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Research Article

New Family of Iterative Methods for Solving Nonlinear Models

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We introduce a new family of iterative methods for solving mathematical models whose governing equations are nonlinear in nature. The new family gives several iterative schemes as special cases. We also give the convergence analysis of our proposed methods. In order to demonstrate the improved performance of newly developed methods, we consider some nonlinear equations along with two complex mathematical models. The graphical analysis for these models is also presented.

1. Introduction

Solving nonlinear equations is one of the important problems in mathematical sciences, especially in numerical analysis. There is a vast literature available to find the solution of nonlinear equations; see, for example, [1–24] and references therein. The construction of numerical methods is usually based on diverse techniques such as Taylor series, quadrature formulas, homotopy perturbation, and decomposition. One of the most powerful and well-known techniques for finding the solution of nonlinear equations is Newton's method which converges quadratically [19].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \ n = 0, 1, 2, \ldots. \tag{1}$$

To improve the efficiency, several modified higher order methods have been presented in the literature by using different techniques [1–6, 9–12, 14–20, 22–24]. Recently, some useful methods have been introduced in [7, 8, 13, 21]. Abbasbandy [1] used Adomian decomposition method (ADM) [2] to find the simple root of nonlinear equations. It is worth mentioning here that the involvement of higher order derivatives of Adomian polynomials is a major weakness of ADM. This weakness was eradicated by Daftardar-Gejji and Jafari [5], when they introduced a new decomposition technique. The said decomposition technique is quite simple as compared to the ADM as it does not need the higher order derivatives of functions. This technique has extensively been used to develop some useful algorithms for solving nonlinear equations [4, 5, 10, 18]. In 2015, Noor et al. [18], using the same decomposition technique along with the idea of coupled system, proposed two fourth-order iterative methods ([18], Algorithms 2.12 and 2.13) with efficiency index of 1.2600 each.

In this work, using the quadrature formula along with the fundamental law of calculus, truncating the series of $f(x)$ at quadratic level and decomposition technique of [5], we construct a family of new iterative methods for solving nonlinear equations. As special cases, we propose two fourth-order and two sixth-order methods. The number of evaluations per iteration for the third-order methods is 4 and 5; the fourth-order methods 5 and 6; and sixth-order methods 8 and 9; thus the efficiency indices of our methods are 1.3161, 1.2457, 1.3195, 1.2600, 1.2510, and 1.2203. The convergence criteria of newly constructed families are also presented. In order to demonstrate the validity and better efficiency of our proposed methods, we solve the nonlinear equations arising in the population model and in the motion of a particle on an inclined plane [18]. We also present the graphical analysis for the endorsement of numerical results.

2. Iterative Methods

Let α be the simple root of the nonlinear equation:

$$f(x) = 0. \tag{2}$$

We rewrite (2) in the form of the following coupled system using the quadrature formula and fundamental law of calculus:

$$\begin{aligned} f(x) \approx f(\gamma) &+ \left[\sum_{i=1}^{p} w_i f'(\gamma + \tau_i (x-\gamma))\right](x-\gamma) \\ &+ \left[\sum_{i=1}^{p} w_i f''(\gamma + \tau_i (x-\gamma))\right]\frac{(x-\gamma)^2}{2} \\ &+ g(x) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} g(x) = f(x) &- f(\gamma) \\ &- \left[\sum_{i=1}^{p} w_i f'(\gamma + \tau_i (x-\gamma))\right](x-\gamma) \\ &- \left[\sum_{i=1}^{p} w_i f''(\gamma + \tau_i (x-\gamma))\right]\frac{(x-\gamma)^2}{2}, \end{aligned} \tag{4}$$

where γ denotes the initial approximation sufficiently close to the exact root α of (2), τ_i are knots in $[0, 1]$, and w_i are the weights satisfying

$$\sum_{i=1}^{p} w_i = 1. \tag{5}$$

Equation (3) can be written as follows:

$$x = \gamma - \frac{2(f(\gamma) + g(x))}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i (x-\gamma)) + (x-\gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i (x-\gamma))} \tag{6}$$

$$= c + N(x), \tag{7}$$

where

$$c = \gamma, \tag{8}$$

$$N(x) = -\frac{2(f(\gamma) + g(x))}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i (x-\gamma)) + (x-\gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i (x-\gamma))}. \tag{9}$$

Here $N(x)$ is a nonlinear operator which can be decomposed as follows using the decomposition technique mainly due to Daftardar-Gejji and Jafari [5].

$$\begin{aligned} N(x) &= N\left(\sum_{i=0}^{\infty} x_i\right) \\ &= N(x_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right\}. \end{aligned} \tag{10}$$

The purpose of the above decomposition is to find the solution of (2) in the form of a series:

$$x = \sum_{i=0}^{\infty} x_{i.} \tag{11}$$

Combining (7), (10), and (11), we get

$$\begin{aligned} x &= \sum_{i=0}^{\infty} x_i \\ &= c + N(x_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right\}. \end{aligned} \tag{12}$$

Thus, we have the following iterative scheme:

$$\begin{aligned} x_0 &= c, \\ x_1 &= N(x_0), \\ x_2 &= N(x_0 + x_1) - N(x_0), \\ &\vdots \\ x_{m+1} &= N(x_0 + x_1 + \cdots + x_m) \\ &\quad - N(x_0 + x_1 + \cdots + x_{m-1}), \quad m = 1, 2, \ldots. \end{aligned} \tag{13}$$

Therefore, we have

$$x_1 + x_2 + \cdots + x_{m+1} = N(x_0 + x_1 + \cdots + x_m),$$

$$m = 1, 2, \ldots. \tag{14}$$

Since $x_0 = c$, (11) gives

$$x = c + \sum_{i=1}^{\infty} x_i. \tag{15}$$

From (8) and (13), we have

$$x_0 = c = \gamma. \tag{16}$$

From (4), one can easily compute

$$g(x_0) = 0. \tag{17}$$

Using (9), (13), and (16), we have

$$x_1 = N(x_0) = -\frac{2(f(\gamma) + g(x_0))}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 - \gamma)) + (x_0 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 - \gamma))}. \tag{18}$$

Using condition (5) and (17), the above equation yields

$$x_1 = -\frac{f(\gamma)}{f'(\gamma)}. \tag{19}$$

We note that x is approximated as

$$x = \lim_{m\to\infty} X_m, \quad \text{where } X_m = x_0 + x_1 + \cdots + x_m. \tag{20}$$

For $m = 1$, in (20) and using (16) and (19), we have

$$x \approx X_1 = x_0 + x_1 = \gamma - \frac{f(\gamma)}{f'(\gamma)}. \tag{21}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (2).

Algorithm 1. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \ n = 0, 1, 2, \ldots, \tag{22}$$

which is the well-known Newton's method for solving nonlinear (2). Now from (21), we have

$$x_0 + x_1 - \gamma = -\frac{f(\gamma)}{f'(\gamma)}. \tag{23}$$

From (4) and using (23), the following can easily be obtained:

$$\begin{aligned} g(x_0 + x_1) &= f(x_0 + x_1) - f(\gamma) \\ &\quad - \left[\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right](x_0 + x_1 - \gamma) \\ &\quad - \left[\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))\right] \\ &\quad \cdot \frac{(x_0 + x_1 - \gamma)^2}{2}. \\ &= f(x_0 + x_1) - f(\gamma) \\ &\quad + \left[\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right]\left(\frac{f(\gamma)}{f'(\gamma)}\right) \\ &\quad - \left[\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))\right]\left(\frac{f(\gamma)^2}{2f'(\gamma)^2}\right). \end{aligned} \tag{24}$$

Similarly, from (9), by using (14) and (24), we have

$$\begin{aligned} x_1 + x_2 = N(x_0 + x_1) &= -\frac{2(f(\gamma) + g(x_0 + x_1))}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma)) + (x_0 + x_1 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))} \\ &= -\frac{f(\gamma)}{f'(\gamma)} - \frac{2f(x_0 + x_1)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma)) + (x_0 + x_1 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))}. \end{aligned} \tag{25}$$

For $m = 2$, in (20) and by using (13), (14), (16), and (25), we get

$$\begin{aligned} x \approx X_2 &= x_0 + x_1 + x_2 = c + N(x_0 + x_1) \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{2f(x_0 + x_1)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma)) + (x_0 + x_1 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))}. \end{aligned} \tag{26}$$

Using above relation, we can suggest the following two-step iterative method for solving nonlinear equation (2).

Algorithm 2. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - \frac{2f(y_n)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(y_n - x_n)) + (y_n - x_n)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(y_n - x_n))},$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots. \tag{27}$$

From (26), we obtained

$$x_0 + x_1 + x_2 - \gamma = \frac{f(\gamma)}{f'(\gamma)} - \frac{2f(x_0 + x_1)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma)) + (x_0 + x_1 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))}. \tag{28}$$

From (4) and by using (28), we have

$$\begin{aligned} g(x_0 + x_1 + x_2) &= f(x_0 + x_1 + x_2) - f(\gamma) \\ &\quad - \left[\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma))\right] \\ &\quad \cdot (x_0 + x_1 + x_2 - \gamma) \\ &\quad - \left[\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma))\right] \\ &\quad \cdot \frac{(x_0 + x_1 + x_2 - \gamma)^2}{2}. \end{aligned} \tag{29}$$

From (9), by using (14) and (29), we obtain

$$\begin{aligned} x_1 + x_2 + x_3 &= N(x_0 + x_1 + x_2) \\ &= -\frac{2(f(\gamma) + g(x_0 + x_1 + x_2))}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma)) + (x_0 + x_1 + x_2 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma))} \\ &= (x_0 + x_1 + x_2 - \gamma) \\ &\quad - \frac{2f(x_0 + x_1 + x_2)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma)) + (x_0 + x_1 + x_2 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma))}. \end{aligned} \tag{30}$$

For $m = 3$, in (20), we have

$$\begin{aligned} x \approx X_3 &= x_0 + x_1 + x_2 + x_3 = c + N(x_0 + x_1 + x_2) \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{2f(x_0 + x_1)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma)) + (x_0 + x_1 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 - \gamma))} \\ &\quad - \frac{2f(x_0 + x_1 + x_2)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma)) + (x_0 + x_1 + x_2 - \gamma)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(x_0 + x_1 + x_2 - \gamma))}. \end{aligned} \tag{31}$$

Using this formulation, we suggest the following three-step iterative method for solving nonlinear equations (2).

Algorithm 3. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$\begin{aligned} x_{n+1} &= z_n - \frac{2f(z_n)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(z_n - x_n)) + (z_n - x_n)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(z_n - x_n))}, \\ z_n &= y_n - \frac{2f(y_n)}{2\sum_{i=1}^{p} w_i f'(\gamma + \tau_i(y_n - x_n)) + (y_n - x_n)\sum_{i=1}^{p} w_i f''(\gamma + \tau_i(y_n - x_n))}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots \end{aligned} \tag{32}$$

According to our knowledge, Algorithms 2 and 3 are new to solve nonlinear equation (2).

2.1. Some Special Cases of Algorithm 2. Now, we present some special cases of Algorithm 2. For $p = 1, w_1 = 1$ and $\tau_1 = 0$, Algorithm 2 reduces to the following iterative method for solving nonlinear equations.

Algorithm 4. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme

$$\begin{aligned} x_{n+1} &= y_n - \frac{2f(y_n)}{2f'(x_n) + (y_n - x_n) f''(x_n)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{33}$$

Using $p = 1, w_1 = 1$ and $\tau_1 = 1$, Algorithm 2 reduces to the following iterative method.

Algorithm 5. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$\begin{aligned} x_{n+1} &= y_n - \frac{2f(y_n)}{2f'(y_n) + (y_n - x_n) f''(y_n)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{34}$$

Now, using $p = 1, w_1 = 1$ and $\tau_1 = 1/2$, Algorithm 2 reduces to the following iterative method.

Algorithm 6. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$\begin{aligned} x_{n+1} &= y_n \\ &\quad - \frac{2f(y_n)}{2f'((x_n + y_n)/2) + (y_n - x_n) f''((x_n + y_n)/2)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{35}$$

Now by using $p = 2, w_1 = 1/4, w_2 = 3/4, \tau_1 = 0$ and $\tau_2 = 2/3$, Algorithm 2 reduces to the following iterative method for solving $f(x) = 0$.

Algorithm 7. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$\begin{aligned} x_{n+1} &= y_n - \frac{8f(y_n)}{2f'(x_n) + 6f'((x_n + 2y_n)/3) + (y_n - x_n) f''(x_n) + 3(y_n - x_n) f''((x_n + 2y_n)/3)} \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{36}$$

To the best of our knowledge, Algorithms 4–7 are new iterative methods for solving nonlinear equation (2).

2.2. Some Special Cases of Algorithm 3. Now, we present some special cases of Algorithm 3. For $p = 1, w_1 = 1$ and $\tau_1 = 0$, Algorithm 3 reduces to the following iterative method for solving nonlinear equations.

Algorithm 8. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$\begin{aligned} x_{n+1} &= z_n - \frac{2f(z_n)}{2f'(x_n) + (z_n - x_n) f''(x_n)} \\ z_n &= y_n - \frac{2f(y_n)}{2f'(x_n) + (y_n - x_n) f''(x_n)} \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{37}$$

Using $p = 1, w_1 = 1$ and $\tau_1 = 1$, Algorithm 3 reduces to the following iterative method for solving $f(x) = 0$.

Algorithm 9. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = z_n - \frac{2f(z_n)}{2f'(z_n) + (z_n - x_n) f''(z_n)}$$

$$z_n = y_n - \frac{2f(y_n)}{2f'(y_n) + (y_n - x_n) f''(y_n)}$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{38}$$

Now, using $p = 1$, $w_1 = 1$ and $\tau_1 = 1/2$, Algorithm 3 reduces to the following iterative method for solving $f(x) = 0$.

Algorithm 10. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = z_n - \frac{2f(z_n)}{2f'((z_n + x_n)/2) + (z_n - x_n) f''((z_n + x_n)/2)},$$
$$z_n = y_n - \frac{2f(y_n)}{2f'((y_n + x_n)/2) + (y_n - x_n) f''((y_n + x_n)/2)},$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{39}$$

Using $p = 2$, $w_1 = 1/4$, $w_2 = 3/4$, $\tau_1 = 0$, and $\tau_2 = 2/3$, Algorithm 2 reduces to the following iterative method for solving $f(x) = 0$.

Algorithm 11. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = z_n - \frac{8f(z_n)}{2f'(x_n) + 6f'((x_n + 2z_n)/3) + (z_n - x_n) f''(x_n) + 3(z_n - x_n) f''((x_n + 2z_n)/3)},$$
$$z_n = y_n - \frac{8f(y_n)}{2f'(x_n) + 6f'((x_n + 2y_n)/3) + (y_n - x_n) f''(x_n) + 3(y_n - x_n) f''((x_n + 2y_n)/3)}, \tag{40}$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

To the best of our knowledge, Algorithms 8–11 are new iterative methods for solving nonlinear equation (2).

3. Convergence Analysis

In this section, convergence criteria of newly suggested methods are studied in the form of the following theorem.

Theorem 12. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ on an open interval I has a simple root $\alpha \in I$. Let $f(x)$ be sufficiently differentiable in the neighborhood of α; then the convergence orders of the methods defined by Algorithms 2–11 are* 3, 4, 3, 3, 4, 4, 4, 4, 6 *and* 6, *respectively.*

Proof. Let α be a simple zero of $f(x)$. Since f is sufficiently differentiable, the Taylor series expansions of $f(x_n)$, $f'(x_n)$, and $f''(x_n)$ about α are given by

$$f(x_n) = f'(\alpha)\left\{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O\left(e_n^6\right)\right\}, \tag{41}$$

$$f'(x_n) = f'(\alpha)\left\{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_4^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O\left(e_n^5\right)\right\}, \tag{42}$$

$$f''(x_n) = f'(\alpha)\left\{2c_2 + 6c_3 e_n + 12c_4 e_4^2 + 20c_5 e_n^3 + 30c_6 e_n^4 + O\left(e_n^5\right)\right\}, \tag{43}$$

where $e_n = x_n - \alpha$ and $c_j = (1/j!)(f^{(j)}(\alpha)/f'(\alpha))$, $j = 2, 3, \ldots$.

From (41) and (42), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right) e_n^3 + \left(-4c_2^3 + 7c_2 c_3 - 3c_4\right) e_n^4 + \left(8c_2^4 - 20c_2^2 c_3 + 10c_2 c_4 + 6c_3^2 - 4c_5\right) e_n^5 + O\left(e_n^6\right). \tag{44}$$

Using (44), we have

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 - 2\left(c_2^2 - c_3\right) e_n^3 + \left(4c_2^3 - 7c_2 c_3 + 3c_4\right) e_n^4 + \left(-8c_2^4 + 20c_2^2 c_3 - 10c_2 c_4 - 6c_3^2 + 4c_5\right) e_n^5 + O\left(e_n^6\right). \tag{45}$$

From (45), we get

$$f(y_n) = c_2 e_n^2 - 2\left(c_2^2 - c_3\right) e_n^3 + \left(5c_2^3 - 7c_2 c_3 + 3c_4\right) e_n^4 + \left(-12c_2^4 + 24c_2^2 c_3 - 10c_2 c_4 - 6c_3^2 + 4c_5\right) e_n^5 + O\left(e_n^6\right). \tag{46}$$

Now, the Taylor's series expansions of $\sum_{i=1}^{p} w_i f'(x_n + \tau_i(z_n - x_n))$ and $\sum_{i=1}^{p} w_i f''(x_n + \tau_i(z_n - x_n))$ are given by

$$\begin{aligned}\sum_{i=1}^{p} w_i f'\left(x_n + \tau_i\left(y_n - x_n\right)\right) &= 1 - 2\sum_{i=1}^{p} w_i\left(\tau_i c_2 - c_2\right) e_n \\ &+ \sum_{i=1}^{p} w_i\left(2\tau_i c_2^2 + 3\tau_i^2 c_3 - 6\tau_i c_3 + 3c_3\right) e_n^2 \\ &+ \sum_{i=1}^{p} w_i\left(-4\tau_i c_2^3 - 6\tau_i^2 c_2 c_3 - 4\tau_i^3 c_4 + 10\tau_i c_2 c_3\right. \\ &\left. + 12\tau_i^2 c_4 - 12\tau_i c_4 + 4c_4\right) e_n^3 + O\left(e_n^4\right),\end{aligned} \tag{47}$$

$$\begin{aligned}\sum_{i=1}^{p} w_i f''\left(x_n + \tau_i\left(y_n - x_n\right)\right) &= 2c_2 + \sum_{i=1}^{p} w_i\left(-6\tau_i c_3\right. \\ &\left. + 6c_3\right) e_n + \sum_{i=1}^{p} w_i\left(6\tau_i c_2 c_3 + 24\tau_i^2 c_4 - 48\tau_i c_4 + 24c_4\right) \\ &\cdot e_n^2 + \sum_{i=1}^{p} w_i\left(-12\tau_i c_2^2 c_3 - 48\tau_i^2 c_2 c_4 - 120\tau_i^3 c_5\right. \\ &\left. + 48\tau_i c_2 c_4 + 12\tau_i c_3^2 + 360\tau_i^2 c_5 - 360\tau_i c_5 + 120c_5\right) e_n^3 \\ &+ O\left(e_n^4\right).\end{aligned} \tag{48}$$

Using (45), (47), and (48), we obtain the error term for Algorithm 2 as follows:

$$\begin{aligned}z_n &= y_n - \frac{2f\left(y_n\right)}{\sum_{i=1}^{p} w_i f'\left(x_n + \tau_i\left(y_n - x_n\right)\right) + \left(y_n - x_n\right)\sum_{i=1}^{p} w_i f''\left(x_n + \tau_i\left(y_n - x_n\right)\right)} \\ &= \alpha + \sum_{i=1}^{p} w_i\left(-2c_2^2\tau_i + c_2^2\right) e_n^3 + \sum_{i=1}^{p} w_i\left(-4\tau_i^2 c_2^3 + 10\tau_i c_2^3 + 3\tau_i^2 c_2 c_3 - 3c_2^3 - 7\tau_i c_2 c_3 + 2c_2 c_3\right) e_n^4 + O\left(e5\right).\end{aligned} \tag{49}$$

From (49), we have

$$\begin{aligned}f\left(z_n\right) &= \sum_{i=1}^{p} w_i\left(-2c_2^2\tau_i + c_2^2\right) e_n^3 + \sum_{i=1}^{p} w_i\left(-4\tau_i^2 c_2^3\right. \\ &\left. + 10\tau_i c_2^3 + 3\tau_i^2 c_2 c_3 - 3c_2^3 - 7\tau_i c_2 c_3 + 2c_2 c_3\right) e_n^4 \\ &+ O\left(e5\right).\end{aligned} \tag{50}$$

The Taylor's series expansions of $\sum_{i=1}^{p} w_i f'(x_n + \tau_i(z_n - x_n))$ and $\sum_{i=1}^{p} w_i f''(x_n + \tau_i(z_n - x_n))$ are given by

$$\begin{aligned}\sum_{i=1}^{p} w_i f'\left(x_n + \tau_i\left(z_n - x_n\right)\right) &= 1 - 2\sum_{i=1}^{p} w_i\left(\tau_i c_2 - c_2\right) e_n \\ &+ \sum_{i=1}^{p} w_i\left(3\tau_i^2 c_3 - 6\tau_i c_3 + 3c_3\right) e_n^2 + \sum_{i=1}^{p} w_i\left(-4\tau_i^2 c_2^3\right. \\ &\left. + 2\tau_i c_2^3 - 4\tau_i^3 c_4 + 12\tau_i^2 c_4 - 12\tau_i c_4 + 4c_4\right) e_n^3 \\ &+ O\left(e_n^4\right), \\ \sum_{i=1}^{p} w_i f''\left(x_n + \tau_i\left(z_n - x_n\right)\right) &= 2 + \sum_{i=1}^{p} w_i\left(-4\tau_i c_2 - 2c_2\right) \\ &\cdot e_n + \sum_{i=1}^{p} w_i\left(6\tau_i^2 c_3 - 6\tau_i c_3\right) e_n^2 + \sum_{i=1}^{p} w_i\left(-8\tau_i^2 c_2^3\right. \\ &\left. + 2\tau_i c_2^3 - 8\tau_i^3 c_4 + 2c_2^3 + 24\tau_i c_4 - 16c_4\right) e_n^3 + O\left(e_n^4\right).\end{aligned} \tag{51}$$

Using (49) and (51), the error term for Algorithm 3 is obtained as

$$\begin{aligned}x_{n+1} &= z_n - \frac{2f\left(z_n\right)}{\sum_{i=1}^{p} w_i f'\left(x_n + \tau_i\left(z_n - x_n\right)\right) + \left(z_n - x_n\right)\sum_{i=1}^{p} w_i f''\left(x_n + \tau_i\left(z_n - x_n\right)\right)} \\ &= \alpha + \sum_{i=1}^{p} w_i\left(4\tau_i^2 c_2^3 - 4\tau_i c_2^3 + c_2^3\right) e_n^4 + O\left(e_n^5\right).\end{aligned} \tag{52}$$

Now, we prove the convergence orders of the special cases of Algorithms 2 and 3. □

The Taylor's series expansion of $2f'(x_n) + (y_n - x_n) f''(x_n)$ is given by

$$\begin{aligned}&2f'\left(x_n\right) + \left(y_n - x_n\right) f''\left(x_n\right) \\ &\quad = 2 + 2c_2 e_n + 2c_2^2 e_n^2 + \left(-4c_2^3 + 10c_2 c_3 - 4c_4\right) e_n^3 \\ &\qquad + \left(8c_2^4 - 26c_2^2 c_3 + 18c_2 c_4 + 12c_3^2 - 10c_5\right) e_n^4\end{aligned}$$

$$+ O\left(e_n^5\right). \tag{53}$$

Using (45), (46), and (53), the error term for Algorithm 4 is obtained as

$$\begin{aligned} z_n &= y_n - \frac{2f(y_n)}{2f'(x_n) + (y_n - x_n) f''(x_n)} \\ &= \alpha + c_2^2 e_n^3 + \left(-3c_2^3 + 2c_2c_3\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{54}$$

Now, expending $2f'(y_n) + (y_n - x_n)f''(y_n)$ by Taylor's series, we get

$$\begin{aligned} &2f'(y_n) + (y_n - x_n) f''(y_n) \\ &\quad = 2 - 2c_2e_n + 6c_2^2e_n^2 + \left(-12c_2^3 + 6c_2c_3\right) e_n^3 \\ &\qquad + \left(24c_2^4 - 18c_2^2c_3 + 18c_2c_4 - 12c_3^2\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{55}$$

Using equations (45), (46), and (55), the error term for Algorithm 5 is obtained as

$$\begin{aligned} z_n &= y_n - \frac{2f(y_n)}{2f'(y_n) + (y_n - x_n) f''(y_n)} \\ &= \alpha - c_2^2 e_n^3 + \left(3c_2^3 - 2c_2c_3\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{56}$$

The Taylor's series expansions of $f'((x_n+y_n)/2)$ and $f''((x_n+y_n)/2)$ are given by

$$\begin{aligned} f'\left(\frac{x_n+y_n}{2}\right) &= 1 + c_2e_n + \left(c_2^2 + \frac{3}{4}c_3\right) e_n^2 + \left(-2c_2^3 \right. \\ &\quad \left. + \frac{7}{2}c_2c_3 + \frac{1}{2}c_4\right) e_n^3 + \left(4c_2^4 - \frac{37}{4}c_2^2c_3 + \frac{9}{2}c_2c_4 + 3c_3^2 \right. \\ &\quad \left. + \frac{5}{16}c_5\right) e_n^4 + O\left(e_n^5\right), \\ f''\left(\frac{x_n+y_n}{2}\right) &= 2c_2 + 3c_3e_n + \left(3c_2c_3 + 6c_4\right) e_n^2 \\ &\quad + \left(-6c_2^2c_3 + 12c_2c_4 + 6c_3^2 + 15c_5\right) e_n^3 + \left(12c_2^2c_3 \right. \\ &\quad \left. - 18c_2^2c_4 - 21c_2c_3^2 + 45c_2c_5 + 33c_3c_4 + 45c_6\right) e_n^4 \\ &\quad + O\left(e_n^5\right). \end{aligned} \tag{57, 58}$$

Using (45), (46), (57), and (58), we obtain the error term for Algorithm 6 as follows:

$$\begin{aligned} z_n &= y_n \\ &\quad - \frac{2f(y_n)}{2f'((x_n+y_n)/2) + (y_n - x_n) f''((x_n+y_n)/2)} \\ &= \alpha + \left(c_2^3 - \frac{3}{4}c_2c_3\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{59}$$

Expanding $f'((x_n+2y_n)/3)$ and $f''((x_n+2y_n)/3)$ by Taylor's series, we have

$$\begin{aligned} f'\left(\frac{x_n+2y_n}{3}\right) &= 1 + \frac{2}{3}c_2e_n + \left(\frac{4}{3}c_2^2 + \frac{1}{3}c_3\right) e_n^2 \\ &\quad + \left(-\frac{8}{3}c_2^3 + 4c_2c_3 + \frac{4}{27}c_4\right) e_n^3 + \left(\frac{16}{3}c_2^4 - \frac{32}{3}c_2^2c_3 \right. \\ &\quad \left. + \frac{44}{9}c_2c_4 + \frac{8}{3}c_3^2 + \frac{5}{81}c_5\right) e_n^4 + O\left(e_n^5\right), \\ f''\left(\frac{x_n+2y_n}{3}\right) &= 2c_2 + 2c_3e_n + \left(4c_2c_3 + \frac{8}{3}c_4\right) e_n^2 \\ &\quad + \left(-8c_2^2c_3 + 8c_3^2 + \frac{32}{3}c_2c_4 + \frac{40}{9}c_5\right) e_n^3 + \left(16c_2^3c_3 \right. \\ &\quad \left. - 28c_2c_3^2 + \frac{100}{3}c_3c_4 - \frac{32}{3}c_2^2c_4 + \frac{80}{3}c_2c_5 + \frac{80}{9}c_6\right) e_n^4 \\ &\quad + O\left(e_n^5\right). \end{aligned} \tag{60}$$

Using (42), (43), (45), (46), and (60), the error term for Algorithm 7 is obtained as

$$\begin{aligned} z_n &= y_n - \frac{8f(y_n)}{2f'(x_n) + 6f'((x_n+2y_n)/3) + (y_n - x_n) f''(x_n) + 3(y_n - x_n) f''((x_n+2y_n)/3)} \\ &= \alpha + \left(c_2^3 - \frac{1}{2}c_2c_3\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{61}$$

Using (54), we have

$$f(z_n) = c_2^2 e_n^3 + \left(-3c_2^3 + 2c_2c_3\right) e_n^4 + O\left(e_n^5\right). \tag{62}$$

Using (42), (43), (54), and (62), the error term for Algorithm 8 is obtained as

$$\begin{aligned} x_{n+1} &= z_n - \frac{2f(z_n)}{2f'(x_n) + (z_n - x_n) f''(x_n)} \\ &= \alpha + c_2^3 e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{63}$$

Now, from (56), we have

$$f(z_n) = -c_2^2 e_n^3 + \left(3c_2^3 - 2c_2c_3\right) e_n^4 + O\left(e_n^5\right),$$

$$f'(z_n) = 1 - 2c_2^3 e_n^3 + \left(6c_2^4 - 4c_2^2 c_3\right) e_n^4 + O\left(e_n^5\right),$$
$$f''(z_n) = 2c_2 - 6c_2^2 c_3 e_n^3 + \left(18c_2^3 c_3 - 12c_2 c_3^2\right) e_n^4 + O\left(e_n^5\right). \tag{64}$$

Using (56) and (64), the error term for Algorithm 9 is obtained as

$$x_{n+1} = z_n - \frac{2f(z_n)}{2f'(z_n) + (z_n - x_n) f''(z_n)} = \alpha + c_2^3 e_n^4 + O\left(e_n^5\right). \tag{65}$$

Using (58), we get

$$f(z_n) = \left(c_2^3 - \frac{3}{4}c_2 c_3\right) e_n^4 + \left(-4c_2^4 + 7c_2^2 c_3 - \frac{5}{2}c_2 c_4 - \frac{3}{2}c_3^2\right) e_n^5 + O\left(e_n^6\right). \tag{66}$$

Taylor's series expansions of $f'((z_n + x_n)/2)$ and $f''((z_n + x_n)/2)$ are given by

$$f'\left(\frac{z_n + x_n}{2}\right) = 1 + c_2 e_n + \frac{3}{4}c_3 e_n^2 + \frac{1}{2}c_4 e_n^3 + \left(c_2^4 - \frac{3}{4}c_2^2 c_3 + \frac{5}{16}c_5\right) e_n^4 + O\left(e_n^5\right),$$
$$f''\left(\frac{z_n + x_n}{2}\right) = 2c_2 + 3c_3 e_n + 6c_4 e_n^2 + 15c_5 e_n^3 + \left(3c_2^3 c_3 - \frac{9}{4}c_2 c_3^2 + 45c_6\right) e_n^4 + O\left(e_n^5\right). \tag{67}$$

Therefore, using (59), (66), and (67), the error term for Algorithm 10 is obtained as

$$x_{n+1} = z_n - \frac{2f(z_n)}{2f'((z_n + x_n)/2) + (z_n - x_n) f''((z_n + x_n)/2)} = \alpha + \left(-\frac{3}{4}c_2^3 c_3 + \frac{9}{16}c_2 c_3^2\right) e_n^6 + O\left(e_n^7\right). \tag{68}$$

Using (61), we have

$$f(z_n) = \left(c_2^3 - \frac{1}{2}c_2 c_3\right) e_n^4 + \left(-4c_2^4 + 6c_2^2 c_3 - \frac{25}{18}c_2 c_4 - c_3^2\right) e_n^5 + O\left(e_n^6\right). \tag{69}$$

Taylor's series expansions of $f'((x_n + 2z_n)/3)$ and $f''((x_n + 2z_n)/3)$ are given by

$$f'\left(\frac{x_n + 2z_n}{3}\right) = 1 + \frac{2}{3}c_2 e_n + \frac{1}{3}c_3 e_n^2 + \frac{4}{27}c_4 e_n^3 + \left(\frac{4}{3}c_2^4 - \frac{2}{3}c_2^2 c_3 + \frac{5}{81}c_5\right) e_n^4 + O\left(e_n^5\right),$$
$$f''\left(\frac{x_n + 2z_n}{3}\right) = 2c_2 + 2c_3 e_n + \frac{8}{3}c_4 e_n^2 + \frac{40}{9}c_5 e_n^3 + \left(4c_2^3 c_3 - 2c_2 c_3^2 + \frac{80}{9}c_6\right) e_n^4 + O\left(e_n^5\right). \tag{70}$$

Using (61), (69), and (70), the error term for Algorithm 11 is obtained as

$$x_{n+1} = z_n - \frac{8f(z_n)}{2f'(x_n) + 6f'((x_n + 2z_n)/3) + (z_n - x_n) f''(x_n) + 3(z_n - x_n) f''((x_n + 2z_n)/3)} = \alpha + \left(\frac{3}{4}c_2^5 - \frac{7}{8}c_2^3 c_3 + \frac{1}{4}c_2 c_3^2\right) e_n^6 + O\left(e_n^7\right). \tag{71}$$

This complete the proof.

4. Numerical Examples

In this section, we demonstrate the validity and efficiency of our proposed iterative schemes by considering the nonlinear equations obtained from population model and the motion of a particle on an inclined plane, that is,

$$1{,}564{,}000 = 1{,}000{,}000 e^{\lambda} + \frac{435{,}000}{\lambda}\left(e^{\lambda} - 1\right), \tag{72}$$

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right). \tag{73}$$

We take $x_0 = 1.5$ for (72) and $x_0 = -1.2$ for (73) as initial guess for computer program. Tables 1 and 2 and Figures 1 and 2 give the comparison of our newly proposed iterative methods, that

Table 1: Numerical results for population model (equation (72))

Method	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$
NM	7	0.1009979296857497890	$5.933183e-16$	$1.104193e-18$
NR1	6	0.1009979296857497889	$2.329650e-34$	$6.369050e-20$
NR2	6	0.1009979296857497889	$1.115803e-29$	$1.138639e-17$
NR3	4	0.1009979296857497890	$5.933183e-16$	$3.778475e-19$
NR4	4	0.1009979296857497890	$5.933183e-16$	$1.398260e-18$
AG1	4	0.1009979296857497889	$2.632474e-83$	$1.401588e-22$
AG2	4	0.1009979296857497889	$1.002432e-77$	$3.228512e-21$
AG3	3	0.1009979296857497900	$1.406532e-12$	$1.744956e-11$
AG4	3	0.1009979296857497900	$1.406532e-12$	$3.152777e-12$

Table 2: Numerical results for motion of a particle (equation (73))

Method	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$
NM	10	−0.3170617745729570950	$1.543287e-22$	$1.557967e-19$
NR1	9	−0.3170617745729570950	$6.413654e-28$	$4.163537e-14$
NR2	9	−0.3170617745729570950	$1.888111e-24$	$2.182328e-12$
NR3	6	−0.3170617745729570950	$1.543287e-22$	$2.116367e-38$
NR4	6	−0.3170617745729570950	$1.543287e-22$	$2.709996e-37$
AG1	5	−0.3170617745729570950	$1.743432e-44$	$5.976014e-12$
AG2	5	−0.3170617745729570950	$2.080693e-42$	$1.938938e-11$
AG3	4	−0.3170617745729571000	$3.332577e-19$	$7.533267e-33$
AG4	4	−0.3170617745729571000	$3.332577e-19$	$6.625240e-24$

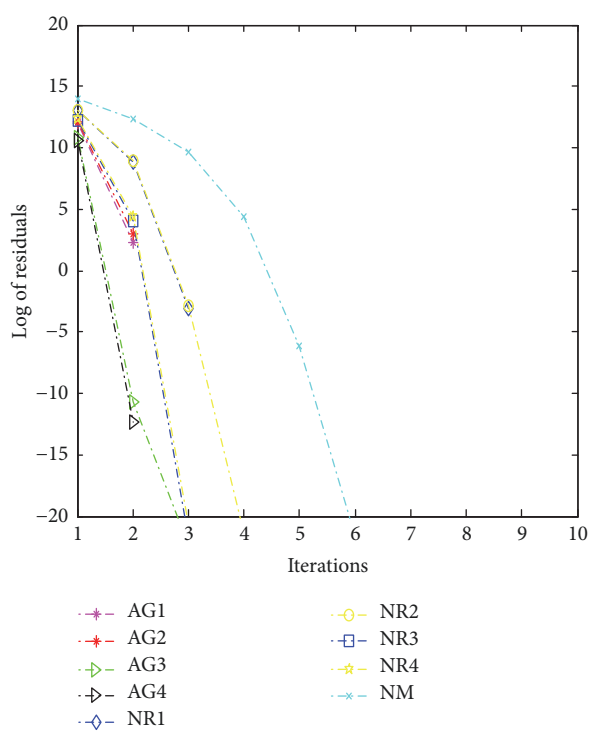

Figure 1: Log of residuals for equation (72).

is, Algorithm 6 (AG 1), Algorithm 7 (AG 2), Algorithm 10 (AG 3), and Algorithm 11 (AG 4) with standard Newton's method (NM, (1)), Noor et al. methods ([18], Algorithm 2.6) (NR 1), ([18], Algorithm 2.7) (NR 2), ([18], Algorithm 2.12) (NR 3), and ([18], Algorithm 2.13) (NR 4). In Table 3, we present numerical results for various nonlinear equations which also indicate that our methods are more efficient.

We use Maple18 for all numerical computations and MATLAB 9 to plot the graphs. For the computer programs, we use the following stopping criteria:

$$\left(\left|x_{n+1} - x_n\right| + \left|f\left(x_n\right)\right|\right) < \epsilon, \quad \text{where } \epsilon = 10^{-15}. \tag{74}$$

5. Conclusions

We have introduced a new family of iterative methods (Algorithms 2 and 3), based on decomposition technique, for solving nonlinear equations using coupled system of equations. Several new iterative methods have been established as special cases of newly established family. We have explored the convergence criteria of our new methods and investigated for convergence order and efficiency index. We present the comparative study both numerically and graphically (Tables 1 and 2 and Figures 1 and 2) of our newly constructed methods with some known methods by considering two real life models. In Table 3, we present numerical results by considering various nonlinear equations.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Table 3

$f(x)$	$x\_\circ$	Method	Number of iterations	$x[k]$	$f(x\_n)$	δ
		NM	7	3.0000000000000000007	$9.100000e-18$	$3.263396e-10$
		NR1	7	3.0000000000000000007	$9.105537e-18$	$3.263396e-10$
		NR2	7	3.0000000000000000007	$9.105537e-18$	$3.263396e-10$
		NR3	5	3.0000000000000000000	$0.000000e+00$	$1.415161e-15$
$e^{(}x^2+7x-30)-1$	2.9	NR4	5	3.0000000000000000000	$0.000000e+00$	$3.366886e-27$
		AG1	4	3.0000000000000000000	$2.921644e-79$	$3.555566e-21$
		AG2	4	3.0000000000000000000	$8.793703e-51$	$4.351958e-14$
		AG3	3	3.0000000000000000000	$0.000000e+00$	$1.645921e-14$
		AG4	3	3.0000000000000000000	$0.000000e+00$	$3.392389e-14$
		NM	7	2.2041177331716202960	$3.305103e-20$	$1.295280e-10$
		NR1	7	2.2041177331716202960	$1.864247e-22$	$7.456850e-12$
		NR2	7	2.2041177331716202960	$3.743730e-20$	$1.033112e-10$
		NR3	5	2.2041177331716202960	$3.305103e-20$	$1.194680e-24$
x^2-e^x+x+2	1.65	NR4	5	2.2041177331716202960	$3.305103e-20$	$7.482519e-36$
		AG1	4	2.2041177331716202960	$2.534748e-56$	$1.035700e-14$
		AG2	4	2.2041177331716202960	$3.405915e-48$	$1.073168e-12$
		AG3	3	2.2041177331716203000	$1.464912e-17$	$3.775506e-15$
		AG4	3	2.2041177331716203000	$1.464912e-17$	$2.401455e-11$
		NM	8	−1.8414056604369606378	$2.472561e-19$	$8.481033e-21$
		NR1	7	−1.8414056604369606378	$1.469668e-22$	$5.161458e-12$
		NR2	8	−1.8414056604369606378	$1.750316e-40$	$5.508062e-21$
		NR3	5	−1.8414056604369606378	$2.472561e-19$	$5.885759e-25$
$(x+2)e^{(}-x)-1$	−1.35	NR4	5	−1.8414056604369606378	$2.472561e-19$	$3.160149e-36$
		AG1	4	−1.8414056604369606378	$1.123854e-58$	$2.235422e-15$
		AG2	4	−1.8414056604369606378	$2.782588e-49$	$4.782839e-13$
		AG3	3	−1.8414056604369606000	$2.007912e-16$	$6.603435e-17$
		AG4	3	−1.8414056604369606000	$2.007912e-16$	$1.073115e-11$
		NM	6	0.9977265554138634538	$8.696898e-21$	$7.838761e-13$
		NR1	5	0.9977265554138634538	$1.302709e-21$	$3.286611e-11$
		NR2	6	0.9977265554138634538	$3.176527e-37$	$4.745782e-19$
		NR3	4	0.9977265554138634538	$8.696898e-21$	$5.066551e-25$
$2x^2-1/2\quad \sin x-1/2\pi$	0.5	NR4	4	0.9977265554138634538	$8.696898e-21$	$3.934532e-25$
		AG1	4	0.9977265554138634538	$1.775184e-99$	$2.198399e-25$
		AG2	4	0.9977265554138634538	$3.133051e-99$	$2.528325e-25$
		AG3	3	0.9977265554138634500	$1.410674e-17$	$5.046616e-21$
		AG4	3	0.9977265554138634500	$1.410674e-17$	$4.537399e-21$
		NM	8	0.9716662861175176121	$2.044833e-20$	$2.890826e-20$
		NR1	7	0.9716662861175176121	$1.855259e-30$	$3.505396e-16$
		NR2	7	0.9716662861175176121	$3.234098e-23$	$1.380792e-12$
		NR3	5	0.9716662861175176121	$2.044833e-20$	$5.272663e-24$
$8x^3-3x^2\quad \sin x-5$	0.5	NR4	5	0.9716662861175176121	$2.044833e-20$	$4.036720e-35$
		AG1	4	0.9716662861175176121	$1.832541e-52$	$5.721104e-14$
		AG2	4	0.9716662861175176121	$7.018615e-46$	$2.443339e-12$
		AG3	3	0.9716662861175176100	$3.426161e-17$	$5.515718e-14$
		AG4	3	0.9716662861175176100	$3.426161e-17$	$4.503689e-11$

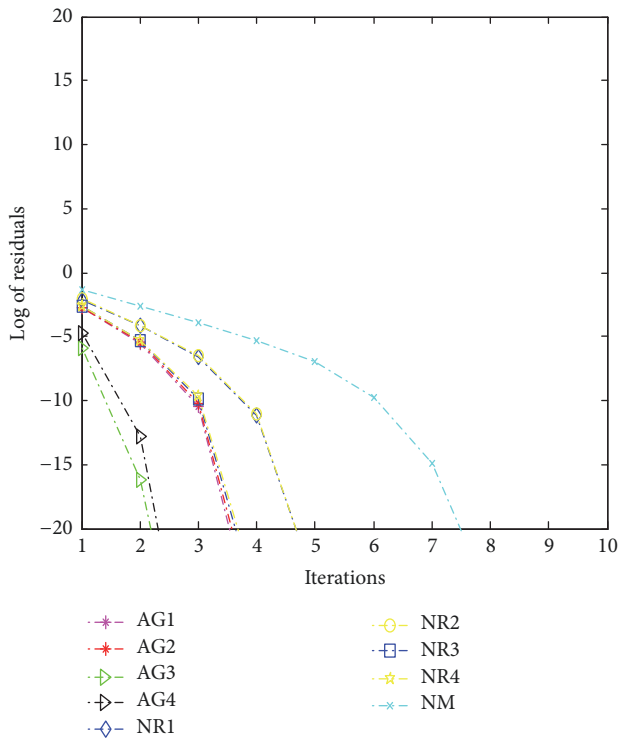

Figure 2: Log of residuals for equation (73).

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