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Research Article

Multiple Criteria Decision Making Based on Probabilistic Interval-Valued Hesitant Fuzzy Sets by Using LP Methodology

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Probabilistic interval-valued hesitant fuzzy sets (PIVHFSs) are an extension of interval-valued hesitant fuzzy sets (IVHFSs) in which each hesitant interval value is considered along with its occurrence probability. These assigned probabilities give more details about the level of agreeness or disagreeness. PIVHFSs describe the belonging degrees in the form of interval along with probabilities and thereby provide more information and can help the decision makers (DMs) to obtain precise, rational, and consistent decision consequences than IVHFSs, as the correspondence of unpredictability and inaccuracy broadly presents in real life problems due to which experts are confused to assign the weights to the criteria. In order to cope with this problem, we construct the linear programming (LP) methodology to find the exact values of the weights for the criteria. Furthermore these weights are employed in the aggregation operators of PIVHFSs recently developed. Finally, the LP methodology and the actions are then applied on a certain multiple criteria decision making (MCDM) problem and a comparative analysis is given at the end.

1. Introduction

Vagueness usually appears in real life issues, like risk management, intelligent computations, or applications in the field of engineering, etc. Dealing with the decision making issues, the building of the mathematical models for the uncertainty has its own prominence. Although we have the opportunity to get plentiful information, it is still hard to dispense with the vagueness absolutely. Often the existing huge data sets include uncertain information. Subsequently, the demonstrating of hesitancy has invited an academic challenge. In 1965 Zadeh [1] introduced the notion of fuzzy sets in order to exhibit the subjective indeterminacy and uncertainty. Fuzzy sets have induced a new spirit of exploration because it discusses the ambiguity, vagueness, and hesitation directly. Fuzzy sets have created the comforts for the professionals in the area of decision making. Recently, fuzzy sets are very much used in management sciences, statistics, social sciences, and many other areas. In the present decade, the decision makers (DMs) face more complex situations. In order to overcome such complex problems, DMs use the fuzzy sets and some extensions of fuzzy sets to express their vague and undefined options [2, 3]. Zadeh [4] presented the idea of interval-valued fuzzy sets (IVFSs) in 1975, an augmentation of fuzzy sets in which the values of the belonging degrees are given in the form of interval of numbers instead of the numbers. IVFSs gave a more satisfactory depiction of vulnerability than conventional fuzzy sets. Atanassov [5] constructed some valuable operations for interval-valued fuzzy sets due to which evaluations become easier. Gorzalczany [6] introduced a method of inference in approximate reasoning based on IVFSs.

Hesitancy appears mostly everywhere in our life. It is difficult to decide in real life to choose one of the best alternatives with identical features. Due to ambiguity and uncertainty in the information, specialists are having difficulties in making decisions. To overcome the reluctance, Torra [7] presented the idea of hesitant fuzzy sets (HFSs) as an extension of ordinary fuzzy sets to manage those situations in which several values are possible for a membership function. From the last decade, HFSs play a dominant role in decision making [8, 9]. Xia and Xu [10] presented the basic element of HFSs that is hesitant fuzzy element (HFE) which plays a very important role in developing aggregations. Various extensions of HFSs have been presented, for example, hesitant triangular fuzzy set [11], hesitant multiplicative numbers [12], dual hesitant fuzzy sets [13], and convex hesitant fuzzy sets [14]. In numerous decision making issues, because of deficiency in giving data, it might be troublesome for DMs to precisely measure their feelings with a crisp number, but they can be denoted by an interval number which belongs to $[0, 1]$. In various convoluted real life determinations, it is rather very hard for the DMs to give right values for the belonging degrees of definite elements to a specified set, but a range of values fit in zero to one. These problems increase further when these notifications are present in a set of intervals. In order to overcome such problems, Chen et al. [15, 16] presented the idea of interval-valued hesitant fuzzy sets (IVHFSs), which is the extended form of HFSs. IVHFSs permit the belonging level of a component to have various possible interval values in $[0, 1]$ rather than real numbers and therefore can deal better with intrinsic hesitancy and instability in the human decision making process. Meng et al. gave the notion of correlation coefficients of IVHFSs [17]. This implies that it is extremely essential to present the idea of interval-valued hesitant fuzzy sets (IVHFSs), which allow the belonging degrees of a component to an offered set to have a couple of various interval values.

Multiple criteria decision making (MCDM) is a discipline which supports the DMs to settle on an ideal decision from alternatives in the light of multiple criteria [18, 19]. In the present decade, various MCDM strategies have been developed [20–22] and have been used in different fields, for example, selection of suppliers [23, 24] and development of managing energy projects [25]. Present MCDM techniques are typically based on the expected utility theory which describes that where DMs are supposed to be totally rational. Qiaoping and Jiewen [26] suggested a method based on TOPSIS and entropy-weighted method to solve the multiattribute decision making (MADM) problems under hesitant fuzzy situation in which the weights of the attribute are completely unknown. Zhang et al. [27] constructed multiple criteria decision analysis based on shapely fuzzy measures and interval-valued hesitant fuzzy linguistic numbers.

In real-world applications, there exist many uncertainties and hesitancies which are ordered as stochastic and nonstochastic [28]. Usually, stochastic hesitancies can be taken exactly by probabilistic modeling [29]. In any case, probabilistic models and the traditional, fuzzy set theory are only useful to prepare one part of the vagueness. Hence it would be valuable to integrate the probability theory with the fuzzy set theory [30, 31]. Therefore, a few ideas and techniques are proposed, for example, probability measures of fuzzy events [32], fuzzy random set [33], fuzzy random variable [34, 35], nonstationary fuzzy sets [36], and fuzzy model with probability-based rule weights [37]. In a general sense, two sorts of combination standard caused these strategies. One supposition is to bring the fuzzy hesitancy into the measurable structure and another supposition is to bring the stochastic hesitancies into the fuzzy framework. In light of the second supposition, the probabilistic fuzzy set (PFS) is proposed and created by bringing the probabilistic theory into the conventional fuzzy set depicted by center and width [38]. Thus the fuzzy evaluations in the traditional fuzzy set turn into the stochastic factors described by the secondary probability density function (PDF), which makes it ready to detain both stochastic and nonstochastic hesitancies. Pang et al. [39] intended a novel concept called probabilistic linguistic term set which is an extension of the existing tools and put forward some basic effective rules and aggregation operators for probabilistic linguistic term set. As of late, in view of the probabilistic fuzzy set, the probability fuzzy logic system is proposed which has been applied for stochastic modeling and control [38] and function approximation problem [40]. Because of the deformities of hesitant fuzzy sets (HFSs), Zhu and Xu [41] extended hesitant fuzzy sets to probability hesitant fuzzy sets (PHFSs) to increase the modeling capacity and developed the concept of probability hesitant fuzzy preference relation. Xu and Zhou [42] built the consensus among a group of decision makers under the hesitant probabilistic fuzzy element (HPFE) environment. Zhang et al. [43] presented the actions and integrations of probabilistic hesitant fuzzy information to make a decision. Recently, Hao et al. [44] established the probabilistic dual hesitant fuzzy set and used it into risk evaluation. Li and Wang [45] anticipated the extended QUALIFLEX method under probability hesitant fuzzy environment. Ding et al. [46] built up an intelligent way to deal with probabilistic hesitant fuzzy multiattribute group decision making with partial weight data, in which the appraisals given by the experts to options over criteria are stated by probabilistic hesitant fuzzy elements and the weight data on criteria is mostly known. Wu et al. gave the idea of probability hesitant fuzzy preference relations [47]. Liao et al. [48] presented some measures of probabilistic interval-valued intuitionistic hesitant fuzzy set (PIVIHFS) and applied it in reducing excessive medical examinations. Bashir et al. [49] introduced hesitant probabilistic fuzzy preference relations (HPFPRs) and established few algorithms to obtain the stability of HPFPRs. Most of the articles have been written in 2017, which also show that the PHFS is a new and interesting field of research.

Linear programming (LP) is one of the most useful techniques in the operational research, which finds the best available solution under the given constraints. The ideal solution of an LP is based only on a restricted number of constraints; thereby, most of the composed data has a slight influence on the solution. It is valuable to consider the awareness of DMs about the parameters as fuzzy information. The idea of decision in fuzzy framework was presented by Zadeh and Bellman [50] in 1970 for first time. Tanaka et al. [51] developed a novel technique to solve the fuzzy mathematical programming problems. Feuring and Buckley [52] construct another strategy to evaluate the solution in fuzzy environment with the help of LP problem by altering the objective function into a linear multiobjective problem. Maleki [53] developed a technique for solving LP problem with ambiguous constraints by utilizing ranking function. Wang and Chen [54] construct a novel MCDM technique based on linear programming methodology for interval-valued intuitionistic fuzzy values (IVIFVs). Su et al. [55] developed an input-output LP model to study energy-economic retrieval flexibility of an economy. Aliyev [56] offered an interval LP where the uncertain locality is termed by interval numbers. Aliaga et al and Jun el al. presented the decision structure for entropies and linear systems [57, 58].

Inspired by the application of the LP model used in different fields, simplicity in calculation and the ample information are attained by the PIVHFSs. For the novelty of this paper, we have applied the LP technique to obtain the exact weights of criteria under some constraints and then employed these weights of criteria to find out the best choice from the given alternatives. The bigger the measure of assessment data incorporates, the more noteworthy the impact of measurement properties to choice outcomes will be.

The rest of the paper is managed in the following way: In Section 2, we briefly describe the core ideas of hesitant fuzzy sets, probabilistic hesitant fuzzy set, IVHFSs, PIVHFSs, and an LP model which is constructed to evaluate the weights of criteria and used in the aggregation operators of PIVHFSs for MCDM. In Section 3, based on LP model an MCDM method is proposed. Section 4 comprises a practical example to analyze the experimental results of the proposed technique and a brief comparison is described in Section 5. The conclusions are discussed in Section 6.

2. Some Fundamental Concepts

Definition 1 (see [7]). Let X be a fixed set, a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of $[0,1]$. To be easily understood, Xia and Xu [10] express the HFS by a mathematical symbol:

$H = \{\langle x, h_{H(x)}\rangle : x \in X\}$, where $h_{H(x)}$ is a set of some different values in $[0,1]$ denoting the possible membership degrees of the element $x \in X$ to the set H.

Definition 2 (see [41]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a source set, then a PHFS P on X can be expressed by $P = \{< x, h_x(p_x) >| \ x \in X\}$, where both h_x and p_x are two sets of some values in $[0,1]$. h_x represents the possible belonging degrees of the element x in X whereas p_x is a set of probabilities associated with h_x. For convenience, we call $h_x(p_x)$ probabilistic hesitant fuzzy element (PHFE), represented as $h(p)$ and specified by $h(p) = \{\gamma_i(p_i) \mid i = 1,2,3,\ldots,|h(p)|\}$, where p_i is the probability of the belonging degree γ_i, $\gamma_i(p_i)$ is called a term of PHFE, $|h(p)|$ is the number of all belonging degrees, and $\sum_{i=1}^{|h(p)|} p_i = 1$.

Definition 3 (see [15, 16]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a source set and $I[0,1]$ be the set of all closed subintervals of $[0,1]$. An interval-valued hesitant fuzzy set (IVHFS) on X is $\widetilde{H} = \{\langle x, \widetilde{h}_{\widetilde{H}(x)}\rangle : x \in X\}$, where $\widetilde{h}_{\widetilde{H}(x)} : X \longrightarrow [0,1]$ represents all possible interval membership degrees of the element $x \in X$ to the set $\widetilde{H}$. Normally, $\widetilde{h}_{\widetilde{H}(x)}$ is called an interval-valued hesitant fuzzy element (IVHFE) represented by $\widetilde{h} = \{\widetilde{\gamma}^1, \widetilde{\gamma}^2, \widetilde{\gamma}^3, \ldots, \widetilde{\gamma}^{\#\widetilde{h}}\}$ where $\widetilde{\gamma}^\lambda (\lambda = 1,2,3,\ldots,\#\widetilde{h})$ is an interval number represented by $\widetilde{\gamma}^\lambda = [\widetilde{\gamma}^{\lambda L}, \widetilde{\gamma}^{\lambda U}] \in I[0,1]$. The IVHFE can be expressed to HFE if $\widetilde{\gamma}^{\lambda L} = \widetilde{\gamma}^{\lambda U} (\lambda = 1,2,3,\ldots,\#\widetilde{h})$ and the symbol $\#\widetilde{h}$ denotes the number of element in $\widetilde{h}$.

Definition 4. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, then a PIVHFSs $\widetilde{P}$ on X can be defined as $\widetilde{P} = \{< x_i, \widetilde{h}_{x_i}(p_{x_i}) >| \ x_i \in X\}$ where $i = 1,2,3,\ldots,n$. The function $\widetilde{h}_{x_i} : X \longrightarrow I[0,1]$ is a set of intervals belonging to $[0,1]$, which is described by the probability distribution p_{x_i}, where $\widetilde{h}_{x_i}$ represents the possible belonging degree of element x_i in X to $\widetilde{P}$. For convenience, $\widetilde{h}_{xi}(p_{x_i})$ is called a PIVHFE and represented as $\widetilde{h}(p)$ and is specified by $\widetilde{h}(p) = \{\widetilde{\gamma}_i(p_{x_i}) \mid i = 1,2,3,\ldots,|\widetilde{h}(p)|\}$, where p_{x_i} is the probability of the belonging degree $\widetilde{\gamma}_i$, $\widetilde{\gamma}_i(p_{x_i})$ is said to be a term of the PIVHFE, and $|\widetilde{h}(p)|$ is the number of all distinct belonging degrees such that $\sum_{i=1}^{|\widetilde{h}(p)|} p_{x_i} = 1$ and $\widetilde{\gamma}_i(p_{x_i}) = [\widetilde{\gamma}^l, \widetilde{\gamma}^u](p_{x_i})$. Clearly, the PIVHFE can preserve much more appraisal material from the DMs than the IVHFE, and exhausting PIVHFE in the place of HFE to precise the DMs favorites is more consistent and realistic. In many real-world presentations, some DMs cannot provide complete information; for example, they do not have plenty capability; then $\sum_{i=1}^{|\widetilde{h}(p)|} p_{x_i} < 1$; it means that there exists partial illiteracy. Moreover, if $\sum_{i=1}^{|\widetilde{h}(p)|} p_{x_i} = 0$, it represents the ample unawareness.

Example 5. Let $X = \{x_1, x_2\}$ be a source set and

$\widetilde{h}_1(p_1) = \{[0.2,0.3](0.3), [0.4,0.5](0.2), [0.5,0.6](0.1), [0.7,0.8](0.4)\}$ and

$\widetilde{h}_2(p_2) = \{[0.3,0.4](0.1), [0.4,0.5](0.9)\}$ be two PIVHFEs of $x_i (i = 1,2)$ to a set $\widetilde{P}$, respectively. Then $\widetilde{h}(P)$ can be considered as a PIVHFS,

$$\widetilde{P} = \{\langle x_1, \{[0.2,0.3](0.3), [0.4,0.5](0.2), [0.5,0.6](0.1), [0.7,0.8](0.4)\}\rangle, \langle x_2, \{[0.3,0.4](0.1), [0.4,0.5](0.9)\rangle\} \quad (1)$$

If we ignore the probabilities of the possible values in a PIVHFE, then the possible values are with the same probability; in this case, PIVHFE becomes IVHFE and further if intervals are single valued, then it is probabilistic hesitant fuzzy element (PHFE).

Definition 6. Let $\widetilde{h}(p) = \cup_{\widetilde{\gamma}\in\widetilde{h}}\{[\widetilde{\gamma}^l, \widetilde{\gamma}^u](p)\}$, $\widetilde{h}_1(p) = \cup_{\widetilde{\gamma}_1\in\widetilde{h}_1}\{[\widetilde{\gamma}_1^l, \widetilde{\gamma}_1^u](p)\}$ and

$\widetilde{h}_2(p) = \cup_{\widetilde{\gamma}_2\in\widetilde{h}_2}\{[\widetilde{\gamma}_2^l, \widetilde{\gamma}_2^u](p)\}$ be three PIVHFEs, then the basic operations on PIVHFEs are defined as

(1)

$$\tilde{h}_1(p)\otimes\tilde{h}_2(p)=\bigcup_{\tilde{\gamma}_{1m}\in\tilde{h}_1,\tilde{\gamma}_{2n}\in\tilde{h}_2}\left\{\left[\tilde{\gamma}^l_{1m}\cdot\tilde{\gamma}^l_{2n},\tilde{\gamma}^u_{1m}\cdot\tilde{\gamma}^u_{2n}\right]\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{\tilde{h}_1(p)}p_{1m}\cdot\sum_{n=1}^{\tilde{h}_2(p)}p_{2n}}\right)\right\}; \quad (2)$$

(2)

$$\tilde{h}_1(p)\oplus\tilde{h}_2(p)=\bigcup_{\tilde{\gamma}_{1m}\in\tilde{h}_1,\tilde{\gamma}_{2n}\in\tilde{h}_2}\left\{\left[\tilde{\gamma}^l_{1m}+\tilde{\gamma}^l_{2n}-\tilde{\gamma}^l_{1m}\cdot\tilde{\gamma}^l_{2n},\tilde{\gamma}^u_{1m}+\tilde{\gamma}^u_{2n}-\tilde{\gamma}^u_{1m}\cdot\tilde{\gamma}^u_{2n}\right]\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{\tilde{h}_1(p)}p_{1m}\cdot\sum_{n=1}^{\tilde{h}_2(p)}p_{2n}}\right)\right\}; \quad (3)$$

(3)

$$\varphi\tilde{h}(p)=\bigcup_{\tilde{\gamma}\in\tilde{h}}\left\{\left[1-\left(1-\tilde{\gamma}^l\right)^{\varphi},1-\left(1-\tilde{\gamma}^u\right)^{\varphi}\right](p_k):\tilde{\gamma}\in\tilde{h}\right\}, \quad \varphi>0; \quad (4)$$

(4)

$$\tilde{h}^{\varphi}(p)=\bigcup_{\tilde{\gamma}\in\tilde{h}}\left\{\left[\left(\tilde{\gamma}^l\right)^{\varphi},\left(\tilde{\gamma}^u\right)^{\varphi}\right](p_k)\right\}, \quad \varphi>0; \quad (5)$$

(5)

$$\tilde{h}^c(p)=\bigcup_{\tilde{\gamma}\in\tilde{h}}\left\{\left[1-\tilde{\gamma}^u,1-\tilde{\gamma}^l\right](p_k):\tilde{\gamma}\in\tilde{h}\right\}. \quad (6)$$

It is clear from the definition of PIVHFE that the number of interval values in various PIVHFEs is normally distinct and the interval values in PIVHFE are mostly inoperative. Suppose that $\tilde{h}_1(p)$ and $\tilde{h}_2(p)$ are two PIVHFEs with $|\tilde{h}_1(p)|$ and $|\tilde{h}_2(p)|$, the number of intervals in the PIVHFEs, respectively; we can line up the interval numbers in an increasing or decreasing order by using the definitions below.

Definition 7. For a given PIVHFE $\tilde{h}(p)$ if $\sum_{i=1}^{|\tilde{h}(p)|}p_{x_i}<1$, then its connected PIVHFE $\hat{h}(p)$ can be defined as $\hat{h}(p)=\{\tilde{\gamma}_i(\hat{p}_{x_i})\mid i=1,2,3\ldots,|\tilde{h}(p)|\}$, where $\hat{p}_{x_i}=p_{x_i}/\sum_{i=1}^{|\tilde{h}(p)|}p_{x_i}$, $i=1,2,3,\ldots,|\tilde{h}(p)|$.

Example 8. Let $\tilde{h}_1(p)=[0.7,0.8](0.2),[0.4,0.5](0.4),[0.3,0.4](0.3)$ and $\tilde{h}_2(p)=[0.8,0.9](0.5),[0.6,0.7](0.5)$ be two PIVHFEs; then by definition,

$\hat{h}_1(p)=\{[0.7,0.8](0.222),[0.4,0.5](0.444),[0.3,0.4](0.333)\}$ and

$\hat{h}_2(p)=\{[0.8,0.9](0.5),[0.6,0.7](0.5)\}$.

Definition 9. Let $\tilde{h}(p)$ be an PIVHFE; then its score value is defined as follows:

$S(\tilde{h}(p))=(\sum_{k=1}^{|\tilde{h}(p)|}\tilde{\gamma}_k(p_{x_k})/\sum_{k=1}^{|\tilde{h}(p)|}p_{x_k}$ is said to be the score function of $\tilde{h}(p)$. For any two PIVHFEs, if $S(\tilde{h}_1(p))=S(\tilde{h}_2(p))$, then $\tilde{h}_1(p)=\tilde{h}_2(p)$ and if $S(\tilde{h}_1(p))>S(\tilde{h}_2(p))$, then $\tilde{h}_1(p)>\tilde{h}_2(p)$.

Thus we can easily rank the alternatives by using the score values.

Example 10. Let $\tilde{h}_i(p)$ $(i=1,2,3)$ be three PIVHFEs, $\tilde{h}_1(p)=\{[0.8,0.9](0.5),[0.7,0.8](0.4)\}$, $\tilde{h}_2(p)=\{[0.8,0.9](0.5),[0.6,0.7](0.5)\}$, $\tilde{h}_3(p)=\{[0.7,0.8](0.5),[0.5,0.6](0.4)\}$, the score values of three PIVHFEs be $S(\tilde{h}_1(p))=2.3261$; $S(\tilde{h}_2(p))=2.0564$; $S(\tilde{h}_3(p))=1.7267$.

Theorem 11. *Let $\tilde{h}_1(p)$, $\tilde{h}_2(p)$, and $\tilde{h}_3(p)$ be three PIVHFEs, and let η be a positive number; then*

(1) $\tilde{h}_1(p)\oplus\tilde{h}_2(p)=\tilde{h}_2(p)\oplus\tilde{h}_1(p)$;
(2) $\tilde{h}_1(p)\oplus\tilde{h}_2(p))\oplus\tilde{h}_3(p)=\tilde{h}_1(p)\oplus(\tilde{h}_2(p)\oplus\tilde{h}_3(p))$;
(3) $\tilde{h}_1(p)\otimes\tilde{h}_2(p)=\tilde{h}_2(p)\otimes\tilde{h}_1(p)$;
(4) $(\tilde{h}_1(p)\otimes\tilde{h}_2(p))\otimes\tilde{h}_3(p)=\tilde{h}_1(p\otimes(\tilde{h}_2(p)\otimes\tilde{h}_3(p))$;
(5) $(\tilde{h}_1(p)\otimes\tilde{h}_2(p))^{\eta}=\tilde{h}_1^{\eta}(p)\otimes\tilde{h}_2^{\eta}(p)$.

Proof. (1)

$$\begin{aligned}\tilde{h}_1(p)\oplus\tilde{h}_2(p)&=\bigcup_{\tilde{\gamma}_{1m}\in\tilde{h}_1,\tilde{\gamma}_{2n}\in\tilde{h}_2}\left\{\left[\tilde{\gamma}_{1m}+\tilde{\gamma}_{2n}-\tilde{\gamma}_{1m}\cdot\tilde{\gamma}_{2n}\right]\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\tilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\tilde{h}_2(p)|}p_{2n}}\right)\right\}\\&=\bigcup_{\tilde{\gamma}_{2n}\in\tilde{h}_2,\tilde{\gamma}_{1m}\in\tilde{h}_1,}\left\{\left[\tilde{\gamma}_{2n}+\tilde{\gamma}_{1m}-\tilde{\gamma}_{2n}\cdot\tilde{\gamma}_{1m}\right]\cdot\left(\frac{p_{2n}\cdot p_{1m}}{\sum_{n=1}^{|\tilde{h}_2(p)|}p_{2n}\cdot\sum_{m=1}^{|\tilde{h}_1(p)|}p_{1m}}\right)\right\}=\tilde{h}_2(p)\oplus\tilde{h}_1(p)\end{aligned} \quad (7)$$

(2)

$$\begin{aligned}\left(\tilde{h}_1(p)\oplus\tilde{h}_2(p)\right)\oplus\tilde{h}_3(p)&=\bigcup_{\tilde{\gamma}_{1m}\in\tilde{h}_1,\tilde{\gamma}_{2n}\in\tilde{h}_2}\left\{\left[\tilde{\gamma}_{1m}+\tilde{\gamma}_{2n}-\tilde{\gamma}_{1m}\cdot\tilde{\gamma}_{2n}\right]\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\tilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\tilde{h}_2(p)|}p_{2n}}\right)\right\}\oplus\tilde{h}_3(p)\\&=\bigcup_{\tilde{\gamma}_{1m}\in\tilde{h}_1,\tilde{\gamma}_{2n}\in\tilde{h}_2,\tilde{\gamma}_{3k}\in\tilde{h}_3}\left\{\left[\tilde{\gamma}_{1m}+\tilde{\gamma}_{2n}-\tilde{\gamma}_{1m}\cdot\tilde{\gamma}_{2n}+\tilde{\gamma}_{3k}-\tilde{\gamma}_{3k}\left(\tilde{\gamma}_{1m}+\tilde{\gamma}_{2n}-\tilde{\gamma}_{1m}\cdot\tilde{\gamma}_{2n}\right)\right]\right.\end{aligned}$$

$$\cdot\left(\frac{p_{1m}\cdot p_{2n}\cdot p_{3k}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{k=1}^{|\widetilde{h}_3(p)|}p_{3k}}\right)\Bigg\}$$
$$=\bigcup_{\widetilde{\gamma}_{2n\in\widetilde{h}_2},\widetilde{\gamma}_{3k}\in\widetilde{h}_3,\widetilde{\gamma}_{1m\in h_1}}\Bigg\{\left[\left(\widetilde{\gamma}_{2n}+\widetilde{\gamma}_{3k}-\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{3k}+\widetilde{\gamma}_{1m}\right)\right.$$
$$\left.-\widetilde{\gamma}_{1m}\cdot\left(\widetilde{\gamma}_{2n}+\widetilde{\gamma}_{3k}-\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{3k}\right)\right]$$
$$\cdot\left(\frac{p_{2n}\cdot p_{3k}\cdot p_{1m}}{\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{k=1}^{|\widetilde{h}_3(p)|}p_{3k}\cdot\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}}\right)\Bigg\}$$
$$=\bigcup_{\widetilde{\gamma}_{2n\in\widetilde{h}_2},\widetilde{\gamma}_{3k}\in\widetilde{h}_3}\Bigg\{\left[\left(\widetilde{\gamma}_{2n}+\widetilde{\gamma}_{3k}-\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{3k}\right)\right]$$
$$\cdot\left(\frac{p_{2n}\cdot p_{3k}}{\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{k=1}^{|\widetilde{h}_3(p)|}p_{3k}}\right)\Bigg\}\oplus\widetilde{h}_1(p)=\left(\widetilde{h}_2(p)\right.$$
$$\left.\oplus\widetilde{h}_3(p)\right)\oplus\widetilde{h}_1(p) \tag{8}$$

(3)

$$\widetilde{h}_1(p)\otimes\widetilde{h}_2(p)=\bigcup_{\widetilde{\gamma}_{1m\in\widetilde{h}_1},\widetilde{\gamma}_{2n\in\widetilde{h}_2}}\Bigg\{\left[\widetilde{\gamma}_{1m}\cdot\widetilde{\gamma}_{2n}\right]$$
$$\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}}\right)\Bigg\}$$
$$=\bigcup_{\widetilde{\gamma}_{2n\in\widetilde{h}_2},\widetilde{\gamma}_{1m\in\widetilde{h}_1},}\Bigg\{\left[\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{1m}\right]$$
$$\cdot\left(\frac{p_{2n}\cdot p_{1m}}{\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}}\right)\Bigg\}=\widetilde{h}_2(p)\otimes\widetilde{h}_1(p) \tag{9}$$

(4)

$$\left(\widetilde{h}_1(p)\otimes\widetilde{h}_2(p)\right)\otimes\widetilde{h}_3(p)=\bigcup_{\widetilde{\gamma}_{1m\in\widetilde{h}_1},\widetilde{\gamma}_{2n\in\widetilde{h}_2}}\Bigg\{\left[\widetilde{\gamma}_{1m}\cdot\widetilde{\gamma}_{2n}\right]$$
$$\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\widetilde{h}_1(p)|}p_{2n}}\right)\Bigg\}\otimes\widetilde{h}_3(p)$$
$$=\bigcup_{\widetilde{\gamma}_{1m\in\widetilde{h}_1},\widetilde{\gamma}_{2n\in\widetilde{h}_2},\widetilde{\gamma}_{3k}\in h_3}\Bigg\{\left[\widetilde{\gamma}_{1m}\cdot\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{3k}\right]$$
$$\cdot\left(\frac{p_{1m}\cdot p_{2n}\cdot p_{3k}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{k=1}^{|\widetilde{h}_3(p)|}p_{3k}}\right)\Bigg\}$$
$$=\bigcup_{\widetilde{\gamma}_{2n\in\widetilde{h}_2},\widetilde{\gamma}_{3k}\in\widetilde{h}_3,\widetilde{\gamma}_{1m\in h_1}}\Bigg\{\left[\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{3k}\cdot\widetilde{\gamma}_{1m}\right]$$
$$\cdot\left(\frac{p_{2n}\cdot p_{3k}\cdot p_{1m}}{\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{k=1}^{|\widetilde{h}_3(p)|}p_{3k}\cdot\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}}\right)\Bigg\}$$
$$=\bigcup_{\widetilde{\gamma}_{2n\in\widetilde{h}_2},\widetilde{\gamma}_{3k}\in\widetilde{h}_3}\Bigg\{\left[\widetilde{\gamma}_{2n}\cdot\widetilde{\gamma}_{3k}\right]$$
$$\cdot\left(\frac{p_{2n}\cdot p_{3k}}{\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}\cdot\sum_{k=1}^{|\widetilde{h}_3(p)|}p_{3k}}\right)\Bigg\}\otimes\widetilde{h}_1(p)=\left(\widetilde{h}_2(p)\right.$$
$$\left.\otimes\widetilde{h}_3(p)\right)\otimes\widetilde{h}_1(p) \tag{10}$$

(5)

$$\left(\widetilde{h}_1(p)\otimes\widetilde{h}_2(p)\right)^\eta=\bigcup_{\widetilde{\gamma}_{1m\in\widetilde{h}_1},\widetilde{\gamma}_{2n\in\widetilde{h}_2}}\Bigg(\Bigg\{\left[\widetilde{\gamma}_{1m}\cdot\widetilde{\gamma}_{2n}\right]$$
$$\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}p_{1m}\cdot\sum_{n=1}^{|\widetilde{h}_2(p)|}p_{2n}}\right)\Bigg\}\Bigg)^\eta$$
$$=\bigcup_{\widetilde{\gamma}_{1m}\in\widetilde{h}_1,\widetilde{\gamma}_{2n}\in\widetilde{h}_2}\Bigg\{\left[\widetilde{\gamma}_{1m}\cdot\widetilde{\gamma}_{2n}\right]^\eta\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}\cdot\sum_{n=1}^{|\widetilde{h}_2(p)|}}\right)\Bigg\}$$
$$=\bigcup_{\widetilde{\gamma}_{1m}\in\widetilde{h}_1,\widetilde{\gamma}_{2n}\in\widetilde{h}_2}\Bigg\{\left[\widetilde{\gamma}_{1m}\right]^\eta\cdot\left[\widetilde{\gamma}_{2n}\right]^\eta$$
$$\cdot\left(\frac{p_{1m}\cdot p_{2n}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}\cdot\sum_{n=1}^{|\widetilde{h}_2(p)|}}\right)\Bigg\}=\bigcup_{\widetilde{\gamma}_{1m\in\widetilde{h}_1}}\Bigg\{\left[\widetilde{\gamma}_{1m}\right]^\eta$$
$$\cdot\left(\frac{p_{1m}}{\sum_{m=1}^{|\widetilde{h}_1(p)|}}\right)\Bigg\}\cdot\bigcup_{\widetilde{\gamma}_{2n\in\widetilde{h}_2}}\Bigg\{\left[\widetilde{\gamma}_{2n}\right]^\eta\left(\frac{p_{2n}}{\sum_{n=1}^{|\widetilde{h}_2(p)|}}\right)\Bigg\}$$
$$=\widetilde{h}_1^\eta(p)\otimes\widetilde{h}_2^\eta(p). \tag{11}$$

□

Definition 12 (see [59]). A linear programming model is formulated as follows:

$$\begin{aligned}
\text{Maximize}\quad & Z=d_1x_1+d_2x_2+d_3x_3+\cdots+d_nx_n\\
\text{Subject to}\quad & b_{11}x_1+b_{12}x_2+b_{13}x_3+\cdots+b_{1n}x_n\le c_1\\
& b_{21}x_1+b_{22}x_2+b_{23}x_3+\cdots+b_{2n}x_n\le c_2\\
& \vdots
\end{aligned}$$

$$b_{m1}x_1 + b_{m2}x_2 + b_{m3}x_3 + \cdots + b_{mn}x_n \leq c_m$$
$$x_1, x_2, \ldots, x_n \geq 0, \tag{12}$$

where m denotes the number of constraints and n denotes the number of decision variables $x_1, x_2, \ldots, x_n$. A solution $(x_1, x_2, \ldots, x_n)$ is called feasible if it satisfies all of the constraints. The purpose of the LP methodology is to find the optimal values of the decision variables $x_1, x_2, \ldots, x_n$ for maximizing the linear function Z.

In order to facilitate the DMs, Wu et al. [60] presented some aggregation actions such as the generalized probabilistic interval-valued hesitant fuzzy ordered weighted averaging (GPIVHFOWA) operator and the generalized probabilistic interval-valued hesitant fuzzy ordered weighted geometric (GPIVHFOWG) operator.

3. Multiple Criteria Decision Making with Probabilistic Interval-Valued Hesitant Fuzzy Sets

In this segment, we should use the probabilistic interval-valued hesitant fuzzy weighted aggregation operators to build up the multiple criteria decision making issues. In order to solve multiple criteria decision making issues under probabilistic interval-valued hesitant fuzzy situation, let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria. The ranking order for the criteria is $C_1 \succ C_2 \succ \ldots \succ C_n$. Criteria C_j show a higher significance than C_s, if $j < s$. If the decision makers offer some interval values for the alternative A_i $(i = 1, 2, 3, \ldots, m)$ under the criteria C_j $(j = 1, 2, 3, \ldots, n)$, these values can be considered as an PIVHFE $\widetilde{h}_{ij}(p)$ $(i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n)$. In some cases where two decision makers give the identical value, then the value emerges only once in $\widetilde{h}_{ij}(p)$. The weights given by the experts are conditionally known and can be found by using the LP model defined in Definition 12. On the basis of two actions, PIVHFOWA and PIVHFOWG for probabilistic interval-valued hesitant fuzzy multiple criteria decision making technique can be proceed as follows.

Step 1. Based on LP model described in Definition 12 , obtain the exact weights for the criteria.

Step 2. Based on action operators GPIVHFOWA and PIVHFOWG presented by Wu et al. [60], calculate the probabilistic values $P_{(A_i,C_j)}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ of each alternative against the given criteria.

Step 3. Based on Definition 9 , calculate score values $S_{P_{(A_i,C_j)}}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ of the probabilistic values $P_{(A_i,C_j)}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ obtained in Step 1 to rank the alternatives.

Step 4. Calculate the average of score values of each alternative obtained in Step 3.

Step 5. Rank all the alternatives and choose the best one.

4. Illustrative Example

In order to reinforce theoretical education and encourage the structure of the teaching body, the school of management in a university wants to appoint outstanding educationist. This presentation has raised extraordinary consideration from the school, the university rector, dean of management school, and human resource officials who make a board of decision makers, which will assume the entire liability for this introduction. They made austere assessment for four applicants of E_i $(i = 1, 2, 3, 4)$ under the criteria C_j $(j = 1, 2, 3, 4)$, C_1 is the ethics; C_2 is the research competency; C_3 is the teaching skill; C_4 is the educational experience. The weights of the criteria given by the experts are completely unknown:

$$\begin{aligned}
\textit{Maximize}\quad & W = 0.2w_1^* + 0.1w_2^* + 0.15w_3^* \\
\textit{Subjectto}:\quad & -0.3w_1^* + 0.2w_2^* + 0.5w_3^* \leq 0.55; \\
& 0.2w_1^* - 0.1w_2^* + 0.2w_3^* \leq 0.26; \\
& 0.1w_1^* + 0.2w_2^* - 0.3w_3^* \leq 0.3; \\
& w_1^* + w_2^* + w_3^* = 1; \\
& 0.1 \leq w_1^* \leq 0.2; \\
& 0.2 \leq w_2^* \leq 0.4; \\
& 0.3 \leq w_3^* \leq 0.4.
\end{aligned} \tag{13}$$

The four possible applicants of E_i $(i = 1, 2, 3, 4)$ are to be assessed with the help of probabilistic interval-valued hesitant fuzzy data given in Tables 1–3 by the three decision experts under the above four criteria, the probabilistic interval-valued hesitant fuzzy decision data is shown in the following.

Step 1. By applying the LP model defined in Definition 12 on (13), we get the exact weights for the criteria as $w_1^* = 0.2$; $w_1^* = 0.4$; $w_1^* = 0.4$.

Step 2. Based on action operators GPIVHFOWA and PIVHFOWG presented by Wu et al. [60], calculate the probabilistic values $P_{(E_i,C_j)}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ of each alternative against the given criteria as

$$\begin{aligned}
P(E_1, C_1) = \{ & [0.5427, 0.6435]\,(0.0375), \\
& [0.6272, 0.7298]\,(0.0375), [0.5871, 0.6896]\,(0.0750), \\
& [0.6634, 0.7648]\,(0.0750), [0.6193, 0.5924]\,(0.0375), \\
& [0.6969, 0.6677]\,(0.0375), [0.7703, 0.6320]\,(0.0625), \\
& [0.7361, 0.7000]\,(0.0625), [0.8000, 0.6607]\,(0.1250), \\
& [0.6534, 0.7703]\,(0.1250), [0.7175, 0.8259]\,(0.0625),
\end{aligned}$$

Table 1: The decision making information given by the expert 1.

	C_1	C_2
E_1	{[0.5, 0.6](0.25), [0.7, 0.8](0.5), [0.8, 0.9](0.25)}	{[0.5, 0.6](0.6), [0.6, 0.8](0.4)}
E_2	{[0.5, 0.6](0.5), [0.7, 0.8](0.5)}	{[0.4, 0.5](0.3), [0.6, 0.7](0.3), [0.8, 0.9](0.4)}
E_3	{[0.6, 0.7](0.6), [0.8, 0.9](0.4)}	{[0.2, 0.3](0.2), [0.4, 0.5](0.4), [0.5, 0.6](0.4)}
E_4	{[0.7, 0.8](0.5), [0.8, 0.85](0.3), [0.8, 0.9](0.2)}	{[0.6, 0.7](0.6), [0.7, 0.8](0.2)}
	C_3	C_4
E_1	{[0.5, 0.6](0.8), [0.7, 0.8](0.2)}	{[0.3, 0.4](0.4), [0.4, 0.5](0.6)}
E_2	{[0.4, 0.5](0.3), [0.6, 0.7](0.3), [0.7, 0.8](0.4)}	{[0.5, 0.6](0.6), [0.7, 0.8](0.2)}
E_3	{[0.5, 0.6](0.2), [0.6, 0.7](0.8)}	{[0.4, 0.5](0.4), [0.6, 0.7](0.6)}
E_4	{[0.6, 0.7](0.4), [0.7, 0.8](0.3), [0.8, 0.9](0.3)}	{[0.4, 0.5](0.5), [0.6, 0.7](0.5)}

Table 2: The decision making information given by the expert 2.

	C_1	C_2
E_1	{[0.5, 0.6](0.5), [0.7, 0.8](0.5)}	{[0.6, 0.7](0.7), [0.7, 0.8](0.2), [0.8, 0.9](0.1)}
E_2	{[0.4, 0.5](0.4), [0.6, 0.7](0.6)}	{[0.5, 0.6](0.7), [0.7, 0.8](0.3)}
E_3	{[0.5, 0.6](0.7), [0.6, 0.7](0.2), [0.7, 0.8](0.1)}	{[0.5, 0.6](0.6), [0.7, 0.8](0.2)}
E_4	{[0.6, 0.7](0.4), [0.7, 0.8](0.4)}	{[0.5, 0.6](0.5), [0.7, 0.8](0.1), [0.8, 0.9](0.2)}
	C_3	C_4
E_1	{[0.4, 0.5](0.3), [0.6, 0.7](0.5)}	{[0.4, 0.5](0.3), [0.6, 0.7](0.4), [0.7, 0.8](0.3)}
E_2	{[0.5, 0.6](0.3), [0.7, 0.8](0.4), [0.8, 0.9](0.3)}	{[0.5, 0.6](0.5), [0.6, 0.7](0.5)}
E_3	{[0.3, 0.4](0.4), [0.5, 0.6](0.6)}	{[0.5, 0.6](0.5), [0.6, 0.7](0.2), [0.8, 0.9](0.2)}
E_4	{[0.6, 0.7](0.5), [0.7, 0.8](0.3), [0.8, 0.9](0.2)}	{[0.4, 0.5](0.4), [0.6, 0.7](0.6)}

Table 3: The decision making information given by the expert 3.

	C_1	C_2
E_1	{[0.6, 0.7](0.3), [0.7, 0.8](0.5), [0.8, 0.9](0.2)}	{[0.5, 0.6](0.4), [0.7, 0.8](0.4)}
E_2	{[0.6, 0.7](0.5), [0.7, 0.8](0.3)}	{[0.5, 0.6](0.3), [0.6, 0.7](0.4), [0.7, 0.8](0.3)}
E_3	{[0.5, 0.6(0.4)], [0.6, 0.7](0.4)}	{[0.5, 0.6](0.5), [0.6, 0.7](0.5)}
E_4	{[0.4, 0.5](0.3), [0.6, 0.7](0.4), [0.7, 0.8](0.3)}	{[0.5, 0.6](0.3), [0.6, 0.7](0.3), [0.7, 0.8](0.4)}
	C_3	C_4
E_1	{[0.6, 0.7](0.6), [0.7, 0.8](0.4)}	{[0.6, 0.65](0.4), [0.7, 0.75](0.2), [0.75, 0.8](0.4)}
E_2	{[0.5, 0.6](0.3), [0.6, 0.7](0.7)}	{[0.6, 0.7](0.8), [0.8, 0.9](0.2)}
E_3	{[0.6, 0.7](0.5), [0.7, 0.8](0.5), [0.8, 0.9](0.5)}	{[0.6, 0.65](0.2), [0.7, 0.75](0.4), [0.75, 0.8](0.4)}
E_4	{[0.6, 0.7](0.2), [0.7, 0.8](0.6)}	{[0.6, 0.7](0.4), [0.7, 0.8](0.6)}

$$[0.6871, 0.8000]\,(0.0625)\,, [0.7449, 0.8484]\,(0.0250)\,,$$
$$[0.7115, 0.7298]\,(0.0250)\,, [0.6896, 0.7952]\,(0.050)\,,$$
$$[0.7703, 0.7234]\,(0.050)\,, [0.8259, 0.8259]\,(0.0250)\,,$$
$$[0.7648, 0.8680]\,(0.0250)\}$$
$$S_{P(E_1,C_1)} = 2.0679$$
$$P\left(E_1, C_2\right) = \{[0.5427, 0.6435]\,(0.2100)\,,$$
$$[0.5924, 0.6969]\,(0.0600)\,, [0.6534, 0.7703]\,(0.0300)\,,$$
$$[0.5627, 0.6896]\,(0.1400)\,, [0.6102, 0.7361]\,(0.0400)\,,$$
$$[0.6686, 0.8000]\,(0.0200)\,, [0.6272, 0.7298]\,(0.2100)\,,$$
$$[0.6677, 0.7703]\,(0.0600)\,, [0.7175, 0.8259]\,(0.0300)\,,$$
$$[0.6435, 0.7648]\,(0.1400)\,, [0.6822, 0.8000]\,(0.0400)\,,$$
$$[0.7298, 0.8484]\,(0.0200)\}$$
$$S_{P(E_1,C_2)} = 2.6674$$
$$P\left(E_1, C_3\right) = \{[0.5081, 0.6102]\,(0.1800)\,,$$
$$[0.5817, 0.6822]\,(0.0450)\,, [0.5559, 0.6607]\,(0.1200)\,,$$
$$[0.6224, 0.7234]\,(0.0300)\,, [0.5616, 0.6686\,(0.3000)\,,$$
$$[0.6272, 0.7298]\,(0.0750)\,, [0.6041, 0.7115]\,(0.2000)\,,$$
$$[0.6634, 0.7648]\,(0.0500)\}$$
$$S_{P(E_1,C_3)} = 2.0429$$
$$P\left(E_2, C_1\right) = \{[0.5081, 0.6102]\,(0.1250)\,,$$

$$
\begin{aligned}
&[0.5817, 0.6822]\,(0.1250), [0.5559, 0.6607]\,(0.0750),\\
&[0.6224, 0.7234]\,(0.0750), [0.5616, 0.6686]\,(0.1875),\\
&[0.6272, 0.7298]\,(0.1875), [0.6041, 0.7115]\,(0.1125),\\
&[0.6634, 0.7648]\,(0.1125)\}\\
&S_{P(E_2,C_1)} = 1.8948\\
&P(E_2, C_2) = \{[0.4814, 0.5817]\,(0.0630),\\
&[0.5773, 0.6830]\,(0.0270), [.5218, 0.6224]\,(0.0840),\\
&[0.6102, 0.7138]\,(0.0360), [0.5837, 0.6969]\,(0.0630),\\
&[0.66070.7703]\,(0.0270), [0.5257, 0.6272]\,(0.0270),\\
&[0.6134, 0.7175]\,(0.0840), [0.5627, 0.6634]\,(0.0360),\\
&[0.6435, 0.7449]\,(0.1120), [0.6193, 0.7298]\,(0.0270),\\
&[0.68960.7952]\,(0.0840), [0.5773, 0.6830]\,(0.0630),\\
&[0.6554, 0.7598]\,(0.0360), [0.6102, 0.7138]\,(0.0840),\\
&[0.6822, 0.7831]\,(0.0480), [0.6607, 0.7703]\,(0.0630),\\
&[0.72340.8259]\,(0.0360)\}\\
&S_{P(E_2,C_2)} = 1.7105\\
&P(E_2, C_3) = \{[0.4814, 0.5817]\,(0.0270),\\
&[0.5773, 0.6830]\,(0.0270), [0.6406, 0.7598]\,(0.0360),\\
&[0.5218, 0.6224]\,(0.0360), [0.6102, 0.7138]\,(0.0360),\\
&[0.5257, 0.6272]\,(0.0360), [0.6134, 0.7175]\,(0.0270),\\
&[0.6712, 0.7859]\,(0.0270), [0.5627, 0.6634]\,(0.0480),\\
&[0.6435, 0.7449]\,(0.0630), [0.6686, 0.7831]\,(0.0840),\\
&[0.5486, 0.6518]\,(0.0630), [0.6320, 0.7361]\,(0.0630),\\
&[0.6871, 0.8000]\,(0.1120), [0.6969, 0.8067]\,(0.0630),\\
&[0.5871, 0.6896]\,(0.0840), [0.6634, 0.7648]\,(0.0840),\\
&[0.7138, 0.8217]\,(0.0840)\}\\
&S_{P(E_2,C_3)} = 2.2677\\
&P(E_2, C_4) = \{[0.5427, 0.6435]\,(0.3000),\\
&[0.5817, 0.6822]\,(0.1000), [0.6224, 0.7234]\,(0.0750),\\
&[0.6102, 0.7138]\,(0.0250), [0.6534, 0.7703]\,(0.3000),\\
&[0.6830, 0.7952]\,(0.1000), [0.7138, 0.8217]\,(0.0750),\\
&[0.6435, 0.7449]\,(0.0250)\}\\
&S_{P(E_2,C_4)} = 2.5318
\end{aligned}
\tag{14}
$$

Similarly we can calculate the remaining probabilistic values $P_{(E_i,C_j)}$ of each alternative against the given criteria.

Step 3. Based on Definition 9, calculate score values $S_{P(E_i,C_j)}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ of the probabilistic values $P_{(E_i,C_j)}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ obtained in Step 1 as

$$
\begin{aligned}
S_{P(E_1,C_4)} &= 3.4610;\\
S_{P(E_3,C_1)} &= 2.4574;\\
S_{P(E_3,C_2)} &= 2.1566;\\
S_{P(E_3,C_3)} &= 1.8793;\\
S_{P(E_3,C_4)} &= 4.0163;\\
S_{P(E_4,C_1)} &= 3.0807;\\
S_{P(E_4,C_2)} &= 3.3164;\\
S_{P(E_4,C_3)} &= 3.3068;\\
S_{P(E_4,C_4)} &= 1.8880.
\end{aligned}
\tag{15}
$$

Tables 4 and 5 show all the score values by applying PIVHFOWA and PIVHFOWG actions, respectively.

Step 4. Table 6 shows the average score values of each alternative obtained in Step 2.

Step 5. The ranking order of the alternatives in accordance with the average of score values are obtained in Step 3, $E_4 \succ E_3 \succ E_1 \succ E_2$ and $E_4 \succ E_2 \succ E_1 \succ E_3$, respectively, which shows that E_4 is the best choice.

Table 7 shows the score values of PIVHFEs obtained by using definition of score function given by Ye [61].

The average score values of each alternative are 0.0823, 0.0668, 0.0792, 0.0977. We see that these results gave the ranking order as $E_4 \succ E_1 \succ E_3 \succ E_2$; that is, the best alternative is E_4.

In order to convert the PIVHFEs to IVHFEs, exclude the probabilities from the PIVHFEs in Tables 1–3. Table 8 shows the score values of IVHFEs of each applicant by applying the score function described in [61].

The average score values of each alternative are 0.0736, 0.0700, 0.1163, 0.0859. We see that these results gave the ranking order as $E_3 \succ E_4 \succ E_2 \succ E_1$; that is, the best alternative is E_3.

5. Comparative Analysis

Keeping in mind the end goal to approve the achievability of the proposed decision making technique, a comparative study is conducted using other method based on the same illustrative example. We can see that the ranking order obtained from the two proposed operators PIVHFWA and PIVHFWG is $E_4 \succ E_3 \succ E_1 \succ E_2$ and $E_4 \succ E_2 \succ E_1 \succ E_3$, respectively, which shows that E_4 is the most favorable choice. Based on Table 7, the ranking order of the alternatives is $E_4 \succ E_1 \succ E_3 \succ E_2$, which shows that E_4 is the best one. Though, if we eliminate the probabilities, the PIVHFEs

Table 4: The score values of alternatives by using PIVHFOWA.

	C_1	C_2	C_3	C_4
E_1	2.0679	2.6674	2.0429	3.4610
E_2	1.8948	1.7105	2.2677	2.5318
E_3	2.4574	2.1566	1.8793	4.0163
E_4	3.0807	3.3164	3.3068	1.8880

Table 5: The score values of alternatives by using PIVHFOWG.

	C_1	C_2	C_3	C_4
E_1	1.8991	1.1673	1.1412	1.0472
E_2	2.3072	1.0303	1.4085	1.2153
E_3	1.1412	0.8371	1.0853	0.7734
E_4	3.0807	1.0701	3.3549	0.9725

Table 6: The average score values.

	E_1	E_2	E_3	E_4
Average score of PIVHFOWA	2.5598	2.2349	2.6274	2.8980
Average score of PIVHFOWG	1.3137	1.5153	0.9593	2.1196

Table 7: The score values of PIVHFEs of alternatives against each criterion.

	C_1	C_2	C_3	C_4
E_1	0.0574	0.1215	0.0819	0.0685
E_2	0.0889	0.0475	0.0719	0.0590
E_3	0.1021	0.0630	0.0783	0.0735
E_4	0.0460	0.1266	0.1004	0.1180

Table 8: The score values of IVHFEs of each criterion.

	C_1	C_2	C_3	C_4
E_1	0.0819	0.0719	0.0783	0.0622
hline E_2	0.0685	0.0590	0.0735	0.0789
hline E_3	0.0611	0.2223	0.1277	0.0541
hline E_4	0.0947	0.0583	0.0630	0.1274

in Tables 1–3 will change to IVHFEs; at that point, utilizing the conventional IVHFEs to calculate the score values of the four applicants, the results can be obtained in Table 8. The average score values obtained from Table 8 show ranking order as $E_3 \succ E_4 \succ E_2 \succ E_1$; that is, E_3 is the best choice which is different from the results obtained by using PIVHFEs. From the above discussion, we may conclude that the more information about the criteria causes the stability in the results. However, PIVHFS approach has a limitation that the aggregation operators do not work properly if the DMs assign 0 or 1 probability to the alternative against any criteria in their evaluations. It reveals that the conventional IVHFEs have a genuine loss of data. In fact, in the assessment, our new constructed approach of probabilistic interval-valued hesitant fuzzy data is more reliable. It seems that the probability provided by the DMs is the main factor which affects the ranking order. Figure 1 represents graphically the ranking order of the four alternatives obtained from the above techniques.

6. Conclusions

There are assorted varieties of decision making issues which might be ordered into two classifications multitarget decision making issues that decision maker ought to build up an approach that contains the greatest incentive by taking least assets and multiple criteria decision making issues that decision maker should choose one alternative from various available alternatives so that it has the most utility. In the present paper, we have proposed PIVHFSs and PIVHFEs. The PIVHFEs can efficiently decrease the loss of information which makes the decision making procedure more consistent.

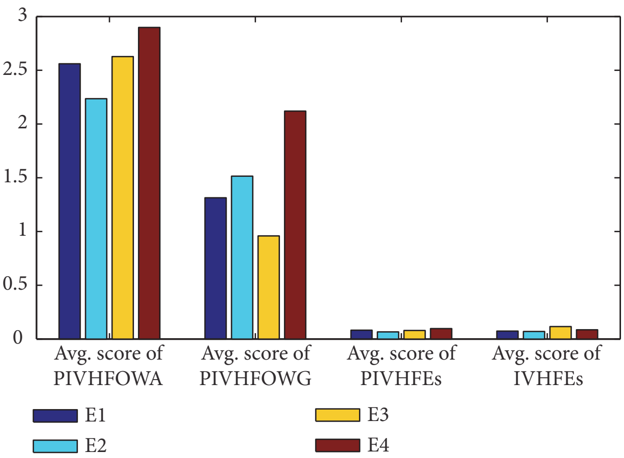

Figure 1: Ranking of alternatives.

We have introduced two aggregation operators PIVHFOWA and PIVHFOWG and examined some properties of PIVHFEs. At the end, an example about the faculty hiring for the school of management in a university was given to establish the practicality and effectiveness of the developed approaches. As indicated by the calculation above, it can be discovered that relatively every score, even the last ranking order of the two strategies, is similar. But the result found by the score function defined in [61] is not predictable with our proposed approaches. It shows that the conventional IVHFSs have a genuine loss of data without any doubt, and comparatively, our new approach of PIVHFSs is more consistent. It is predictable that forthcoming studies may also apply probability along with a diversity of famed sets like neutrosophic sets [62], pythagorean fuzzy set [63], and interval-valued fuzzy soft sets [64] to discourse several MCDM issues.

Data Availability

Our research is not based in any data set.

Conflicts of Interest

The authors declare no conflicts of interest.

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