Research Article

Dynamics Analysis of a Stochastic Delay Gilpin-Ayala Model with Markovian Switching

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This paper considers a stochastic delay Gilpin-Ayala model with Markovian switching. Using Lyapunov method, we show existence and uniqueness of global positive solution. Then, by using Chebyshev's inequality, M -matrix method, and BDG's inequality, stochastic permanence and asymptotic estimations of solutions are studied. Finally, numerical simulations illustrate the theoretical results. Our results generalize and improve the existing results.

1. Introduction

Logistic equation is one of the most important and most widely used mathematical models to describe the growth of biological species. In [1], Gilpin and Ayala introduced the following generalized logistic equation (Gilpin-Ayala model):

$$\frac{dx(t)}{dt} = x(t) \left(r - ax^\lambda(t) \right), \quad (1)$$

where $x(t)$ stands for the population size; r, a , and λ are positive constants. For more details, see [1, 2].

Population systems are often subject to environmental noise and there are different types of noise. Therefore, stochastic population systems have been studied extensively. See [3–9] and the references therein. In [8], by using Itô's formula and the exponential martingale inequality, the authors studied the stationary distribution, ergodicity, and extinction of the following stochastic Gilpin-Ayala model:

$$dx(t) = x(t) \left(r - ax^\lambda(t) \right) dt + \sigma_1 x(t) dB_1(t) + \sigma_2 x^{1+\lambda}(t) dB_2(t). \quad (2)$$

Further, [9] obtained the stationary distribution and ergodicity of model (2) and improved the corresponding results in [8]. In addition, there is another type of environment

noise, namely, colour noise, which can be demonstrated as a switching between two or more regimes of environment. See [10–18] and the references therein. Reference [11] considered the following stochastic Gilpin-Ayala model under regime switching:

$$dx(t) = x(t) \left[r(\gamma(t)) - a(\gamma(t)) x^\lambda(t) \right] dt + \sigma_1(\gamma(t)) x(t) dB_1(t) + \sigma_2(\gamma(t)) x^{1+\theta}(t) dB_2(t), \quad (3)$$

where there exist $\lambda > 0, \theta > 0$, and studied persistence, extinction, nonpersistence, and global attractivity of (3). Further, [12] obtained the lower-growth rate and the upper-growth rate of the positive solution of (3). Reference [13] considered the following stochastic Gilpin-Ayala population model with Markovian switching:

$$dx(t) = x(t) \left[r(\gamma(t)) - a(\gamma(t)) x^{\lambda(\gamma(t))}(t) \right] dt + \sigma_1(\gamma(t)) x(t) dB(t), \quad (4)$$

where the Gilpin-Ayala parameter λ is also allowed to switch. In [13], the authors established the global stability of the trivial equilibrium state of (4). In addition, they obtained extinction, persistence, and existence of a stationary distribution of (4).

On the other hand, many population systems depend on not only present states but also past states. See [19–26]. In [26], the authors considered a delay logistic model under regime switching

$$\begin{aligned} dx(t) &= x(t) \\ &\cdot [r(\gamma(t)) - a(\gamma(t))x(t) + b(\gamma(t))x(t-\tau)] dt \\ &+ \sigma_1(\gamma(t))x(t) dB(t). \end{aligned} \quad (5)$$

They obtained stochastically ultimate boundedness, stochastic permanence, and extinction as well as asymptotic estimations of solutions.

Motivated by the works mentioned above, in this paper, we consider a more general delay stochastic Gilpin-Ayala model under Markovian switching

$$\begin{aligned} dx(t) &= x(t) \\ &\cdot [r(\gamma(t)) - a(\gamma(t))x^\lambda(t) + b(\gamma(t))x^\lambda(t-\tau)] dt \\ &+ \sigma_1(\gamma(t))x(t) dB_1(t) + \sigma_2(\gamma(t))x^{1+\theta}(t) dB_2(t), \end{aligned} \quad (6)$$

with the initial conditions

$$x(s) = \varphi(s) > 0, \quad s \in [-\tau, 0]; \quad \gamma(0) = i, \quad (7)$$

where $\tau > 0$, $\varphi \in C([-\tau, 0], R_+)$; $R_+ = (0, +\infty)$; $\lambda > 0$, $\theta > 0$; $r, a : S \rightarrow R_+$; $b, \sigma_1, \sigma_2 : S \rightarrow R$; $S = \{1, 2, \dots, m\}$.

Throughout this paper, let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a complete probability space with filtration $\{F_t\}_{t \geq 0}$, where F_t is right continuous and F_0 contains all P -null sets. The one-dimensional Brownian motions $B_1(t)$ and $B_2(t)$ are defined in this space.

Let $\gamma(t)$ be a right-continuous Markov chain on the probability space, taking values in S , and with infinitesimal generator $Q = (q_{ij}) \in R^{m \times m}$ given by

$$\begin{aligned} P\{\gamma(t+\delta) = j \mid \gamma(t) = i\} \\ = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } i = j, \end{cases} \end{aligned} \quad (8)$$

where $\delta > 0$, $q_{ij} \geq 0$ is the transitions rate from i to j for $i \neq j$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$ for each $i, j \in S$. We assume that the Markov chain $\gamma(t)$ is irreducible, and $B_1(t), B_2(t)$, and $\gamma(t)$ are independent under this condition; the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \dots, \pi_m) \in R^{1 \times m}$, which can be determined by solving the following equation:

$$\pi Q = 0, \quad (9)$$

subject to $\sum_{i=1}^m \pi_i = 1$ and $\pi_i > 0, i \in S$.

For convenience, for any $f : S \rightarrow R$, denote

$$\begin{aligned} \hat{f} &= \max_{i \in S} \{f(i)\}, \\ \check{f} &= \min_{i \in S} \{f(i)\}, \\ \bar{f} &= \max_{i \in S} \{|f(i)|\}. \end{aligned} \quad (10)$$

For (6), we give the following conditions:

- (A1) $\sigma_2(i) \neq 0$;
- (A2) $\bar{b} - a(i) < \sigma_2(i) = 0$;
- (A3) $\bar{b} - a(i) = \sigma_2(i) = 0, \lambda < 1/2, r(i) < (1/4)\sigma_1^2(i)$;
- (A4) $\bar{b} - a(i) = \sigma_2(i) = 0, \lambda = 1/2, \bar{b} + a(i) + r(i) \leq (1/4)\sigma_1^2(i)$;
- (A5) $\bar{b} - a(i) < (1/4)\sigma_2^2(i)$;
- (A6) $\bar{b} - a(i) = (1/4)\sigma_2^2(i), \lambda < 1/2, r(i) < (1/4)\sigma_1^2(i)$;
- (A7) $\bar{b} - a(i) = (1/4)\sigma_2^2(i), \lambda = 1/2, \bar{b} + a(i) + (1/2)\sigma_2^2(i) + r(i) \leq (1/4)\sigma_1^2(i)$;
- (A8) $\bar{b} < a(i)$;
- (A9) $\bar{b} = a(i), \lambda < 2\theta + 1/2, \sigma_2(i) \neq 0$;
- (A10) $\bar{b} = a(i), \lambda < 2\theta + 1/2, \sigma_2(i) = 0, \lambda < 1/2, r(i) < (1/4)\sigma_1^2(i)$;
- (A11) $\bar{b} = a(i), \lambda = 2\theta + 1/2, \bar{b} + a(i) < (1/4)\sigma_2^2(i)$;
- (A12) $\bar{b} = a(i), \lambda = 2\theta + 1/2, \bar{b} + a(i) = (1/4)\sigma_2^2(i), 2\theta < 1/2, r(i) < (1/4)\sigma_1^2(i)$;
- (A13) $\bar{b} = a(i), \lambda = 2\theta + 1/2, \bar{b} + a(i) = (1/4)\sigma_2^2(i), 2\theta = 1/2, r(i) + (1/2)\sigma_2^2(i) < (1/4)\sigma_1^2(i)$.
- (H1) $\lambda < 2\theta$; and for every $i \in S$, one of conditions (A1)-(A4) holds;
- (H2) $\lambda = 2\theta$; and for every $i \in S$, one of conditions (A5)-(A7) holds;
- (H3) $\lambda > 2\theta$; and for every $i \in S$, one of conditions (A8)-(A13) holds;
- (H4) $\lambda < 2\theta$; and for every $i \in S$, (A1) or (A4) holds;
- (H5) $\lambda = 2\theta, \bar{b} < a(i) + (1/4)\sigma_2^2(i), i \in S$;
- (H6) $\lambda > 2\theta, \bar{b} < \check{a}$;
- (H7) there exists $u \in S$ such that for any $i \in S \setminus \{u\}, q_{iu} > 0$;
- (H8) $\sum_{i=1}^m \pi_i \beta(i) > 0$, where $\beta(i) = r(i) - (1/2)\sigma_1^2(i), i \in S$;
- (H9) $\bar{b} < \check{a}$.

2. Uniqueness of Global Positive Solution

As $x(t)$ in (6) denotes population size at time t , it should be nonnegative. Hence, we shall consider the existence and uniqueness of global positive solution of (6).

Theorem 1. *If one of conditions (H1)-(H3) holds, then, for any initial value $\varphi \in C([-\tau, 0], R_+)$, $i \in S$, there exists a unique solution $x(t)$ to (6) on $[-\tau, +\infty)$ and $x(t)$ will remain in R_+ with probability 1.*

Proof. Since the coefficients of (6) are locally Lipschitz continuous, then there is a unique maximal local solution $x(t)$ on $[-\tau, \tau_e)$, where τ_e is the explosion time. Let $k_0 > 0$ satisfying

$$\frac{1}{k_0} < \min_{-\tau \leq t \leq 0} \varphi(t) \leq \max_{-\tau \leq t \leq 0} \varphi(t) < k_0. \quad (11)$$

For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k \right) \right\}. \quad (12)$$

Here, set $\inf \emptyset = \infty$. Obviously, τ_k is increasing as $k \rightarrow \infty$ and $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., then $x(t) \in R_+$, for $t \geq 0$ and $\tau_e = \infty$ a.s. Thus, to prove this theorem, we need to prove $\tau_\infty = \infty$ a.s. Let

$$V(x) = \sqrt{x} - 1 - \frac{1}{2} \ln x, \quad x > 0. \quad (13)$$

Clearly, $V(x) \geq 0$ for any $x > 0$. Applying Itô's formula to $V(x)$, we have

$$\begin{aligned} dV(x(t)) &= LV(x(t), x(t-\tau), \gamma(t)) dt \\ &+ \frac{\sqrt{x(t)} - 1}{2} [\sigma_1(\gamma(t)) dB_1(t) \\ &+ \sigma_2(\gamma(t)) x^\theta(t) dB_2(t)], \end{aligned} \quad (14)$$

where $LV : R_+ \times R_+ \times S \rightarrow R$ is defined by

$$\begin{aligned} LV(x, y, i) &= -\frac{1}{4} \left(\frac{1}{2} x^{1/2} - 1 \right) [\sigma_1^2(i) + \sigma_2^2(i) x^{2\theta}] \\ &+ \frac{1}{2} (x^{1/2} - 1) [r(i) - a(i) x^\lambda] \\ &+ \frac{1}{2} b(i) (x^{1/2} - 1) y^\lambda. \end{aligned} \quad (15)$$

By Young's inequality,

$$\begin{aligned} LV(x, y, i) &\leq -\frac{1}{4} \left(\frac{1}{2} x^{1/2} - 1 \right) [\sigma_1^2(i) + \sigma_2^2(i) x^{2\theta}] \\ &+ \frac{1}{2} (x^{1/2} - 1) [r(i) - a(i) x^\lambda] + \frac{1}{2} \bar{b} x^{1/2} y^\lambda \\ &+ \frac{1}{2} \bar{b} y^\lambda \\ &\leq -\frac{1}{4} \left(\frac{1}{2} x^{1/2} - 1 \right) [\sigma_1^2(i) + \sigma_2^2(i) x^{2\theta}] \\ &+ \frac{1}{2} (x^{1/2} - 1) [r(i) - a(i) x^\lambda] \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{2} \bar{b} \left(\frac{1/2}{1/2 + \lambda} x^{1/2+\lambda} + \frac{\lambda}{1/2 + \lambda} y^{1/2+\lambda} \right) \\ &+ \frac{1}{2} \bar{b} y^\lambda \\ &= -\frac{1}{4} \left(\frac{1}{2} x^{1/2} - 1 \right) [\sigma_1^2(i) + \sigma_2^2(i) x^{2\theta}] \\ &+ \frac{1}{2} (x^{1/2} - 1) [r(i) - a(i) x^\lambda] + \frac{(1/2) \bar{b}}{1 + 2\lambda} x^{1/2+\lambda} \\ &+ \frac{\bar{b} \lambda}{1 + 2\lambda} x^{1/2+\lambda} + \frac{1}{2} \bar{b} x^\lambda \\ &+ \frac{\bar{b} \lambda}{1 + 2\lambda} (y^{1/2+\lambda} - x^{1/2+\lambda}) + \frac{1}{2} \bar{b} (y^\lambda - x^\lambda) \\ &\leq F(x, i) + \frac{\lambda \bar{b}}{1 + 2\lambda} (y^{1/2+\lambda} - x^{1/2+\lambda}) \\ &+ \frac{\bar{b}}{2} (y^\lambda - x^\lambda), \end{aligned} \quad (16)$$

where

$$\begin{aligned} F(x, i) &= -\frac{1}{4} \left(\frac{1}{2} x^{1/2} - 1 \right) [\sigma_1^2(i) + \sigma_2^2(i) x^{2\theta}] \\ &+ \frac{1}{2} r(i) (x^{1/2} - 1) + \frac{1}{2} (\bar{b} + a(i)) x^\lambda \\ &+ \frac{1}{2} (\bar{b} - a(i)) x^{1/2+\lambda}. \end{aligned} \quad (17)$$

If one of conditions (H1)-(H3) holds, then $F(x, i)$ is bounded above; i.e., there exists $M_1 > 0$ such that $F(x, i) \leq M_1, x \in R_+, i \in S$. Substituting these into (14) yields

$$\begin{aligned} dV(x(t)) &\leq \left[M_1 \right. \\ &+ \frac{\lambda \bar{b}}{1 + 2\lambda} (x^{1/2+\lambda}(t-\tau) - x^{1/2+\lambda}(t)) \\ &+ \frac{\bar{b}}{2} (x^\lambda(t-\tau) - x^\lambda(t)) \left. \right] dt + \frac{1}{2} (x^{1/2}(t) - 1) \\ &\cdot [\sigma_1(\gamma(t)) dB_1(t) + \sigma_2(\gamma(t)) x^\theta(t) dB_2(t)]. \end{aligned} \quad (18)$$

For any $T > 0$ and $t \in [0, T]$, integrating both sides of (18) from 0 to $\tau_k \wedge t$ and taking the expectations yields

$$\begin{aligned} EV(x(\tau_k \wedge t)) &\leq V(\varphi(0)) + E \int_0^{\tau_k \wedge t} \left[M_1 \right. \\ &+ \frac{\lambda \bar{b}}{1 + 2\lambda} (x^{1/2+\lambda}(s-\tau) - x^{1/2+\lambda}(s)) \\ &+ \frac{\bar{b}}{2} (x^\lambda(s-\tau) - x^\lambda(s)) \left. \right] ds. \end{aligned} \quad (19)$$

In addition,

$$\begin{aligned} E \int_0^{\tau_k \wedge t} x^{1/2+\lambda}(s-\tau) ds \\ \leq \int_{-\tau}^0 \varphi^{1/2+\lambda}(s) ds + E \int_0^{\tau_k \wedge t} x^{1/2+\lambda}(s) ds, \end{aligned} \quad (20)$$

and

$$\begin{aligned} E \int_0^{\tau_k \wedge t} x^\lambda(s-\tau) ds \leq \int_{-\tau}^0 \varphi^\lambda(s) ds \\ + E \int_0^{\tau_k \wedge t} x^\lambda(s) ds. \end{aligned} \quad (21)$$

Substituting (20) and (21) into (19) gives

$$\begin{aligned} EV(x(\tau_k \wedge t)) \leq V(\varphi(0)) + M_1 T \\ + \frac{\lambda \bar{b}}{1+2\lambda} \int_{-\tau}^0 \varphi^{1/2+\lambda}(s) ds \end{aligned}$$

Note that $x(\tau_k) = k$ or $x(\tau_k) = 1/k$. From (22), for any $T > 0$,

$$\begin{aligned} V(\varphi(0)) + M_1 T + \frac{\lambda \bar{b}}{1+2\lambda} \int_{-\tau}^0 \varphi^{1/2+\lambda}(s) ds \\ + \frac{\bar{b}}{2} \int_{-\tau}^0 \varphi^\lambda(s) ds \geq EV(x(\tau_k \wedge T)) \\ \geq E(I_{(\tau_k \leq T)} V(x(\tau_k \wedge T))) \\ = E(I_{(\tau_k \leq T)} V(x(\tau_k))) \\ \geq E\left[I_{(\tau_k \leq T)} \left(V(k) \wedge V\left(\frac{1}{k}\right)\right)\right] \\ = \left(V(k) \wedge V\left(\frac{1}{k}\right)\right) E(I_{(\tau_k \leq T)}) \\ = \left(V(k) \wedge V\left(\frac{1}{k}\right)\right) P(\tau_k \leq T), \end{aligned} \quad (23)$$

which implies

$$\limsup_{k \rightarrow \infty} P(\tau_k \leq T) \leq \lim_{k \rightarrow \infty} \frac{V(\varphi(0)) + M_1 T + (\lambda \bar{b}/(1+2\lambda)) \int_{-\tau}^0 \varphi^{1/2+\lambda}(s) ds + (\bar{b}/2) \int_{-\tau}^0 \varphi^\lambda(s) ds}{V(k) \wedge V(1/k)} = 0. \quad (24)$$

Then, $\lim_{k \rightarrow \infty} P(\tau_k \leq T) = 0$. Further, $P(\tau_\infty \leq T) = 0$. Since $T > 0$ is arbitrary, we must have $P(\tau_\infty < \infty) = 0$. That is, $\tau_\infty = \infty$ a.s. Therefore, Theorem 1 holds. $\square$

Corollary 2. Assume that (H9) holds. Then, for any initial value $\varphi \in C([- \tau, 0]; R_+)$, $\iota \in S$, there exists a unique solution $x(t)$ to (6) on $[- \tau, +\infty)$ and $x(t)$ will remain in R_+ with probability 1.

Remark 3. If $\gamma(t) \equiv \zeta \in S$, $t \in [0, +\infty)$; $r(\zeta) = r$, $a(\zeta) = a$, $b(\zeta) = 0$, $\sigma_1(\zeta) = \sigma_1$, $\sigma_2(\zeta) = \sigma_2$, $\theta = \lambda$, then (6) transforms to (2). Clearly, if $b(\zeta) = 0$, $\theta = \lambda$, then (H1) holds. Hence, Theorem 1 generalizes Lemma 2 in [9].

Remark 4. If $b(i) = 0$, $i \in S$, then (6) reduces to (3). Clearly, if $b(i) = 0$, $i \in S$, then one of conditions (H1)-(H3) holds. Hence, Theorem 1 generalizes Theorem 1 in [11].

3. Stochastic Permanence

Definition 5. Equation (6) is stochastically permanent if for any $\varepsilon \in (0, 1)$, there exist constants $H = H(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ such that for any solution $x(t)$ of (6),

$$\begin{aligned} \liminf_{t \rightarrow +\infty} P\{x(t) \leq H\} \geq 1 - \varepsilon, \\ \liminf_{t \rightarrow +\infty} P\{x(t) \geq \delta\} \geq 1 - \varepsilon. \end{aligned} \quad (25)$$

Lemma 6 (Lemma 3, [12]). If (H7) and (H8) hold, then there exists a constant $\alpha > 0$ such that the matrix

$$A(\alpha) = \text{diag}(\xi_1(\alpha), \xi_2(\alpha), \dots, \xi_m(\alpha)) - Q \quad (26)$$

is a nonsingular M -matrix, where $\xi_i(\alpha) = (1 + \theta)\alpha\beta(i) - (1/2)\alpha^2(1 + \theta)^2\sigma_1^2(i)$, $i \in S$.

Lemma 7 (Theorem 2.10, [27]). If $A = (a_{ij})_{n \times n} \in Z^{n \times n}$, then the following results are equivalent.

- (1) A is nonsingular M -matrix.
- (2) A is semipositive; that is, there is $x \gg 0$ in R^n satisfying $Ax \gg 0$.

Lemma 8. Assume that one of the following conditions holds:

- (i) $0 < p < 1$ and (H4) holds
- (ii) $p > 0$, $\bar{b} < a(i) + (1/2)(1 - p)\sigma_2^2(i)$, $i \in S$ and (H5) holds
- (iii) $p > 0$ and (H6) holds

For any initial value $\varphi \in C([- \tau, 0]; R_+)$, $\iota \in S$, let $x(t)$ be the solution of (6) with the initial conditions (7). Then there exists $K_1(p) > 0$ such that

$$\limsup_{t \rightarrow +\infty} E(x^p(t)) \leq K_1(p). \quad (27)$$

Proof. From Theorem 1, if one of conditions (H4)-(H6) holds, then (6) exists as a unique global positive solution

$x(t)$. Now, we prove Lemma 3 if (ii) holds. For $i \in S$, let

$$\mu(i) = \begin{cases} \frac{1}{\tau} \ln \frac{(\lambda + p)a(i) - p\bar{b} + (1/2)(\lambda + p)(1 - p)\sigma_2^2(i)}{\lambda\bar{b}}, & \bar{b} > 0 \\ 1, & \bar{b} = 0. \end{cases} \quad (28)$$

It follows from (ii) that $\mu(i) > 0, i \in S$. Let $\beta > 0$ satisfying $\beta < \min_{i \in S} \{\mu(i)\}$. Applying Itô's formula to $U = e^{\beta t} x^p$ yields

$$\begin{aligned} dU(x(t), t) &= LU(x(t), x(t - \tau), t, \gamma(t)) dt \\ &\quad + p\sigma_1(\gamma(t)) e^{\beta t} x^p(t) dB_1(t) \\ &\quad + p\sigma_2(\gamma(t)) e^{\beta t} x^{p+\theta}(t) dB_2(t), \end{aligned} \quad (29)$$

where $LU : R_+ \times R_+ \times [0, +\infty) \times S \rightarrow R$ is defined by

$$\begin{aligned} LU(x, y, t, i) &= e^{\beta t} \left(\beta + pr(i) + \frac{1}{2}p(p-1)\sigma_1^2(i) \right) x^p \\ &\quad - pa(i) e^{\beta t} x^{\lambda+p} + \frac{1}{2}p(p-1) e^{\beta t} \sigma_2^2(i) x^{p+2\theta} \\ &\quad + pb(i) e^{\beta t} x^p y^\lambda. \end{aligned} \quad (30)$$

Applying Young's inequality and (H5), we have

$$\begin{aligned} LU(x, y, t, i) &\leq e^{\beta t} \left(\beta + pr(i) + \frac{1}{2}p(p-1)\sigma_1^2(i) \right) x^p \\ &\quad - pa(i) e^{\beta t} x^{\lambda+p} + \frac{1}{2}p(p-1) e^{\beta t} \sigma_2^2(i) x^{p+2\theta} \\ &\quad + p\bar{b}e^{\beta t} \left(\frac{px^{\lambda+p}}{\lambda+p} + \frac{\lambda y^{\lambda+p}}{\lambda+p} \right) \\ &= e^{\beta t} F_1(x, i) + \frac{p\bar{b}\lambda}{\lambda+p} e^{\beta t} (-e^{\beta\tau} x^{\lambda+p} + y^{\lambda+p}), \end{aligned} \quad (31)$$

where

$$\begin{aligned} F_1(x, i) &= \left(\beta + pr(i) + \frac{1}{2}p(p-1)\sigma_1^2(i) \right) x^p + \frac{1}{2} \\ &\quad \cdot p(p-1)\sigma_2^2(i) x^{p+2\theta} + \left(\frac{\bar{b}p^2}{\lambda+p} - pa(i) \right. \\ &\quad \left. + \frac{\bar{b}p\lambda}{\lambda+p} e^{\beta\tau} \right) x^{\lambda+p} = \left(\beta + pr(i) \right. \\ &\quad \left. + \frac{1}{2}p(p-1)\sigma_1^2(i) \right) x^p + \left(\frac{1}{2}p(p-1)\sigma_2^2(i) \right. \\ &\quad \left. + \frac{\bar{b}p^2}{\lambda+p} - pa(i) + \frac{\bar{b}p\lambda}{\lambda+p} e^{\beta\tau} \right) x^{\lambda+p}. \end{aligned} \quad (32)$$

From condition (ii) and $\beta < \min_{i \in S} \{\mu(i)\}$, one can show that

$$\frac{1}{2}p(p-1)\sigma_2^2(i) + \frac{\bar{b}p^2}{\lambda+p} - pa(i) + \frac{\bar{b}p\lambda}{\lambda+p} e^{\beta\tau} < 0, \quad (33)$$

$i \in S$.

Hence, there exists $M_2 = M_2(p) > 0$ such that $F_1(x, i) \leq M_2, x \in R_+, i \in S$. Thus,

$$\begin{aligned} LU(x, y, t, i) &\leq M_2 e^{\beta t} \\ &\quad + \frac{p\bar{b}\lambda}{\lambda+p} e^{\beta t} (-e^{\beta\tau} x^{\lambda+p} + y^{\lambda+p}). \end{aligned} \quad (34)$$

In addition,

$$\begin{aligned} \int_0^t e^{\beta s} x^{\lambda+p}(s - \tau) ds &\leq e^{\beta\tau} \int_{-\tau}^0 \varphi^{\lambda+p}(s) ds \\ &\quad + e^{\beta\tau} \int_0^t e^{\beta s} x^{\lambda+p}(s) ds. \end{aligned} \quad (35)$$

From (29), (34), and (35), we have

$$\begin{aligned} e^{\beta t} E(x^p(t)) &\leq \varphi^p(0) + \frac{M_2}{\beta} (e^{\beta t} - 1) \\ &\quad + \frac{p\bar{b}\lambda}{\lambda+p} e^{\beta\tau} \int_{-\tau}^0 \varphi^{\lambda+p}(s) ds, \end{aligned} \quad (36)$$

which yields

$$\limsup_{t \rightarrow +\infty} E(x^p(t)) \leq \frac{M_2}{\beta} =: K_1(p). \quad (37)$$

That is, (27) holds. If (i) or (iii) holds, the proof is similar and omitted. Hence, Lemma 8 holds. $\square$

Remark 9. If $\lambda = 1, \sigma_2(i) = 0, i \in S$ and (H6) holds, then condition (A1') in [26] holds. Hence, Lemma 8 generalizes Theorem 3.3 in [26].

Remark 10. If $\lambda > 2\theta, p > 0, b(i) = 0, i \in S$, then (H6) holds. Hence, Lemma 8 improves and generalizes Lemma 6 in [11].

Theorem 11. Assume that (H7) and (H8) and one of conditions (H4)-(H6) hold. If $0 < \theta \leq 1, 0 < \lambda \leq 1 + \theta$ and $b(i) \geq 0, i \in S$, then (6) is stochastically permanent.

Proof. Clearly, if one of conditions (H4)-(H6) holds, then one of conditions (H1)-(H3) holds. From Theorem 1, for any given $\varphi \in C([- \tau, 0], R_+)$, $\iota \in S$, let x be the solution of the initial value problem (6) and (7). Define

$$V_1(x) = \frac{1}{x^{1+\theta}}, \quad x \in R_+. \quad (38)$$

Using the generalized Itô's formula, we have

$$\begin{aligned} dV_1(x(t)) &= (1+\theta) V_1(x(t)) [a(\gamma(t)) x^\lambda(t) \\ &\quad - b(\gamma(t)) x^\lambda(t-\tau) - r(\gamma(t))] dt + \frac{1}{2} (1+\theta) (2 \\ &\quad + \theta) V_1(x(t)) \sigma_1^2(\gamma(t)) dt + \frac{1}{2} (1+\theta) (2+\theta) \\ &\quad \cdot \sigma_2^2(\gamma(t)) x^{\theta-1}(t) dt - (1+\theta) V_1(x(t)) \\ &\quad \cdot \sigma_1(\gamma(t)) dB_1(t) - (1+\theta) \sigma_2(\gamma(t)) \\ &\quad \cdot x^{-1}(t) dB_2(t). \end{aligned} \quad (39)$$

From Lemmas 6 and 7, there is $\alpha > 0, w = (p_1, p_2, \dots, p_m)^T \gg 0$ such that $A(\alpha)w \gg 0$. Further, for any $i \in S$,

$$\begin{aligned} p_i \left[(1+\theta) \alpha \beta(i) - \frac{1}{2} (1+\theta)^2 \alpha^2 \sigma_1^2(i) \right] - \sum_{j=1}^m q_{ij} p_j \\ > 0. \end{aligned} \quad (40)$$

Define $V_2(x, i) = p_i(1 + V_1(x))^\alpha$. By the generalized Itô's formula again,

$$\begin{aligned} EV_2(x(t), \gamma(t)) \\ &= V_2(\varphi(0), \iota) \\ &\quad + E \int_0^t LV_2(x(s), x(s-\tau), \gamma(s)) ds, \end{aligned} \quad (41)$$

where $LV_2 : R_+ \times R_+ \times S \rightarrow R$ is defined by

$$\begin{aligned} LV_2(x, y, i) &= \alpha p_i (1+\theta) (1 + V_1(x))^{\alpha-2} \left\{ (1 \right. \\ &\quad + V_1(x)) \left[V_1(x) (a(i) x^\lambda - b(i) y^\lambda - r(i)) \right. \\ &\quad \left. \left. + \frac{1}{2} (2+\theta) \sigma_1^2(i) V_1(x) + \frac{1}{2} (2+\theta) \sigma_2^2(i) x^{\theta-1} \right] \right\} \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{2} (\alpha-1) (1+\theta) \sigma_1^2(i) V_1^2(x) + \frac{1}{2} (\alpha-1) (1+\theta) \\ &\cdot \sigma_2^2(i) x^{-2} \left\} + (1 + V_1(x))^\alpha \sum_{j=1}^m q_{ij} p_j = (1 \right. \\ &+ V_1(x))^{\alpha-2} \left\{ - \left[p_i (1+\theta) \alpha \beta(i) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} p_i (1+\theta)^2 \alpha^2 \sigma_1^2(i) - \sum_{j=1}^m q_{ij} p_j \right] V_1^2(x) \right. \\ &\quad \left. + \left[-p_i (1+\theta) \alpha r(i) + \frac{1}{2} p_i \alpha (1+\theta) (2+\theta) \sigma_1^2(i) \right. \right. \\ &\quad \left. \left. + 2 \sum_{j=1}^m q_{ij} p_j \right] V_1(x) + \sum_{j=1}^m q_{ij} p_j + p_i \alpha (1+\theta) a(i) \right. \\ &\quad \cdot V_1(x) x^{\lambda-1-\theta} + p_i \alpha (1+\theta) a(i) x^{\lambda-1-\theta} + \frac{1}{2} p_i \alpha (1 \\ &\quad + \theta) (2+\theta) \sigma_2^2(i) x^{\theta-1} + \frac{1}{2} p_i \alpha (\alpha-1) (1+\theta)^2 \sigma_2^2(i) \\ &\quad \cdot x^{-2} + \frac{1}{2} p_i \alpha (1+\theta) (2+\theta) \sigma_2^2(i) x^{-2} - p_i \alpha (1 \\ &\quad \left. + \theta) b(i) y^\lambda V_1(x) (1 + V_1(x)) \right\}. \end{aligned} \quad (42)$$

It follows from (40) that there exists $\eta > 0$ such that for any $i \in S$,

$$\begin{aligned} p_i \left[(1+\theta) \alpha \beta(i) - \frac{1}{2} (1+\theta)^2 \alpha^2 \sigma_1^2(i) \right] - \sum_{j=1}^m q_{ij} p_j \\ - \eta p_i > 0. \end{aligned} \quad (43)$$

From the generalized Itô's formula, we have

$$\begin{aligned} E[e^{\eta t} V_2(x(t), \gamma(t))] &= V_2(\varphi(0), \iota) \\ &\quad + E \int_0^t [\eta e^{\eta s} V_2(x(s), \gamma(s)) \\ &\quad + e^{\eta s} LV_2(x(s), x(s-\tau), \gamma(s))] ds. \end{aligned} \quad (44)$$

In addition,

$$\begin{aligned}
 & \eta e^{\eta t} V_2(x, i) + e^{\eta t} L V_2(x, y, i) = e^{\eta t} (1 + V_1(x))^{\alpha-2} \\
 & \cdot \left\{ - \left[p_i (1 + \theta) \alpha \beta(i) - \frac{1}{2} p_i (1 + \theta)^2 \alpha^2 \sigma_1^2(i) \right. \right. \\
 & \left. - \sum_{j=1}^m q_{ij} p_j - \eta p_i \right] V_1^2(x) + \left[-p_i (1 + \theta) \alpha r(i) \right. \\
 & \left. + \frac{1}{2} p_i \alpha (1 + \theta) (2 + \theta) \sigma_1^2(i) + 2 \sum_{j=1}^m q_{ij} p_j + 2 \eta p_i \right] \\
 & \cdot V_1(x) + \eta p_i + \sum_{j=1}^m q_{ij} p_j + p_i \alpha (1 + \theta) a(i) V_1(x) \\
 & \cdot x^{\lambda-1-\theta} + p_i \alpha (1 + \theta) a(i) x^{\lambda-1-\theta} + \frac{1}{2} p_i \alpha (1 + \theta) \\
 & \cdot (2 + \theta) \sigma_2^2(i) x^{\theta-1} + \frac{1}{2} p_i \alpha (\alpha - 1) (1 + \theta)^2 \sigma_2^2(i) x^{-2} \\
 & + \frac{1}{2} p_i \alpha (1 + \theta) (2 + \theta) \sigma_2^2(i) x^{-2} - p_i \alpha (1 + \theta) b(i) \\
 & \cdot y^\lambda V_1(x) (1 + V_1(x)) \left. \right\} \leq e^{\eta t} F_2(x, i), \quad (45)
 \end{aligned}$$

where

$$\begin{aligned}
 F_2(x, i) = (1 + V_1(x))^{\alpha-2} & \left\{ - \left[p_i (1 + \theta) \alpha \beta(i) \right. \right. \\
 & \left. - \frac{1}{2} p_i (1 + \theta)^2 \alpha^2 \sigma_1^2(i) - \sum_{j=1}^m q_{ij} p_j - \eta p_i \right] V_1^2(x) \\
 & + \left[-p_i (1 + \theta) \alpha \check{r} + \frac{1}{2} p_i \alpha (1 + \theta) (2 + \theta) \widehat{\sigma}_1^2 \right. \\
 & \left. + 2 \sum_{j=1}^m q_{ij} p_j + 2 \eta p_i \right] V_1(x) + \eta p_i + \sum_{j=1}^m q_{ij} p_j \\
 & + p_i \alpha (1 + \theta) \widehat{a} V_1(x) x^{\lambda-1-\theta} + p_i \alpha (1 + \theta) \widehat{a} x^{\lambda-1-\theta} \\
 & + \frac{1}{2} p_i \alpha (2 + \theta)^2 \widehat{\sigma}_2^2 x^{\theta-1} + \frac{1}{2} p_i \alpha^2 (1 + \theta)^2 \widehat{\sigma}_2^2 x^{-2} \\
 & \left. + \frac{1}{2} p_i (2 + \theta)^2 \widehat{\sigma}_2^2 x^{-2} \right\}. \quad (46)
 \end{aligned}$$

Further,

$$\lim_{x \rightarrow +\infty} F_2(x, i) = \begin{cases} \eta p_i + \sum_{j=1}^m q_{ij} p_j, & \theta < 1, \lambda < 1 + \theta, \\ \eta p_i + \sum_{j=1}^m q_{ij} p_j + \frac{1}{2} p_i \alpha (2 + \theta)^2 \widehat{\sigma}_2^2, & \theta = 1, \lambda < 1 + \theta, \\ \eta p_i + \sum_{j=1}^m q_{ij} p_j + p_i \alpha (1 + \theta) \widehat{a}, & \theta < 1, \lambda = 1 + \theta, \\ \eta p_i + \sum_{j=1}^m q_{ij} p_j + \frac{1}{2} p_i \alpha (2 + \theta)^2 \widehat{\sigma}_2^2 + p_i \alpha (1 + \theta) \widehat{a}, & \theta = 1, \lambda = 1 + \theta. \end{cases} \quad (47)$$

Then there is $M_3 > 0$ such that $\sup_{x \in R_+, i \in S} F(x, i) \leq M_3$. Thus, for any $i \in S$,

$$\begin{aligned}
 & p_i E \left[e^{\eta t} (1 + V_1(x(t)))^\alpha \right] \\
 & \leq p_i (1 + V_1(\varphi(0)))^\alpha + \frac{M_3 (e^{\eta t} - 1)}{\eta}. \quad (48)
 \end{aligned}$$

Further,

$$\begin{aligned}
 & \limsup_{t \rightarrow +\infty} E [V_1^\alpha(x(t))] \leq \limsup_{t \rightarrow +\infty} E [(1 + V_1(x(t)))^\alpha] \\
 & \leq \frac{M_3}{\eta \min \{p_1, p_2, \dots, p_m\}} \\
 & =: M_4. \quad (49)
 \end{aligned}$$

That is, $\limsup_{t \rightarrow +\infty} E[x^{-\alpha-\alpha\theta}(t)] \leq M_4$. For any $\varepsilon > 0$, let $\delta = (\varepsilon/M_4)^{1/(\alpha+\alpha\theta)}$. From Chebyshev's inequality,

$$\begin{aligned}
 P\{x(t) < \delta\} &= P\{x^{-\alpha-\alpha\theta}(t) > \delta^{-\alpha-\alpha\theta}\} \\
 &\leq \frac{E[x^{-\alpha-\alpha\theta}(t)]}{\delta^{-\alpha-\alpha\theta}} = \delta^{\alpha+\alpha\theta} E[x^{-\alpha-\alpha\theta}(t)]. \quad (50)
 \end{aligned}$$

Further, $\limsup_{t \rightarrow +\infty} P\{x(t) < \delta\} \leq \delta^{\alpha+\alpha\theta} M_4 = \varepsilon$. This yields

$$\liminf_{t \rightarrow +\infty} P\{x(t) \geq \delta\} \geq 1 - \varepsilon. \quad (51)$$

Next we claim that for any $\varepsilon > 0$, there exists $H > 0$ such that $\liminf_{t \rightarrow +\infty} P\{x(t) \leq H\} \geq 1 - \varepsilon$. Let $p \in (0, 1/2)$. From Lemma 8, there exists $K_1(p) > 0$ such that $\limsup_{t \rightarrow +\infty} E(x^p(t)) \leq K_1(p)$.

For any $\varepsilon > 0$, let $H = (\varepsilon/K_1(p))^{-1/p}$. Using Chebyshev's inequality yields

$$P(x(t) > H) = P(x^p(t) > H^p) \leq H^{-p} E(x^p(t)), \quad (52)$$

which means that

$$\limsup_{t \rightarrow +\infty} P\{x(t) > H\} \leq \limsup_{t \rightarrow +\infty} H^{-p} E(x^p(t)) = \varepsilon. \quad (53)$$

Hence,

$$\liminf_{t \rightarrow +\infty} P\{x(t) \leq H\} \geq 1 - \varepsilon. \quad (54)$$

From (51), (54), and Definition 5, Theorem 11 holds. $\square$

4. Asymptotic Estimation

In this section, we study some asymptotic properties of (6).

Lemma 12 ((Theorem 2.13, [27]) (BDG's inequality)). *Let $g \in \mathfrak{L}^2(\mathbb{R}_+; \mathbb{R}^{m \times n})$. Define, for $t \geq 0$, $x(t) = \int_0^t g(s) dB(s)$ and $A(t) = \int_0^t |g(s)|^2 ds$. Then for every $p > 0$, there exist universal positive constants c_p, C_p , which are only dependent on p , such that*

$$c_p E|A(t)|^{p/2} \leq E\left(\sup_{0 \leq s \leq t} |x(s)|^p\right) \leq C_p E|A(t)|^{p/2} \quad (55)$$

for all $t \geq 0$. In particular, one may take

$$\begin{aligned} c_p &= \left(\frac{p}{2}\right)^p, \\ C_p &= \left(\frac{32}{p}\right)^{p/2}, \\ &\text{if } 0 < p < 2; \\ c_p &= 1, \\ C_p &= 4, \\ &\text{if } p = 2; \\ c_p &= (2p)^{-p/2}, \\ C_p &= \left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{p/2}, \\ &\text{if } p > 2. \end{aligned} \quad (56)$$

Theorem 13. *Assume that one of conditions (H4)-(H6) holds. For any $\varphi \in C([- \tau, 0]; \mathbb{R}_+)$, $\iota \in S$, let $x(t)$ be the solution of the initial value problem (6) and (7).*

(i) *If (H4) holds, then*

$$\limsup_{t \rightarrow +\infty} \frac{\ln x(t)}{\ln t} \leq 2 \quad \text{a.s.} \quad (57)$$

(ii) *If (H5) holds and $\bar{b} < a(i) + (1/2)[1 - \max\{2, 2\lambda, 1 + \lambda\}]\sigma_2^2(i)$, $i \in S$ or (H6) holds, then*

$$\limsup_{t \rightarrow +\infty} \frac{\ln x(t)}{\ln t} \leq 1 \quad \text{a.s.} \quad (58)$$

Proof. (i) Assume that (H4) holds. Applying the generalized Itô's formula to $e^t \ln x(t)$ yields

$$\begin{aligned} e^t \ln x(t) - \ln \varphi(0) &= \int_0^t e^s \left[\ln x(s) + \beta(\gamma(s)) \right. \\ &\quad \left. - a(\gamma(s)) x^\lambda(s) + b(\gamma(s)) x^\lambda(s - \tau) \right. \\ &\quad \left. - \frac{1}{2} \sigma_2^2(\gamma(s)) x^{2\theta}(s) \right] ds + N_1(t) + N_2(t), \end{aligned} \quad (59)$$

where $N_1(t) = \int_0^t e^s \sigma_1(\gamma(s)) dB_1(s)$, $N_2(t) = \int_0^t e^s \sigma_2(\gamma(s)) x^\theta(s) dB_2(s)$. For any $\rho > 1$, $\nu > 0$ and $n \in \mathbb{Z}^+$, it follows from exponential martingale inequality that

$$\begin{aligned} P \left\{ \sup_{0 \leq t \leq \nu n} \left\{ N_i(t) - \frac{1}{2} e^{-\nu n} \langle N_i(t), N_i(t) \rangle \right\} \right. \\ \left. > \rho e^{\nu n} \ln n \right\} \leq \frac{1}{n^\rho}, \quad i = 1, 2. \end{aligned} \quad (60)$$

From the Borel-Cantelli Lemma [27], for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega)$ sufficiently large such that for $n_0 > n_0(\omega)$, $i = 1, 2$,

$$N_i(t) \leq \frac{1}{2} e^{-\nu n} \langle N_i(t), N_i(t) \rangle + \rho e^{\nu n} \ln n, \quad (61)$$

$$0 \leq t \leq \nu n.$$

This, together with (59), implies that for any $t \in [\nu(n - 3/2), \nu(n - 1/2)]$ and for almost all $\omega \in \Omega$,

$$\begin{aligned} e^t \ln x(t) &\leq \ln \varphi(0) + \int_0^t e^s \left[\ln x(s) + \beta(\gamma(s)) \right. \\ &\quad \left. - a(\gamma(s)) x^\lambda(s) + b(\gamma(s)) x^\lambda(s - \tau) \right. \\ &\quad \left. + \frac{1}{2} e^{s-\nu n} \sigma_1^2(\gamma(s)) \right. \\ &\quad \left. - \frac{1}{2} (1 - e^{s-\nu n}) \sigma_2^2(\gamma(s)) x^{2\theta}(s) \right] ds + 2\rho e^{\nu n} \\ &\quad \cdot \ln n \leq \ln \varphi(0) + \int_0^t e^s \left[\ln x(s) + \hat{\beta} \right. \\ &\quad \left. - (\check{a} - \bar{b}e^\tau) x^\lambda(s) + \frac{1}{2} e^{s-\nu n} \sigma_1^2(\gamma(s)) \right. \\ &\quad \left. - \frac{1}{2} (1 - e^{s-\nu n}) \sigma_2^2(\gamma(s)) x^{2\theta}(s) \right] ds \\ &\quad + \bar{b}e^\tau \int_{-\tau}^0 \varphi^\lambda(s) ds + 2\rho e^{\nu n} \ln n \leq \ln \varphi(0) \\ &\quad + \int_0^t e^s F_3(x(s)) ds + \bar{b}e^\tau \int_{-\tau}^0 \varphi^\lambda(s) ds + 2\rho e^{\nu n} \\ &\quad \cdot \ln n, \end{aligned} \quad (62)$$

where $F_3(x) = \ln x + \hat{\beta} - (\check{a} - \bar{b}e^\tau) x^\lambda + (1/2) e^{-(1/2)\nu} \sigma_1^2 - (1/2)(1 - e^{-(1/2)\nu}) \sigma_2^2 x^{2\theta}$. Clearly, there exists $M_5 > 0$ such that $F_3(x) \leq$

M_5 for any $x \in R_+$. Therefore, for almost all $\omega \in \Omega$, if $n \geq n_0(\omega)$, $v(n-3/2) \leq t \leq v(n-1/2)$,

$$\frac{\ln x(t)}{\ln t} \leq \frac{\ln \varphi(0)}{e^t \ln t} + M_5 \frac{1-e^{-t}}{\ln t} + \frac{\bar{b} e^\tau \int_{-\tau}^0 \varphi^\lambda(s) ds}{e^t \ln t} + 2\rho e^{-v(n-3/2)} e^{v n} \frac{\ln n}{\ln [v(n-3/2)]}. \quad (63)$$

This implies

$$\limsup_{t \rightarrow +\infty} \frac{\ln x(t)}{\ln t} \leq 2\rho e^{(3/2)v}, \quad (64)$$

which is the required assertion (57) by letting $\rho \rightarrow 1$ and $v \rightarrow 0$.

(ii) Assume that (H5) and $\bar{b} < a(i) + (1/2)[1 - \max\{2, 2\lambda, 1 + \lambda\}]\sigma_2^2(i)$, $i \in S$ or (H6) holds. It follows from Theorem 1 that for any $t \geq \tau$, $x(t) \in R_+$ a.s. This, together with (5), yields

$$\begin{aligned} dx(t) &\leq (\hat{r}x(t) + \bar{b}x(t)x^\lambda(t-\tau))dt \\ &\quad + \sigma_1(\gamma(t))x(t)dB_1(t) \\ &\quad + \sigma_2(\gamma(t))x^{1+\theta}(t)dB_2(t). \end{aligned} \quad (65)$$

Further,

$$\begin{aligned} E \left\{ \sup_{t \leq u \leq t+1} x(u) \right\} &\leq E(x(t)) + \hat{r} \int_t^{t+1} E(x(s)) ds \\ &\quad + \bar{b} \int_t^{t+1} E(x(s)x^\lambda(s-\tau)) ds \\ &\quad + E \left\{ \sup_{t \leq u \leq t+1} \int_t^u \sigma_1(\gamma(s))x(s)dB_1(s) \right\} \\ &\quad + E \left\{ \sup_{t \leq u \leq t+1} \int_t^u \sigma_2(\gamma(s))x^{1+\theta}(s)dB_2(s) \right\}. \end{aligned} \quad (66)$$

Using the BDG's inequality, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} E \left\{ \sup_{t \leq u \leq t+1} \int_t^u \sigma_1(\gamma(s))x(s)dB_1(s) \right\} &\leq C_1 E \left\{ \int_t^{t+1} [\sigma_1(\gamma(s))x(s)]^2 ds \right\}^{1/2} \\ &\leq E \left\{ \frac{1}{2} \sup_{t \leq u \leq t+1} x(u) \cdot 2C_1^2 \overline{\sigma_1^2} \int_t^{t+1} x(s) ds \right\}^{1/2} \\ &\leq \frac{1}{4} E \left(\sup_{t \leq u \leq t+1} x(u) \right) + C_1^2 \overline{\sigma_1^2} \int_t^{t+1} E(x(s)) ds, \end{aligned} \quad (67)$$

and

$$\begin{aligned} E \left\{ \sup_{t \leq u \leq t+1} \int_t^u \sigma_2(\gamma(s))x^{1+\theta}(s)dB_2(s) \right\} &\leq C_2 E \left\{ \int_t^{t+1} [\sigma_2(\gamma(s))x^{1+\theta}(s)]^2 ds \right\}^{1/2} \\ &\leq E \left\{ \frac{1}{2} \sup_{t \leq u \leq t+1} x(u) \cdot 2C_2^2 \overline{\sigma_2^2} \int_t^{t+1} x^{1+2\theta}(s) ds \right\}^{1/2} \\ &\leq \frac{1}{4} E \left(\sup_{t \leq u \leq t+1} x(u) \right) \\ &\quad + C_2^2 \overline{\sigma_2^2} \int_t^{t+1} E(x^{1+2\theta}(s)) ds. \end{aligned} \quad (68)$$

From Hölder's inequality,

$$\begin{aligned} \int_t^{t+1} E(x(s)x^\lambda(s-\tau)) ds &\leq \int_t^{t+1} (E(x^2(s)))^{1/2} (E(x^{2\lambda}(s-\tau)))^{1/2} ds \\ &\leq \frac{1}{2} \int_t^{t+1} E(x^2(s)) ds + \frac{1}{2} \int_t^{t+1} E(x^{2\lambda}(s-\tau)) ds. \end{aligned} \quad (69)$$

Hence,

$$\begin{aligned} E \left\{ \sup_{t \leq u \leq t+1} x(u) \right\} &\leq 2E(x(t)) + 2\hat{r} \int_t^{t+1} E(x(s)) ds \\ &\quad + \bar{b} \int_t^{t+1} E(x^2(s)) ds \\ &\quad + \bar{b} \int_t^{t+1} E(x^{2\lambda}(s-\tau)) ds \\ &\quad + 2C_1^2 \overline{\sigma_1^2} \int_t^{t+1} E(x(s)) ds \\ &\quad + 2C_2^2 \overline{\sigma_2^2} \int_t^{t+1} E(x^{1+2\theta}(s)) ds. \end{aligned} \quad (70)$$

It follows from (70) and Lemma 8 that

$$\begin{aligned} \limsup_{t \rightarrow +\infty} E \left(\sup_{t \leq u \leq t+1} x(u) \right) &\leq (2 + 2\hat{r} + 2C_1^2 \overline{\sigma_1^2}) K_1(1) + \bar{b} (K_1(2) + K_1(2\lambda)) \\ &\quad + 2C_2^2 \overline{\sigma_2^2} K_1(1 + 2\theta). \end{aligned} \quad (71)$$

From (71), there is $M_6 > 0$ such that

$$E \left(\sup_{k \leq t \leq k+1} x(t) \right) \leq M_6, \quad k = 1, 2, \dots \quad (72)$$

For any $\varepsilon > 0$, from Chebyshev's inequality, we have

$$P \left(\sup_{k \leq t \leq k+1} x(t) > k^{1+\varepsilon} \right) \leq \frac{M_6}{k^{1+\varepsilon}}, \quad k = 1, 2, \dots \quad (73)$$

From the Borel-Cantelli Lemma [27], for almost all $\omega \in \Omega$, there exists an integer $k_0(\omega)$ such that for any $k \geq k_0(\omega)$,

$$\sup_{k \leq t \leq k+1} x(t, \omega) \leq k^{1+\varepsilon}. \quad (74)$$

Consequently, for any $\varepsilon > 0$ and almost all $\omega \in \Omega$, if $k \geq k_0(\omega)$ and $k \leq t \leq k+1$, then

$$\frac{\ln x(t, \omega)}{\ln t} \leq \frac{(1+\varepsilon) \ln k}{\ln k} = 1 + \varepsilon. \quad (75)$$

Therefore

$$\limsup_{t \rightarrow +\infty} \frac{\ln x(t)}{\ln t} \leq 1 + \varepsilon \quad \text{a.s.} \quad (76)$$

Letting $\varepsilon \rightarrow 0$, we obtain the desired assertion (58). $\square$

From Theorem 13, it is easy to see that the following results hold.

Corollary 14. Assume that $b(i) = 0, i \in S$. For any $x_0 > 0, t \in S$, let $x(t)$ be the solution of (3) with the initial condition $x(0) = x_0$ and $\gamma(0) = \iota$. (i) If $\lambda < 2\theta$, then (57) holds; (ii) if $\lambda \geq 2\theta$, then (58) holds.

Corollary 15. Assume that $\lambda = 1, \theta = 1/4, \sigma_2(i) = 0, i \in S$. For any $\varphi \in C([- \tau, 0]; R_+)$, let $x(t)$ be the solution of the initial value problem (6) and (7). If $\bar{b} < \bar{a}$, then (58) holds.

Remark 16. Theorem 6 in [12] shows that (57) holds for any $\lambda > 0, \theta > 0$. Hence, Corollary 14 improves Theorem 6 in [12]. In addition, Corollary 15 is corresponding to Lemma 5.1 in [26]. Hence, Theorem 13 generalizes and improves Theorem 6 in [12] and Lemma 5.1 in [26].

Theorem 17. Assume that (H7), (H8), and one of conditions (H4)-(H6) hold. For any $\varphi \in C([- \tau, 0]; R_+)$, $t \in S$, let $x(t)$ be the solution of the initial value problem (6) and (7). If $0 < \theta \leq 1, 0 < \lambda \leq 1 + \theta$ and $b(i) \geq 0, i \in S$, then

$$\liminf_{t \rightarrow +\infty} \frac{\ln x(t)}{\ln t} \geq -\frac{1}{(1+\theta)\alpha} \quad \text{a.s.} \quad (77)$$

Proof. From (39) and the generalized Itô's formula,

$$\begin{aligned} d \left[(1 + V_1(x(t)))^\alpha \right] &\leq \alpha(1+\theta)(1+V_1(x(t)))^{\alpha-2} \\ &\cdot \left\{ \left[-\tilde{\beta} + \frac{1}{2}\alpha(1+\theta)\bar{\sigma}_1^2 + 1 \right] V_1^2(x(t)) + F_4(x) \right\} \\ &- \alpha(1+\theta)(1+V_1(x(t)))^{\alpha-1} \\ &\cdot \left[\sigma_1(\gamma(t)) V_1(x(t)) dB_1(t) \right. \\ &\left. + \sigma_2(\gamma(t)) x^{-1}(t) dB_2(t) \right], \end{aligned} \quad (78)$$

where

$$\begin{aligned} F_4(x) &= -V_1^2(x) + \left[-\tilde{r} + \frac{1}{2}(1+\theta)\bar{\sigma}_1^2 \right] V_1(x) \\ &+ \hat{a}V_1(x)x^{\lambda-1-\theta} + \hat{a}x^{\lambda-1-\theta} \\ &+ \frac{1}{2}(2+\theta)\bar{\sigma}_2^2 x^{\theta-1} + \frac{1}{2}\alpha(1+\theta)\bar{\sigma}_2^2 x^{-2} \\ &+ \frac{1}{2}(1+\theta)\bar{\sigma}_2^2 x^{-2}. \end{aligned} \quad (79)$$

Clearly, there exists $M_7 > 0$ such that $F_4(x) \leq M_7$ for any $x \in (0, +\infty)$. Further,

$$\begin{aligned} d \left[(1 + V_1(x(t)))^\alpha \right] &\leq \alpha(1+\theta)(1+V_1(x(t)))^{\alpha-2} \\ &\cdot \left\{ \left[-\tilde{\beta} + \frac{1}{2}\alpha(1+\theta)\bar{\sigma}_1^2 + 1 \right] V_1^2(x(t)) + M_7 \right\} dt \\ &- \alpha(1+\theta)(1+V_1(x(t)))^{\alpha-1} \\ &\cdot \left[\sigma_1(\gamma(t)) V_1(x(t)) dB_1(t) \right. \\ &\left. + \sigma_2(\gamma(t)) x^{-1}(t) dB_2(t) \right] \leq \alpha(1+\theta) M_8 \left(1 \right. \\ &\left. + V_1(x(t))^\alpha \right) dt - \alpha(1+\theta)(1+V_1(x(t)))^{\alpha-1} \\ &\cdot \left[\sigma_1(\gamma(t)) V_1(x(t)) dB_1(t) \right. \\ &\left. + \sigma_2(\gamma(t)) x^{-1}(t) dB_2(t) \right], \end{aligned} \quad (80)$$

where $M_8 = \max\{-\tilde{\beta} + (1/2)\alpha(1+\theta)\bar{\sigma}_1^2 + 1, M_7\}$ are positive numbers. By (49), there exists $M_9 > 0$ such that

$$E \left[(1 + V_1(x(t)))^\alpha \right] \leq M_9, \quad t \geq 0. \quad (81)$$

Let $\delta > 0$ satisfy

$$\alpha(1+\theta) \left[M_8\delta + 6\sqrt{\delta}(\bar{\sigma}_1 + \bar{\sigma}_2) \right] < \frac{1}{2}. \quad (82)$$

From (80), for any $k \in \{1, 2, \dots\}$,

$$\begin{aligned} E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\} \\ \leq E \left\{ [1 + V_1(x(k-1))] \right. \\ \cdot \delta]^\alpha \left. \right\} \\ + \alpha(1+\theta) M_8 E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} \int_{(k-1)\delta}^t (1 + V_1(x(s)))^\alpha ds \right\} \\ + \alpha(1+\theta) E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} \left| \int_{(k-1)\delta}^t (1 + V_1(x(s)))^{\alpha-1} \sigma_1(\gamma(s)) V_1(x(s)) dB_1(s) \right| \right\} \\ + \alpha(1+\theta) E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} \left| \int_{(k-1)\delta}^t (1 + V_1(x(s)))^{\alpha-1} \sigma_2(\gamma(s)) \right. \right. \\ \cdot x^{-1}(s) dB_2(s) \left. \right| \right\}. \end{aligned} \quad (83)$$

In addition, we have

$$\begin{aligned} E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} \int_{(k-1)\delta}^t (1 + V_1(x(s)))^\alpha ds \right\} \\ \leq E \int_{(k-1)\delta}^{k\delta} (1 + V_1(x(s)))^\alpha ds \\ \leq \delta E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\}. \end{aligned} \quad (84)$$

From the BDG's inequality,

$$\begin{aligned} E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} \left| \int_{(k-1)\delta}^t (1 + V_1(x(s)))^{\alpha-1} \sigma_1(\gamma(s)) V_1(x(s)) dB_1(s) \right| \right\} \\ \leq 6E \left\{ \int_{(k-1)\delta}^{k\delta} (1 + V_1(x(s)))^{2\alpha-2} \sigma_1^2(\gamma(s)) V_1^2(x(s)) ds \right\}^{1/2} \\ \leq 6\bar{\sigma}_1 E \left\{ \int_{(k-1)\delta}^{k\delta} (1 + V_1(x(s)))^{2\alpha} ds \right\}^{1/2} \\ \leq 6\sqrt{\delta}\bar{\sigma}_1 E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\}, \end{aligned} \quad (85)$$

and

$$\begin{aligned} E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} \left| \int_{(k-1)\delta}^t (1 + V_1(x(s)))^{\alpha-1} \sigma_2(\gamma(s)) x^{-1}(s) dB_2(s) \right| \right\} \\ \leq 6E \left\{ \int_{(k-1)\delta}^{k\delta} (1 + V_1(x(s)))^{2\alpha-2} \sigma_2^2(\gamma(s)) x^{-2}(s) ds \right\}^{1/2} \\ \leq 6\bar{\sigma}_2 E \left\{ \int_{(k-1)\delta}^{k\delta} (1 + V_1(x(s)))^{2\alpha} ds \right\}^{1/2} \\ \leq 6\sqrt{\delta}\bar{\sigma}_2 E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\}. \end{aligned} \quad (86)$$

From (83)–(86),

$$\begin{aligned} E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\} \\ \leq E \left\{ [1 + V_1(x(k-1))\delta]^\alpha \right\} + \alpha(1+\theta) \\ \cdot M_8 \delta E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\} + 6\alpha(1+\theta) \\ \cdot \sqrt{\delta}(\bar{\sigma}_1 + \bar{\sigma}_2) E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\} \\ = E \left\{ (1 + V_1(x(k-1))\delta)^\alpha \right\} + \alpha(1+\theta) \\ \cdot [M_8 \delta + 6\sqrt{\delta}(\bar{\sigma}_1 + \bar{\sigma}_2)] \\ \cdot E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\}. \end{aligned} \quad (87)$$

This, together with (81) and (82), yields

$$E \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha \right\} \leq 2M_9. \quad (88)$$

For any $\varepsilon > 0$, using Chebyshev's inequality,

$$P \left\{ \sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t)))^\alpha > (k\delta)^{1+\varepsilon} \right\} \leq \frac{2M_9}{(k\delta)^{1+\varepsilon}}. \quad (89)$$

From the Borel-Cantelli Lemma, for almost all $\omega \in \Omega$, there exists $k_0(\omega) > 0$ such that for any $k > k_0(\omega)$,

$$\sup_{(k-1)\delta \leq t \leq k\delta} (1 + V_1(x(t, \omega)))^\alpha \leq (k\delta)^{1+\varepsilon}, \quad (90)$$

which yields,

$$\frac{\ln(1 + V_1(x(t)))^\alpha}{\ln t} \leq \frac{(1+\varepsilon)\ln(k\delta)}{\ln[(k-1)\delta]} < 1 + \varepsilon \quad \text{a.s.} \quad (91)$$

Therefore

$$\limsup_{t \rightarrow +\infty} \frac{\ln(1 + V_1(x(t)))^\alpha}{\ln t} \leq 1 + \varepsilon \quad \text{a.s.} \quad (92)$$

Letting $\varepsilon \rightarrow 0$, we have $\limsup_{t \rightarrow +\infty} (\ln(1 + V_1(x(t)))^\alpha / \ln t) \leq 1$ a.s. Further,

$$\limsup_{t \rightarrow +\infty} \frac{\ln(x(t))^{-(1+\theta)\alpha}}{\ln t} \leq 1 \quad \text{a.s.} \quad (93)$$

Hence,

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \frac{\ln(x(t))^{-(1+\theta)\alpha}}{\ln t} \\ = -(1+\theta)\alpha \liminf_{t \rightarrow +\infty} \frac{\ln(x(t))}{\ln t} \leq 1 \quad \text{a.s.} \end{aligned} \quad (94)$$

This yields (77). Hence, Theorem 17 holds. $\square$

Theorem 18. Assume that the conditions of Theorem 13 hold. If (H7)–(H9) hold, then for any $\varphi \in C([- \tau, 0]; R_+)$, $\iota \in S$, the solution $x(t)$ of (6) obeys

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x^\lambda(s) ds \leq \frac{1}{\bar{a} - \bar{b}} \sum_{i=1}^m \pi_i \beta(i) \quad \text{a.s.} \quad (95)$$

Proof. From Theorems 13 and 17,

$$\lim_{t \rightarrow +\infty} \frac{\ln x(t)}{t} = 0 \quad \text{a.s.} \quad (96)$$

Using the generalized Itô's formula, we obtain

$$\begin{aligned} \ln x(t) = \ln x_0 + \int_0^t \left[r(\gamma(s)) - \frac{1}{2} \sigma_1^2(\gamma(s)) \right] ds \\ - \int_0^t a(\gamma(s)) x^\lambda(s) ds \\ + \int_0^t b(\gamma(s)) x^\lambda(s - \tau) ds \\ - \frac{1}{2} \int_0^t \sigma_2^2(\gamma(s)) x^{2\theta}(s) ds + M_1(t) \\ + M_2(t), \end{aligned} \quad (97)$$

where $M_1(t) = \int_0^t \sigma_1(\gamma(s)) dB_1(s)$, $M_2(t) = \int_0^t \sigma_2(\gamma(s)) x^\theta(s) dB_2(s)$. The quadratic variation of $M_2(t)$ is

$$\langle M_2(t), M_2(t) \rangle = \int_0^t \sigma_2^2(\gamma(s)) x^{2\theta}(s) ds. \quad (98)$$

From the exponential martingale inequality, for any integer $n \geq 2$,

$$P \left\{ \sup_{0 \leq t \leq n} \left\{ M_2(t) - \frac{1}{2} \langle M_2(t), M_2(t) \rangle \right\} > 2 \ln n \right\} \leq e^{-2 \ln n} = \frac{1}{n^2}. \quad (99)$$

From Borel-Cantelli Lemma, for almost all $\omega \in \Omega$, there exists a random integer n_0 such that for $n \geq n_0(\omega)$,

$$\sup_{0 \leq t \leq n} \left\{ M_2(t) - \frac{1}{2} \langle M_2(t), M_2(t) \rangle \right\} \leq 2 \ln n. \quad (100)$$

Further, for almost all $\omega \in \Omega$, $n \geq n_0(\omega)$, $0 \leq t \leq n$,

$$\begin{aligned} M_2(t) &\leq 2 \ln n + \frac{1}{2} \langle M_2(t), M_2(t) \rangle \\ &= 2 \ln n + \frac{1}{2} \int_0^t \sigma_2^2(\gamma(s)) x^{2\theta}(s) ds. \end{aligned} \quad (101)$$

Note that

$$\begin{aligned} \int_0^t b(\gamma(s)) x^\lambda(s - \tau) ds &\leq \bar{b} \int_0^t x^\lambda(s - \tau) ds \\ &= \bar{b} \int_{-\tau}^{t-\tau} x^\lambda(s) ds \\ &\leq \bar{b} \int_{-\tau}^0 x^\lambda(s) ds + \bar{b} \int_0^t x^\lambda(s) ds. \end{aligned} \quad (102)$$

From (97), for any $n \geq n_0$, $t \in (n-1, n]$

$$\begin{aligned} \frac{\ln x(t)}{t} &\leq \frac{\ln x_0}{t} + \frac{1}{t} \int_0^t \beta(\gamma(s)) ds \\ &\quad + \frac{-\tilde{a} + \bar{b}}{t} \int_0^t x^\lambda(s) ds + \frac{\bar{b}}{t} \int_{-\tau}^0 x^\lambda(s) ds \\ &\quad + \frac{2 \ln n}{n-1} + \frac{M_1(t)}{t}. \end{aligned} \quad (103)$$

Let $t \rightarrow +\infty$, using the strong law of large numbers for martingales and (96), we obtain

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x^\lambda(s) ds \leq \frac{1}{\tilde{a} - \bar{b}} \sum_{i=1}^m \pi_i \beta(i) \quad \text{a.s.} \quad (104)$$

We then obtain assertions (95). Hence, Theorem 18 holds. $\square$

Remark 19. Liu et al. [12] have claimed that, under (H7) and (H8), the solution of (2) obeys

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x^\lambda(s) ds \leq \frac{1}{\tilde{a}} \sum_{i=1}^m \pi_i \beta(i) \quad \text{a.s.} \quad (105)$$

It is easy to see that if $b(i) = 0$, $i \in S$, then (95) becomes (105). Hence, Theorem 18 generalizes Theorem 8 in [12].

Remark 20. If $\lambda = 1$, $\sigma_2(i) = 0$, $i \in S$, then conditions (H7)-(H9) are corresponding to the conditions (A1''), (A2), and (A3) in [26]. Hence, Theorem 18 generalizes Theorem 5.3 in [26].

5. Numerical Simulations

In this section, we make numerical simulations to illustrate our theoretical results. Consider the following examples.

Example 1. In (6), let $\lambda = 2$, $\theta = 1$, $\tau = 1$, $S = \{1, 2\}$, $q_{12} = -q_{11} = 1$, $q_{21} = -q_{22} = 2$, $r(1) = 3$, $a(1) = 4$, $b(1) = 2$, $\sigma_1(1) = 0.2$, $\sigma_2(1) = 0.2$, $r(2) = 1$, $a(2) = 3$, $b(2) = 1.5$, $\sigma_1(2) = 0.15$, $\sigma_2(2) = 0.2$ and initial value $x(s) = 1.5$, $s \in [-1, 0]$, $\gamma(0) = 1$.

Clearly, $\tilde{a} = 3$, $\bar{b} = 2$, $\beta(1) = 2.98$, $\beta(2) \approx 0.989$, $\pi_1 = 2/3$, $\pi_2 = 1/3$, $\pi_1 \beta(1) + \pi_2 \beta(2) \approx 2.316 > 0$. Hence, (H5) and (H7)-(H9) hold. Further, all conditions of Theorems 11 and 18 are satisfied. According to Theorems 11 and 18, (6) is stochastically permanent and its solution $x(t)$ with any positive initial value satisfies

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x^2(s) ds \leq 2.316 \quad \text{a.s.} \quad (106)$$

In Figure 1, (i) left panel shows stochastic trajectories of model (6) with parameters in Example 1; deterministic trajectory of model (6) with $r(i) = 3$, $a(i) = 4$, $b(i) = 2$, $\sigma_1(i) = 0$, $\sigma_2(i) = 0$, $\lambda = 2$, $\tau = 1$, $i = 1, 2$; (ii) right panel shows stochastic trajectories of model (6) with parameters in Example 1; deterministic trajectory of model (6) with $r(i) = 1$, $a(i) = 3$, $b(i) = 1.5$, $\sigma_1(i) = 0$, $\sigma_2(i) = 0$, $\lambda = 2$, $\tau = 1$, $i = 1, 2$. Figure 1 clearly supports Theorems 11 and 18.

Example 2. In (6), let $\lambda = 1.5$, $\theta = 0.7$, $\tau = 1$, $S = \{1, 2\}$, $q_{11} = q_{22} = -1$, $q_{12} = q_{21} = 1$, $r(1) = 0.4$, $a(1) = 0.15$, $b(1) = 0.1$, $\sigma_1(1) = 0.03$, $\sigma_2(1) = 0.013$, $r(2) = 0.3$, $a(2) = 0.2$, $b(2) = 0.09$, $\sigma_1(2) = 0.02$, $\sigma_2(2) = 0.021$ and initial value $x(s) = 1$, $s \in [-1, 0]$, $\gamma(0) = 2$.

Clearly, $\tilde{a} = 0.15$, $\bar{b} = 0.1$, $\beta(1) = 0.39955$, $\beta(2) = 0.2998$, $\pi_1 = 1/2$, $\pi_2 = 1/2$, $\pi_1 \beta(1) + \pi_2 \beta(2) \approx 0.3497 > 0$. Hence, (H6)-(H9) hold. Further, all conditions of Theorems 11 and 18 are satisfied. From Theorems 11 and 18, (6) is stochastically permanent and its solution $x(t)$ with any positive initial value satisfies

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x^{1.5}(s) ds \leq 6.994 \quad \text{a.s.} \quad (107)$$

In Figure 2, (i) left panel shows stochastic trajectories of model (6) with parameters in Example 2; deterministic trajectory of model (6) with $r(i) = 0.4$, $a(i) = 0.15$, $b(i) = 0.1$, $\sigma_1(i) = 0$, $\sigma_2(i) = 0$, $\lambda = 1.5$, $\tau = 1$, $i = 1, 2$; (ii) right panel shows stochastic trajectories of model (6) with parameters in Example 2; deterministic trajectory of model (6) with $r(i) = 0.3$, $a(i) = 0.2$, $b(i) = 0.09$, $\sigma_1(i) = 0$, $\sigma_2(i) = 0$, $\lambda = 2$, $\tau = 1$, $i = 1, 2$. Figure 2 clearly supports Theorems 11 and 18.

FIGURE 1: Stochastic and deterministic trajectories of model (6).

FIGURE 2: Stochastic and deterministic trajectories of model (6).

Example 3. In (6), let $\lambda = 1, \theta = 0.25, \tau = 1, S = \{1, 2\}$, $q_{12} = -q_{11} = 1, q_{21} = -q_{22} = 2, r(1) = 0.2, a(1) = 0.5, b(1) = 0.3, \sigma_1(1) = 0.4, \sigma_2(1) = 0.1, r(2) = 0, a(2) = 0.4, b(2) = 0.2, \sigma_1(2) = 0.5, \sigma_2(2) = 0$ and initial value $x(s) = 1.5, s \in [-1, 0], \gamma(0) = 1$.

Clearly, $\bar{a} = 0.4, \bar{b} = 0.3, \beta(1) = 0.12, \beta(2) = -0.125, \pi_1 = 2/3, \pi_2 = 1/3, \pi_1\beta(1) + \pi_2\beta(2) \approx 0.0383 > 0$. Hence, (H6)-(H8) hold. Hence, all conditions of Theorem 11 are satisfied. According to Theorem 11, (6) is stochastically permanent and its solution $x(t)$ with any positive initial value satisfies

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x(s) ds \leq 0.383 \quad \text{a.s.} \quad (108)$$

Figure 3 clearly supports Theorems 11 and 18.

Assume that the Markov chain $\gamma(t) = 1 \in S$. Then (6) obeys the following subsystem:

$$\begin{aligned} dx(t) = & x(t) [0.2 - 0.5x(t) + 0.3x(t-1)] dt \\ & + 0.4x(t) dB_1(t) + 0.1x^{1.25}(t) dB_2(t). \end{aligned} \quad (109)$$

Similarly, assume that the Markov chain $\gamma(t) = 2 \in S$. Then (6) obeys the following subsystem:

$$\begin{aligned} dx(t) = & x(t) [-0.4x(t) + 0.2x(t-1)] dt \\ & + 0.5x(t) dB_1(t). \end{aligned} \quad (110)$$

For subsystem (109), it is easy to check that all conditions of Theorem 11 are satisfied. Hence, subsystem (109) is stochastically permanent. In addition, for subsystem (110), $-a(2) + |b(2)| = -0.4 + 0.2 = -0.2 < 0, r(2) - 0.5\sigma_1^2(2) = -0.5 \times 0.5^2 =$

FIGURE 3: Stochastic trajectories of model (6) with parameters in Example 3.

FIGURE 4: Stochastic trajectories of subsystems (109) and (110).

$-0.125 < 0$. From Corollary 4.6 in [26], subsystem (110) is stochastically extinct. Figure 4 clearly supports these results.

In Figure 4, (i) left panel shows stochastic trajectory of subsystem (109); (ii) right panel shows stochastic trajectory of subsystem (110).

Example 3 shows that if the overall system consists of permanent subsystems and nonpermanent subsystems, the overall system can be permanent. Hence, switching component can add some interesting behavior to hybrid systems.

6. Conclusions

In this paper, we study the stochastic permanence and asymptotic estimations of solution to a general stochastic delay Gilpin-Ayala system with Markovian switching. Compared with the models in the literatures, (6) provides a more

realistic system of the population dynamics. However, the delays and Markovian switching mechanism make the task more complicated to deal with. We overcome difficulties by constructing suitable Lyapunov functionals and using some analysis technique. Under some suitable conditions, (6) still retains some well properties, for example, the existence of global positive solution and persistence and asymptotic estimations. Further, the results show that switching component can add some interesting behavior to hybrid systems driven by Markov chain. In addition, some interesting topics merit further consideration. One may propose stability distribution, ergodicity, and extinction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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