lambda} \psi(t).
\end{align*}
\]
(39)

The above two inequalities ensure that \(\{(u_n, v_n)\}\) is a Cauchy sequence in \(E\). Since \(E\) is complete, there exists \((u^*, v^*) \in E\) such that \(\lim_{n \to \infty}(u_n, v_n) = (u^*, v^*)\). Therefore, \((u^*, v^*)\) is a fixed point of \(S\) that follows from the continuity of operator \(S\).

**Step 4.** We show that \(S\) has a unique fixed point. Suppose there exist two elements \((u^*, v^*), (u_*, v_*))\) with \(S(u^*, v^*) = (u^*, v^*)\) and \(S(u_*, v_*) = (u_*, v_*)\). By Step 2, there exists \(M > 0\) such that
\[
T\left(\left[u^*(t) - u_*(t)\right], \left|v^*(t) - v_*(t)\right|\right) \leq M \left(\varphi(t), \psi(t)\right), \quad t \in [0, 1].
\]
(40)

Applying the method used in Step 3 again, for \(p \in \mathbb{N}\), we get
\[
\begin{align*}
|u^*(t) - u_*(t)| &\leq M \frac{\lambda^p}{1 - \lambda} \varphi(t), \\
|v^*(t) - v_*(t)| &\leq M \frac{\lambda^p}{1 - \lambda} \psi(t).
\end{align*}
\]
(41)

Hence we get the desired results.

**Example 5.** Consider the differential system
\[
\begin{align*}
x''(t) &= \sin x(t) + \ln \left(1 + y^2(t)\right) + h_1(t), \\
y''(t) &= 2 \arctan x(t) + \sqrt{1 + y^2(t)} + h_2(t), \\
x(0) &= y(0) = 0, \\
x(1) &= \int_0^1 y(t)\, dt, \\
y(1) &= 2 \int_0^1 x(t)\, dt,
\end{align*}
\]
(42)

where \(h_1, h_2 \in C[0, 1]\). We have
\[
\begin{align*}
A(t) &= t, \\
B(t) &= 2t, \\
\alpha [t] &= \frac{1}{2}, \\
\beta [t] &= 1, \\
\kappa &= \frac{1}{2}, \\
\alpha [\phi] &= \frac{1}{24}, \\
\beta [\phi] &= \frac{1}{12}, \\
a_{11} &= \frac{\alpha [t] \beta [\phi]}{\kappa} + \frac{1}{6} = \frac{1}{4}, \\
a_{12} &= \frac{\alpha [\phi]}{\kappa} = \frac{1}{12}, \\
a_{21} &= \frac{\alpha [\phi] \beta [t]}{\kappa} + \frac{1}{6} = \frac{1}{4}, \\
a_{22} &= \frac{\beta [\phi]}{\kappa} = \frac{1}{6}.
\end{align*}
\]
(43)

Let
\[
\begin{align*}
f(t, x, y) &= \sin x + \ln \left(1 + y^2\right) + h_1(t), \\
g(t, x, y) &= 2 \arctan x + \sqrt{1 + y^2} + h_2(t),
\end{align*}
\]
(44)

then
\[
\begin{align*}
|f(t, u_1, v_1) - f(t, u_2, v_2)| &\leq |u_1 - u_2| + |v_1 - v_2|, \\
|g(t, u_1, v_1) - g(t, u_2, v_2)| &\leq 2 |u_1 - u_2| + |v_1 - v_2|,
\end{align*}
\]
(45)

where \(t \in [0, 1], u_1, u_2, v_1, v_2 \in \mathbb{R}\). Hence, there exists a solution \((\lambda, \mu) = (11/12, 3/2)\) of the following inequality system:
\[
(a_{11} a_1 + a_{12} c_1) + (a_{11} b_1 + a_{12} d_1) \mu = \frac{5}{12} + \frac{1}{3} \mu \leq \lambda,
\]

In the following, we give an example to illustrate our theory.
Therefore, according to Theorem 4, the problem (42) has a unique solution.

When the nonlinearity of differential equation and differential system satisfies Lipschitz condition, the usual method to obtain the uniqueness is the well-known Banach’s contraction principle. For this purpose, we should add some restriction on the Lipschitz constants to guarantee the norm of a linear operator related to differential equation and differential system less than 1. Next, we discuss the estimate of the norm of a linear operator related to differential system (42).

Take \( \sigma(t) = 1 \), \( q(t) = 3/2 \). After standard computation, we get

\[
\begin{align*}
\int_0^1 G_1(t, s) \sigma(s) \, ds &= \int_0^1 G_2(t, s) \sigma(s) \, ds = \frac{4t - t^2}{6}, \\
\int_0^1 H_1(t, s) \sigma(s) \, ds &= \frac{t}{6}, \\
\int_0^1 H_2(t, s) \sigma(s) \, ds &= \frac{t}{3}.
\end{align*}
\]

Then

\[
T_1 (\sigma, q) (t) = \int_0^1 G_1 (t, s) (\sigma(s) + q(s)) \, ds + \int_0^1 H_1 (t, s) (2\sigma(s) + q(s)) \, ds = \frac{20t - 5t^2}{12} + \frac{7t}{12} = \frac{27t - 5t^2}{12},
\]

\[
T_2 (\sigma, q) (t) = \int_0^1 G_2 (t, s) (2\sigma(s) + q(s)) \, ds + \int_0^1 H_2 (t, s) (\sigma(s) + q(s)) \, ds = \frac{28t - 7t^2}{12} + \frac{5t}{6} = \frac{38t - 7t^2}{12}.
\]

So it follows from the definition of the norm for linear operator that

\[
\|T\| \geq \frac{\|T(\sigma, q)\|_E}{\|\sigma, q\|_E} = \frac{31}{18} > 1.
\]

Thus Example 5 shows that Theorem 4 provides the same results with weaker conditions.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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