Research Article

Applications of Graph Kannan Mappings to the Damped Spring-Mass System and Deformation of an Elastic Beam

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The purpose of this article is twofold. Firstly, combining concepts of graph theory and of fixed point theory, we will present a fixed point result for Kannan type mappings, in the framework of recently introduced, graphical b -metric spaces. Appropriate examples of graphs validate the established theory. Secondly, our focus is to apply the proposed results to some nonlinear problems which are meaningful in engineering and science. Some open problems are proposed.

1. Introduction

In several recent studies, graph theory engages an influential role especially for metric fixed point theory in numerous aspects. Jachymski [1] gave a unified version of Banach's contraction principle, by considering the graph structure instead of an order structure on a metric space. This result has been extended and generalized by many researchers in different ways (see, e.g., [2–11]). Recently, Shukla et al. in [10] proposed the notion of graphical metric spaces as a graphical version of metric spaces. It is necessary to mention that the triangular inequality in [10] is required to hold only on those elements which are related to each other under the underlying graph structure rather than the whole space. Most recently, Singh et al. as a coauthor in [5, 12] extended the idea given in [10] and introduced graphical b -metric spaces and graphical rectangular b -metric spaces by showing suitable graphs. Kannan [13] was the first mathematician who derived, under some metrical conditions, a unique fixed point for a not necessary continuous mapping. Since then Kannan mappings have been the center of research for many authors from last few decades (see, e.g., [14–19] and the related references

therein). Another charm of Kannan's theorem is that one can characterize the metric completeness in terms of the fixed point of such mappings. For detailed discussion we refer the reader to [20]. Analytically, Kannan mappings play an important role in metric fixed point theory, but, interestingly, one can hardly find literature connecting Kannan type mappings to application part. In this paper efforts have been made to apply the abstract results invoking graph Kannan type mapping to some real world applications, such as damped spring-mass system of an automobile suspension system and nonlinear elastic beam equations, problems which play an indispensable role in mechanics and engineering. At last, the paper is concluded with some open problems which may lead to new results and applications. In the subsequent analysis the basic assumptions on the graphs is that they are directed graphs encompassing nonempty set of edges.

2. Notations and Basic Facts

We will present first some notations and basic concepts which are important for the main part of the paper.

Following Jachymski [1], let us denote by Δ the diagonal of $Y \times Y$, where Y is a nonempty set. Further, we suppose that G is a directed graph possessing no parallel edges and let $\mathcal{U}(G)$ be the set of all vertices of G , such that $\mathcal{U}(G)$ coincides with the set Y . Let $\mathcal{E}(G)$ be the set of all edges of G such that it contains all loops (i.e., $\Delta \subset \mathcal{E}(G)$). We will write that $G = (\mathcal{U}(G), \mathcal{E}(G))$. If we reverse the direction of edges of G , the obtained graph is denoted by G^{-1} . Furthermore, the letter $\check{G}$ denotes a directed graph with symmetric edges. More precisely, we define $\mathcal{E}(\check{G}) := \mathcal{E}(G^{-1}) \cup \mathcal{E}(G)$. Let $v, w \in \mathcal{U}(G)$. A path (or a directed path) of length m between v and w in G is defined to be a sequence $\{x_j\}_{j=0}^m$ of $(m+1)$ vertices with $v = x_0$, $w = x_m$, and $(x_{j-1}, x_j) \in \mathcal{E}(G)$ for $j = 1, 2, \dots, m$. If any two vertices of G contain a path between them, then G is called a connected graph. If there exists a path between every two vertices in an undirected graph G , then G is said to be weakly connected. We call $G^* = (\mathcal{U}(G^*), \mathcal{E}(G^*))$ a subgraph of $G = (\mathcal{U}(G), \mathcal{E}(G))$ if $\mathcal{U}(G^*) \supseteq \mathcal{U}(G)$ and $\mathcal{E}(G^*) \supseteq \mathcal{E}(G)$. We will also use (see Shukla [10]) the following notation:

$$[u]_G^l := \{v \in Y : \text{there is a directed path from } u \text{ to } v \text{ having length } l\}. \quad (1)$$

Further, a relation P on Y is defined as follows:

$$(uPv)_G \quad \text{iff there exists a path directing from } u \text{ to } v \text{ in } G. \quad (2)$$

and we will write $w \in (uPv)_G$ if w is contained in the path $(uPv)_G$. For a sequence $\{x_m\} \in Y$ if $(x_m P x_{m+1})_G$ for all $m \in \mathbb{N}$, then we say that $\{x_m\}$ is a G -termwise connected (in short G -TWC) sequence.

Singh et al. in [21, 22] proposed some notable real world problems from engineering science and invoked their results to find the solution of these problems. Recently in [5], the authors applied the connected graph theory in metric fixed point theory in a very interesting way and introduced graphical b -metric space as a generalization of b -metric space as follows.

Definition 1 (see [5]). A graphical b -metric coefficient $s \geq 1$ on a nonempty set Y is a mapping $b_G : Y \times Y \rightarrow [0, \infty)$ satisfying the following conditions:

- (G_bM1) $b_G(x, y) = 0$ if and only if $x = y$;
- (G_bM2) $b_G(x, y) = b_G(y, x)$ for all $x, y \in Y$;
- (G_bM3) $(xPy)_G, z \in (xPy)_{G_b} \implies b_G(x, y) \leq s[b_G(x, z) + b_G(z, y)]$.

The pair (Y, b_G) is called graphical b -metric space (briefly $GbMS$) space with coefficient s .

Example 2. Let $Y = \{l, m, n, o, p\}$ be endowed with graphical b -metric b_G defined by

$$r_b(u, v) = \begin{cases} \frac{q}{6}; & u \text{ or } v \notin \{m, p\}, u \neq v, \\ q; & u, v \in \{m, p\}, u \neq v, \\ 0; & u = v, \end{cases} \quad (3)$$

FIGURE 1: Graph depicting $GbMS$.

where $q > 0$ is a constant. It is easy to show that (Y, b_G) is a $GbMS$ with coefficient $s = 3$ encompassing the graph $G = (\mathcal{U}(G), \mathcal{E}(G))$ equipped with $\mathcal{U}(G) = Y$ and $\mathcal{E}(G)$ as displayed in Figure 1.

Definition 3 (see [5]). Every open ball $B_{b_G}(y, \gamma)$ centered in y and having radius $\gamma > 0$ in a $GbMS$ is an open set. Moreover, the corresponding topological space (Y, b_G) is T_1 but it is not T_2 .

Definition 4 (see [5]). A sequence $\{x_m\}$ in a $GR_bMS (Y, b_G)$ is said to be a convergent sequence if there exists $y \in Y$ such that $b_G(y_m, y) \rightarrow 0$ as $m \rightarrow \infty$ and it is a Cauchy sequence if $b_G(y_m, y_n) \rightarrow 0$ as $m, n \rightarrow \infty$.

For more details related to the topology of the underlying space and other noteworthy results one may switch to [5]. For related notions see [12].

3. Main Results

Consider $\mathcal{H}_g$ to be a subgraph of G with $\mathcal{E}(\mathcal{H}_g) \supseteq \Delta$ and further assume $\mathcal{H}_g$ to be a weighted graph. Let $y_0 \in Y$ be the initial value of a sequence $\{y_m\}$; we say $\{y_m\}$ is a g -Picard sequence (g -PS) for a mapping $g : Y \rightarrow Y$ if $y_m = gy_{m-1}$ for all $m \in \mathbb{N}$.

Furthermore, we say a graph $\mathcal{H}_g = (\mathcal{U}(\mathcal{H}_g), \mathcal{E}(\mathcal{H}_g))$ satisfies the property $(\mathcal{P})$ [10], if a $\mathcal{H}_g$ -TWC g -PS $\{y_m\}$ converging in Y ensures that there is a limit $w \in Y$ of $\{y_m\}$ and $m_0 \in \mathbb{N}$ such that $(y_m, w) \in \mathcal{E}(\mathcal{H}_g)$ or $(w, y_m) \in \mathcal{E}(\mathcal{H}_g) \forall m > m_0$.

We present now our main definition as follows.

Definition 5. Let (Y, b_G) be a $GbMS$. A self-mapping $g : Y \rightarrow Y$ is said to be $\mathcal{H}_g$ -graph Kannan mapping on (Y, b_G) if

- ($\mathbb{K}_1$) for all $y_1, y_2 \in Y$ if $(y_1, y_2) \in \mathcal{E}(\mathcal{H}_g) \implies (gx_1, gx_2) \in \mathcal{E}(\mathcal{H}_g)$;
- ($\mathbb{K}_2$) there exists $\delta \in [0, 1/2)$, $\forall y_1, y_2 \in Y$ with $(y_1, y_2) \in \mathcal{E}(\mathcal{H}_g)$, we have

$$b_G(gy_1, gy_2) \leq \frac{\delta}{s} [b_G(y_1, gy_1) + b_G(y_2, gy_2)] \quad (4)$$

Example 6. Every Kannan mapping is an $\mathcal{H}_g$ -graph Kannan mapping equipped with the graph $\mathcal{H}_g = G$ defined by $\mathcal{U}(\mathcal{H}_g) = Y$ and $\mathcal{E}(\mathcal{H}_g) = Y \times Y$

FIGURE 2: Kannan mapping $\implies \mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping.

For instance, let $Y = [0, 10]$ via the graphical b -metric b_G defined by

$$r_{G_b}(y_1, y_2) = \begin{cases} |y_1 - y_2|^2 & \text{if } y_1 \neq y_2, \\ 0 & \text{if } y_1 = y_2. \end{cases} \quad (5)$$

Obviously, (Y, b_G) is a $GbMS$ with $s = 2$. The mapping defined by $g(y) = y/(2y+1)$ is a Kannan mapping for $\delta = 12/25$. Now examine the graph $\mathcal{H}_{\mathcal{G}}$ along with $Y = \mathcal{U}(\mathcal{H}_{\mathcal{G}})$ and $\mathcal{G}(\mathcal{H}_{\mathcal{G}}) = \{(y_1, y_2) \in [0, 10] \times [0, 10] : 0 \leq y_2 - y_1\} \cup \Delta$. One can clearly see that g is a $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping; the same is validated in Figure 2.

Example 7. $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping is not necessarily a Kannan mapping.

Let $Y = \{0, 1, 2, 3, 4\}$ be endowed with the graphical b -metric as defined in Example 6. Then (Y, b_G) is a $GbMS$ with the coefficient $s = 2$. Define the mapping $g : Y \rightarrow Y$ by

$$gy = \begin{cases} 0 & \text{if } y \in \{0, 1, 2\}, \\ 1 & \text{if } y \in \{3, 4\}. \end{cases} \quad (6)$$

Now consider the graph $\mathcal{H}_{\mathcal{G}}$ for which $Y = \mathcal{U}(\mathcal{H}_{\mathcal{G}})$ and $\mathcal{G}(\mathcal{H}_{\mathcal{G}}) = \{(0, 1), (0, 2), (0, 4), (1, 2), (1, 4), (2, 4), (3, 4)\} \cup \Delta$. Then, g is $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping for $\delta = 6/25$. Figure 3 illustrates the directed graph associated with $GbMS$.

Notice that g is not a Kannan mapping, since

$$b_G(g0, g3) = 1 > \frac{24}{25} = \delta [b_G(0, g0) + b_G(3, g3)] \quad (7)$$

In view of Examples 6 and 7, the results in this article are effective generalizations of the existing state-of-the-art concerning Kannan mappings.

Now we present our main result concerning $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping as follows.

Theorem 8. Let $g : Y \rightarrow Y$ be $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping on a $\mathcal{H}_{\mathcal{G}}$ -complete $GbMS$ (Y, b_G) . The following conditions hold:

FIGURE 3: Graph depicting $GbMS$.

(a) $\mathcal{G}$ satisfies the property $(\mathcal{P})$;

(b) there exists $y_0 \in Y$ with $gy_0 \in [y_0]_{\mathcal{H}_{\mathcal{G}}}^r$ for some $r \in \mathbb{N}$.

Then, there exists $y' \in Y$ such that the g -PS $\{y_p\}$ with initial value $y_0 \in Y$ is $\mathcal{H}_{\mathcal{G}}$ -TWC and converges to y' .

Proof. Let $y_0 \in Y$ be such that $gy_0 \in [y_0]_{\mathcal{H}_{\mathcal{G}}}^r$, for some $r \in \mathbb{N}$. Since $\{y_p\}$ is a g -PS starting from y_0 ; therefore, there exists a path $\{x_i\}_{i=0}^r$ such that $y_0 = x_0$, $gy_0 = x_r$, and $(x_{i-1}, x_i) \in \mathcal{G}(\mathcal{H}_{\mathcal{G}})$ for $i = 1, 2, \dots, r$. By hypothesis g being $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping, therefore from assertion $(\mathbb{K}_1)$ we have $(gx_{i-1}, gx_i) \in \mathcal{G}(\mathcal{H}_{\mathcal{G}})$ for $i = 1, 2, \dots, r$. This implies that $\{gx_i\}_{i=0}^r$ is a path from $gx_0 = gy_0 = y_1$ to $gx_r = g^2y_0 = y_2$ having length r , and hence $y_2 \in [y_1]_{\mathcal{H}_{\mathcal{G}}}^r$. Pursuing this process, we acquire that $\{g^p x_i\}_{i=0}^r$ is a path from $g^p x_0 = g^p y_0 = y_p$ to $g^p x_r = g^p gy_0 = y_{p+1}$ of length r ; i.e., $\{g^p x_i\}_{i=0}^r$ is a path possessing length r from y_p to y_{p+1} and hence $y_{p+1} \in [y_p]_{\mathcal{H}_{\mathcal{G}}}^r$, for all $p \in \mathbb{N}$. Thus we attain that $\{y_p\}$ is a $\mathcal{H}_{\mathcal{G}}$ -TWC sequence.

Now $(g^p x_{i-1}, g^p x_i) \in \mathcal{G}(\mathcal{H}_{\mathcal{G}})$ for $i = 1, 2, \dots, r$ and $p \in \mathbb{N}$; thanks to $(\mathbb{K}_2)$, we obtain

$$\begin{aligned} b_G(g^p x_{i-1}, g^p x_i) &= b_G(g(g^{p-1} x_{i-1}), g(g^{p-1} x_i)) \\ &\leq \frac{\delta}{s} \{b_G(g^{p-1} x_{i-1}, g(g^{p-1} x_{i-1})) \\ &\quad + b_G(g^{p-1} x_i, g(g^{p-1} x_i))\} \\ &= \frac{\delta}{s} \{b_G(g^{p-1} x_{i-1}, g^{p-1} x_i) + b_G(g^p x_{i-1}, g^p x_i)\} \end{aligned} \quad (8)$$

This implies that

$$b_G(g^p x_{i-1}, g^p x_i) \leq \left(\frac{\delta}{s-\delta}\right) b_G(g^{p-1} x_{i-1}, g^{p-1} x_i) \quad (9)$$

Set $\xi = \delta/(s-\delta)$ and notice that $\xi \in [0, 1/s)$ by putting different values of δ and s . Thus, the above inequality can be written as

$$b_G(g^p x_{i-1}, g^p x_i) \leq \xi b_G(g^{p-1} x_{i-1}, g^{p-1} x_i); \quad \forall \xi \in \left[0, \frac{1}{s}\right) \quad (10)$$

Repeating this process, we get

$$b_G(g^p x_{i-1}, g^p x_i) \leq \xi^p b_G(x_{i-1}, x_i) \quad (11)$$

Since the sequence $\{y_p\}$ is $\mathcal{G}$ -TWC and $\mathcal{G}$ is a subgraph of G , using (11) and the triangle inequality, we obtain

$$\begin{aligned} b_G(y_p, y_{p+1}) &= b_G(g^p y_0, g^{p+1} y_0) = b_G(g^p x_0, g^p x_r) \\ &\leq s [b_G(g^p x_0, g^p x_1) + b_G(g^p x_1, g^p x_r)] \\ &\leq s [b_G(g^p x_0, g^p x_1)] \\ &\quad + s^2 [b_G(g^p x_1, g^p x_2)] \\ &\quad + \dots s^r [b_G(g^p x_{r-1}, g^p x_r)] \\ &\leq s \xi^p [b_G(x_0, x_1)] + s^2 \xi^p [b_G(x_1, x_2)] \\ &\quad + \dots + s^r \xi^p [b_G(x_{r-1}, x_r)] \\ &= s \xi^p \sum_{k=1}^r s^{k-1} b_G(x_{k-1}, x_k) \end{aligned} \quad (12)$$

Set $\mathcal{S}_b^r := \sum_{k=1}^r s^{k-1} b_G(x_{k-1}, x_k)$. Then, inequality (12) reduces to

$$b_G(y_p, y_{p+1}) \leq s \xi^p (\mathcal{S}_b^r). \quad (13)$$

Again, $\{y_p\}$ is $\mathcal{H}_{\mathcal{G}}$ -TWC, for $p, q \in \mathbb{N}, q > p$; we get

$$\begin{aligned} b_G(y_p, y_q) &\leq s [b_G(y_p, y_{p+1}) + b_G(y_{p+1}, y_q)] \\ &\leq s [b_G(y_p, y_{p+1})] + s^2 [b_G(y_{p+1}, y_{p+2})] \\ &\quad + s^2 [b_G(y_{p+2}, y_q)] \\ &\leq s [b_G(y_p, y_{p+1})] + s^2 [b_G(y_{p+1}, y_{p+2})] \\ &\quad + \dots + s^{q-p} [b_G(y_{q-1}, y_q)] \\ &= \sum_{k=p}^{q-1} [s^{k-p+1} b_G(y_k, y_{k+1})] \\ &\leq s \sum_{k=p}^{q-1} [s^{k-p+1} \xi^k \mathcal{S}_b^r] \\ &= s^2 \xi^p \left[\sum_{k=p}^{q-1} (s \xi)^{k-p} \right] \mathcal{S}_b^r \\ &\leq s^2 \xi^p \left[\sum_{k=1}^{\infty} (s \xi)^{k-1} \right] \mathcal{S}_b^r \\ &= s^2 \xi^p \left(\frac{1}{1 - s \xi} \right) \mathcal{S}_b^r. \end{aligned} \quad (14)$$

Since $\xi \in [0, 1/s)$, we infer that $\lim_{p, q \rightarrow \infty} b_G(y_p, y_q) = 0$. Hence $\{y_p\}$ is a Cauchy sequence in Y . Also since Y is $\mathcal{H}_{\mathcal{G}}$ -complete, therefore $\{y_p\}$ converges in Y and, by hypothesis,

there exists some $y' \in Y, p_0 \in \mathbb{N}$ such that $(y_p, y') \in \mathfrak{G}(\mathcal{G})$ or $(y', y_p) \in \mathfrak{G}(\mathcal{H}_{\mathcal{G}}) \forall p > p_0$ and

$$\lim_{p \rightarrow \infty} b_G(y_p, y') = 0, \quad (15)$$

which shows that $\{y_p\}$ converges to y' . $\square$

For the existence of a fixed point of the underlying mapping, Shukla [10, Theorem 3.10] and Chuensupantharat [5, Theorem 3.4] used the condition (S). However, we drop this condition and assume that the subgraph $\mathcal{H}_{\mathcal{G}}$ is weakly connected. This guarantees the fixed point of mapping g along with its uniqueness.

Theorem 9. *If the hypotheses embodied in Theorem 8 are satisfied and, additionally, we suppose that $\mathcal{H}_{\mathcal{G}}$ is weakly connected, then y' is the unique fixed point of g .*

Proof. Theorem 8 ensures that the g -PS $\{y_p\}$ with initial value y_0 converges to $y' \in Y$. Since $\mathcal{H}_{\mathcal{G}}$ is weakly connected, therefore $(y' P g y')_{\mathcal{H}_{\mathcal{G}}}$ or $(g y' P y')_{\mathcal{H}_{\mathcal{G}}}$, and hence we have

$$\begin{aligned} b_G(y', g y') &\leq s [b_G(y', y_p) + b_G(y_p, g y')] ; \\ & \quad y_p \in Y \end{aligned} \quad (16)$$

$$= s [b_G(y', y_p) + b_G(g y_{p-1}, g y')]$$

Utilizing $\mathbb{K}_2$, we obtain

$$\begin{aligned} b_G(y', g y') &\leq s \left[b_G(y', y_p) + \frac{\delta}{s} (b_G(y_{p-1}, y_p) + b_G(y', g y')) \right] \end{aligned} \quad (17)$$

It follows that

$$\begin{aligned} b_G(y', g y') &\leq \left(\frac{s}{1 - \delta} \right) b_G(y', y_p) \\ &\quad + \left(\frac{\delta}{1 - \delta} \right) b_G(y_{p-1}, y_p) \rightarrow 0 \end{aligned} \quad (18)$$

as $p \rightarrow \infty$. Hence $g y' = y'$; therefore, y' is the fixed point of g .

Uniqueness: Let y^* be another fixed point of g . Assume $(y' P y^*)_{\mathcal{H}_{\mathcal{G}}}$; then there exists a sequence $\{y_j\}_{j=0}^r$ such that $y_0 = y', y_r = y^*$ with $(y_j, y_{j+1}) \in \mathfrak{G}(\mathcal{H}_{\mathcal{G}}), j = 0, 1, \dots, r-1$. Since g is $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping, repeated use of $\mathbb{K}_1$ gives $(g^p y_j, g^p y_{j+1}) \in \mathfrak{G}(\mathcal{H}_{\mathcal{G}}), \forall p \in \mathbb{N}$. Making use of $\mathbb{K}_2$ and proceeding along with the same lines as done in Theorem 8, we acquire

$$b_G(g^p y_j, g^p y_{j+1}) \leq \xi^p b_G(y_j, y_{j+1}), \quad \xi \in \left[0, \frac{1}{s} \right). \quad (19)$$

FIGURE 4: Validation of $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping.

Now utilizing (G_bM3) , we have

$$\begin{aligned}
 b_G(g^p y', g^p y^*) &= b_G(g^p y_0, g^p y_r) \\
 &\leq s [b_G(g^p y_0, g^p y_1) + b_G(g^p y_1, g^p y_r)] \\
 &\leq s \sum_{k=1}^r s^{k-1} b_G(g^p y_{k-1}, g^p y_k) \\
 &\leq s \xi^m \sum_{k=1}^r s^{k-1} b_G(y_{k-1}, y_k)
 \end{aligned} \tag{20}$$

Since $y', y^* \in \text{Fix}(g) \implies g^p y' = y$ and $g^p y^* = y^*$. Proceeding limit $p \rightarrow \infty$, we obtain $y' = y^*$. Hence g admits exactly one fixed point. $\square$

To make our results explicit, we propose the following example.

Example 10. Let $Y = \{1/5^n : n \in \mathbb{N}\} \cup \{0\}$ be endowed with a graph $G = \mathcal{H}_{\mathcal{G}}$ such that $\mathcal{U}(\mathcal{G}) = Y$ and $\mathcal{G}(\mathcal{G}) = \Delta \cup \{(y_1, y_2) \in Y \times Y : (y_1 P y_2), y_1 \geq y_2\}$.

Define the graphical b -metric b_G by

$$b_G(y_1, y_2) = \begin{cases} |y_1 - y_2|^2 & \text{if } y_1 \neq y_2, \\ 0 & \text{if } y_1 = y_2. \end{cases} \tag{21}$$

It is evident that (b_G, Y, s) is a $GbMS$ with $s = 25/17$. Let the map $g : Y \rightarrow Y$ be defined by $gy^* = y^*/5$, for all $y^* \in Y$. One can easily find that there exists $y_0 = 1/5$ such that $g(1/5) = 1/25 \in [1/5]_{\mathcal{H}_{\mathcal{G}}}^1$; i.e., $((1/5)P(1/25))_{\mathcal{H}_{\mathcal{G}}}$ and mapping (4) is satisfied for $\delta = 2/5$; thus g is an $\mathcal{H}_{\mathcal{G}}$ -graph Kannan mapping on Y . Figure 4 authenticates the domination of RHS of graph mapping (4) over LHS for $\delta = 2/5$.

By routine calculations, one can see that all the conditions of Theorem 9 are contented and 0 is the desired fixed point of the mapping g . Figure 5 exemplifies the weighted graph for $\mathcal{U}'(\mathcal{H}_{\mathcal{G}}) = \{a, b, c, d, e, f, g\} \subseteq \mathcal{U}(\mathcal{H}_{\mathcal{G}})$, where the value of $b_G(y_1, y_2)$ is equal to the weight of edge (y_1, y_2) and $\{a, b, c, d, e, f, g\} = \{1/2, 1/5^2, 1/5^3, 1/5^4, 1/5^5, 1/5^6, 0\}$.

Remark 11. It is worth noticing that the results given in this article generalize and extend the pioneer results of Kannan

FIGURE 5: Weighted graph for $\mathcal{U}'(\mathcal{G})$ where $b_G(y_1, y_2)$ = weight of edge (y_1, y_2) .

[13] and Bojor [23], in the context of graphical b -metric spaces.

4. Applications

In this section we show the importance and applicability of the obtained results.

Let $Y = C([0, J], \mathbb{R})$ be the set of real continuous functions on $[0, J]$. Consider $\mathcal{A} = \{v \in Y : \inf_{0 \leq r \leq J} v(r) > 0 \text{ and } v(r) \leq 1, r \in [0, J], J > 0\}$; and let the graph $\mathcal{G}$ be defined by $\mathcal{U}(\mathcal{G}) = Y$ and

$$\begin{aligned}
 \mathcal{G}(\mathcal{G}) &= \Delta \cup \{(v, v^*) \in Y \times Y : v, v^* \in \mathcal{A}, v(r) \\
 &\leq v^*(r), \text{ for all } r \in [0, J]\}.
 \end{aligned} \tag{22}$$

4.1. Application to Damped Spring-Mass System. In engineering problems, one of the realistic applications for the “spring-mass system” is an automobile suspension system. Consider the motion of a spring of a car when it moves along a rough and pitted road, where the forcing term is rough road and shock absorbers provide the damping. The external forces under which the system operates may be gravity, ground vibrations, earthquake, tension force, etc. Let m be the mass of the spring and $\mathcal{F}$ is the external force acting on it; then the critical damped motion of this system subjected to the external force $\mathcal{F}$ is governed by the following initial value problem:

$$\begin{aligned}
 m \frac{d^2 v}{dr^2} + \wp \frac{dv}{dr} - m \mathcal{F}(r, v(r)) &= 0 \\
 v(0) &= 0, \\
 v'(0) &= 0;
 \end{aligned} \tag{23}$$

where $\wp > 0$ is the damping constant and $\mathcal{F} : [0, J] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function.

The aforementioned problem is equivalent to the integral equation

$$v(r) = \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, v(z)) dz, \quad r \in [0, J]; \quad (24)$$

where the respective Green's function $\mathcal{R}(r, z)$ with $\mu = \wp/m$ is given by

$$\mathcal{R}(r, z) = \begin{cases} \frac{1 - e^{\mu(r-z)}}{\mu}; & 0 \leq z \leq r \leq J, \\ 0; & 0 \leq r \leq z \leq J. \end{cases} \quad (25)$$

Define graphical metric $b_G : Y \times Y \rightarrow [0, \infty)$ by

$$b_G(v, v^*) = \sup_{0 \leq r \leq J} |v(r) - v^*(r)|^2, \quad (26)$$

for all $v, v^* \in Y$. Then $(Y, r_{\mathcal{G}_b})$ is a $\mathcal{H}_{\mathcal{G}}$ -complete GbMS.

Consider the operator $g : Y \rightarrow Y$ given by

$$gv(r) = \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, v(z)) dz, \quad \forall v \in Y. \quad (27)$$

Then v is a solution of (24), iff v is a fixed point of g .

Now consider the following assertions:

(k_1) $\beta \in C([0, J], \mathbb{R})$ is a lower solution of problem (24); i.e.,

$$\beta(r) \leq \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, \beta(z)) dz; \quad (28)$$

(k_2) $\mathcal{F}(z, \cdot)$ is an increasing function on $(0, 1]$ for every $z \in [0, J]$ and choose μ suitably such that

$$\inf_{0 \leq r \leq J} \mathcal{R}(r, z) > 0; \quad (29)$$

$$0 \leq \sup_{0 \leq r \leq J} (\mathcal{S}(\mu, r))^2 < \frac{1}{2},$$

$$\mathcal{R}(r, z) \mathcal{F}(z, 1) \leq J^{-1};$$

where $\mathcal{S}(\mu, r) = (1 + r\mu - e^{r\mu})/\mu^2$

(k_3) For each $r \in [0, J]$ and $u, v \in Y$, we have

$$|\mathcal{F}(r, u(r)) - \mathcal{F}(r, v(r))| \leq \sqrt{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2}. \quad (30)$$

Theorem 12. Under the assertions (k_1)-(k_3), the existence of solution for integral equation (24) provides a unique solution for initial value problem (23).

Proof. Utilizing the hypothesis of the theorem under consideration, for $(u, v) \in \mathfrak{G}(\mathcal{H}_{\mathcal{G}})$ with $u, v \in Y$, we have

$$\begin{aligned} |gu(r) - gv(r)| &= \left| \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, u(z)) dz \right. \\ &\quad \left. - \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, v(z)) dz \right| \leq \int_0^J \mathcal{R}(r, z) \\ &\quad \cdot |\mathcal{F}(z, u(z)) - \mathcal{F}(z, v(z))| dz \leq \int_0^J \mathcal{R}(r, z) \\ &\quad \cdot \sup_{0 \leq r \leq J} |\mathcal{F}(r, u(r)) - \mathcal{F}(r, v(r))| dz \\ &= \sup_{0 \leq r \leq J} |\mathcal{F}(r, u(r)) - \mathcal{F}(r, v(r))| \int_0^J \mathcal{R}(r, z) dz \\ &\leq \sup_{0 \leq r \leq J} \sqrt{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2} \\ &\quad \cdot \int_0^J \mathcal{R}(r, z) dz = \left(\frac{1 + r\mu - e^{r\mu}}{\mu^2} \right) \\ &\quad \cdot \sup_{0 \leq r \leq J} \sqrt{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2}. \end{aligned} \quad (31)$$

It follows that

$$\begin{aligned} \sup_{0 \leq r \leq J} |gu(r) - gv(r)|^2 &\leq \left\{ \sup_{0 \leq r \leq J} (\mathcal{S}(\mu, r))^2 \right\} \\ &\quad \cdot \sup_{0 \leq r \leq J} \{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2\}. \end{aligned} \quad (32)$$

In view of hypothesis, taking $\sup_{0 \leq r \leq J} (\mathcal{F}(\mu, r))^2 = \delta/s \in [0, 1/2]$, we get

$$b_G(gu, gv) \leq \frac{\delta}{s} [b_G(u, gu) + b_G(v, gv)]. \quad (33)$$

Thus the contractive condition ($\mathbb{K}_2$) of Theorem 9 is established.

Next, for each $u, v \in Y$ such that $(u, v) \in \mathfrak{G}(\mathcal{G})$, we obtain that $u, v \in \mathcal{A}$ and $u(r) \leq v(r)$ for all $r \in [0, J]$; and by condition (k_2), we obtain $\inf_{0 \leq r \leq J} g(u)(r) > 0$,

$$\begin{aligned} g(u)(r) &= \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, u(z)) dz \\ &\leq \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, 1) dz \leq 1 \end{aligned} \quad (34)$$

and

$$\begin{aligned} g(u)(r) &= \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, u(z)) dz \\ &\leq \int_0^J \mathcal{R}(r, z) \mathcal{F}(z, v(z)) dz = g(v)(r). \end{aligned} \quad (35)$$

This amounts to say that $gu(r) \in \mathcal{A}$ and $(g(u)(r), g(v)(r)) \in \mathfrak{G}(\mathcal{G})$. In accordance with condition (k_1), there exists a solution say $\lambda \in \mathcal{A}$ such that $g(\lambda) \in [\lambda]_{\mathcal{H}_{\mathcal{G}}}^1$, so that the

condition (b) of Theorem 9 is verified. Next, it is easy to see that other conditions of Theorem 9 are satisfied. Therefore, Theorem 9 guarantees that g has a unique fixed point and hence problem (23) has a unique solution in Y . $\square$

4.2. Application to Deformation of Elastic Beam. A fourth-order two-point boundary value problem describes the deformations of elastic beam as per the controls at the ends of the beam. Due the nature and characterization of the equilibrium state, these higher order beam equations have significant applications pertaining to mechanics and engineering related to bridges, ships, automobiles, buildings, and aircraft. We refer readers interested in applications of elastic beams and related problems to [24, 25].

Consider the following functional equation describing the deformation of an elastic beam with one end free and other end cantilevered or fixed:

$$\begin{aligned} v''''(r) &= \mathcal{K}(r, v(r), v'(r)), \quad r \in [0, 1]; \\ v(0) &= v'(0) = v''(1) = v'''(1) = 0, \end{aligned} \quad (36)$$

where $\mathcal{K} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function that appears in the study of elastic beam equations.

Solving problem (36) is equivalent to finding $v^* \in Y = C([0, 1], \mathbb{R})$; solution of the integral equation is

$$v(r) = \int_0^1 \mathcal{R}^*(r, z) \mathcal{K}(z, v(z), v'(z)) dz, \quad r \in [0, 1], \quad (37)$$

where $\mathcal{R}^*(r, z)$ is the Green function given by

$$\mathcal{R}^*(r, z) = \frac{1}{6} \begin{cases} z^2(3r - z), & \text{if } 0 \leq z \leq r \leq 1; \\ t^2(3z - r), & \text{if } 0 \leq r \leq z \leq 1. \end{cases} \quad (38)$$

Define graphical metric $b_G : Y \times Y \rightarrow [0, \infty)$ by

$$b_G(v, v^*) = \sup_{0 \leq r \leq 1} |v(r) - v^*(r)|^2, \quad (39)$$

for all $v, v^* \in Y$. Clearly $(Y, r_{\mathcal{G}_b})$ is a $\mathcal{H}_{\mathcal{G}}$ -complete GbMS.

Next theorem provides us a sufficient condition for the existence and uniqueness of solution of problem (36).

Theorem 13. Suppose the following conditions hold:

(1) $\gamma \in C([0, 1], \mathbb{R})$ is a lower solution of the problem (37); i.e.,

$$\gamma(r) \leq \int_0^1 \mathcal{R}^*(r, z) \mathcal{K}(z, \gamma(z), \gamma'(z)) dz, \quad (40)$$

(2) Choose $\kappa > 0$ suitably such that

$$\inf_{0 \leq r \leq 1} \mathcal{R}^*(r, z) > 0;$$

$$0 \leq \sup_{0 \leq t \leq 1} (\mathcal{Q}(\kappa, r))^2 < \frac{1}{2},$$

$$\mathcal{K}(z, 1, 0) \leq 1 \quad (41)$$

where $\mathcal{Q}(\kappa, r) = (r^4 - 4r^3 + 6r^2)/24\kappa$

(3) For each $r \in [0, 1]$ and $u, v \in Y$, we have

$$\begin{aligned} &|\mathcal{K}(r, u(r), u'(r)) - \mathcal{K}(r, v(r), v'(r))| \\ &\leq \frac{1}{\kappa} \sqrt{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2}, \end{aligned} \quad (42)$$

where the operator g is defined in (43).

Then the existence of solution for integral equation (37) provides a unique solution for the functional equation describing the deformation of an elastic beam (36).

Proof. Define the operator $g : Y \rightarrow Y$ given by

$$gv(r) = \int_0^1 \mathcal{R}^*(r, z) \mathcal{K}(z, v(z), v'(z)) dz, \quad (43)$$

$\forall v \in Y$.

Clearly operator $g : Y \rightarrow Y$ is well defined. Utilizing condition hypothesis of the theorem under consideration, for $(u, v) \in \mathfrak{G}(\mathcal{H}_{\mathcal{G}})$ with $u, v \in Y$, we have

$$\begin{aligned} &|gu(r) - gv(r)| \\ &= \left| \int_0^1 \mathcal{R}^*(r, z) \mathcal{K}(z, u(z), u'(z)) dz \right. \\ &\quad \left. - \int_0^1 \mathcal{R}^*(r, z) \mathcal{K}(z, v(z), v'(z)) dz \right| \\ &\leq \int_0^1 \mathcal{R}^*(r, z) \\ &\quad \cdot |\mathcal{K}(z, u(z), u'(z)) - \mathcal{K}(z, v(z), v'(z))| dz \\ &\leq \sup_{0 \leq r \leq 1} |\mathcal{K}(z, u(r), u'(r)) - \mathcal{K}(z, v(r), v'(r))| \\ &\quad \cdot \int_0^1 \mathcal{R}^*(r, z) dz \leq \frac{1}{\kappa} \\ &\quad \cdot \sup_{0 \leq r \leq 1} \sqrt{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2} \\ &\quad \cdot \int_0^1 \mathcal{R}^*(r, z) dz = \left(\frac{r^4 - 4r^3 + 6r^2}{24\kappa} \right) \\ &\quad \cdot \sup_{0 \leq r \leq 1} \sqrt{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2}. \end{aligned} \quad (44)$$

It follows that

$$\begin{aligned} &\sup_{0 \leq r \leq 1} |gu(r) - gv(r)|^2 \leq \left\{ \sup_{0 \leq r \leq 1} (\mathcal{Q}(\kappa, r))^2 \right\} \\ &\quad \cdot \sup_{0 \leq r \leq 1} \{|u(r) - gu(r)|^2 + |v(r) - gv(r)|^2\}. \end{aligned} \quad (45)$$

In view of hypothesis, taking $\{\sup_{0 \leq r \leq 1} (\mathcal{Q}(\kappa, r))^2\} = \delta/s \in [0, 1/2)$, we get

$$b_G(gu, gv) \leq \frac{\delta}{s} [b_G(u, gu) + b_G(v, gv)]. \quad (46)$$

Thus the contractive condition $(\mathbb{K}_2)$ of Theorem 9 is established.

Other conditions of Theorem 9 can be proved along similar lines as done in Theorem 12; for the sake of brevity, we omit the rest part of the proof.

Hence we conclude that the fourth-order boundary value problem (36) describing the deformation of an elastic beam has a unique solution in Y . $\square$

Open Problems

- (i) The governing second-order differential equation for an infinite rod with absorption describing the steady state concentration $m(t)$ dispersing in an absorbing medium is given by

$$-\frac{d^2 m}{dr^2} + \alpha^2 m = g(r, m(r)), \quad -\infty < r < +\infty \quad (47)$$

where $\alpha > 0$, $g(r, m(r))$ is the density of the substance in the steady state, and the whole process is examined in an infinity long tube.

Whether the existence of solution of the above second-order differential equation can be derived from results established in this note?

- (ii) Establish analogue results of Edelstein [26], Hardy-Rogers [27], Meir-Keeler [28], and Reich [29] type contractions in the underlying space.
- (iii) Apply Kannan type graph contractions or equivalent graph contractions to establish the existence of solution of the following beam equation

$$\begin{aligned} v''''(r) g(v) v' + \mathcal{K}(r, v, v', v'') &= e(r), \quad r \in [0, 1]; \\ v(0) &= v'(0) = 0, \\ v'(1) &= v'''(1) = 0, \end{aligned} \quad (48)$$

where $\mathcal{K} : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function and $e(r) \in L^1[0, 1]$.

5. Conclusions

This article implements the idea of graphical structure of Kannan mappings. Based on this structure, we showed that every Kannan mapping is graph Kannan but not conversely. We obtained the fixed point result by dropping the property S as used in [5, 10]. Obtained results are validated by appropriate examples endowed with suitable graphs. Embellishing the applications of established results, two problems are considered: damped spring-mass system and deformation of an elastic beam. The results are important both theoretically and practically for fixed point theorists as well as engineers and physicists working in the related fields.

Data Availability

All the data related to this research work are included in the text.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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