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*Research Article*

# Kamenev Type Oscillatory Criteria for Linear Conformable Fractional Differential Equations

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Using integral average method and properties of conformable fractional derivative, new Kamenev type oscillation criteria are given firstly for conformable fractional differential equations, which improve known results in oscillation theory. Examples are also given to illustrate the effectiveness of the main results.

## 1. Introduction

Fractional differential equations are used in various fields such as physics, mathematics, biology, biomedical sciences, and finance as well as other disciplines (see [1, 2]). In the last few decades, a lot of attention was paid to finding the more suitable definitions of fractional derivatives. There are many definitions in the existent literature, such as the Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald-Letnikov, Hadamard, and Chen derivatives, etc. In 2014, R. Khalil et al. [3] have suggested a new fractional derivative, which is called the conformable derivative. This new definition satisfies almost all the requirements of standard derivative, for example, the conformable derivative of a constant is zero, and the chain rule (see [3–7] and references therein).

In this paper, we consider the linear conformable fractional differential equation

$$\left(p(t)\, y^{(\alpha)}(t)\right)^{(\alpha)} + q(t)\, y(t) = 0, \quad t \geq t_0, \tag{1}$$

where $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$, $0 < \alpha \leq 1$, and $q$ might change signs.

A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

When $\alpha = 1$, we have the following second-order differential equation:

$$\left(p(t)\, y'(t)\right)' + q(t)\, y(t) = 0. \tag{2}$$

There are a lot of papers involving the oscillation for (2) and other linear, nonlinear, damped, and forced differential equations or Hamiltonian systems (see [8–11]) since the foundation work of Wintner [11] (see also for [12–25]). Especially, if $p(t) \equiv 1$, we obtain the second-order linear Hill equation

$$y''(t) + q(t)\, y(t) = 0. \tag{3}$$

In 1978, Kamenev [8] established a new oscillation criterion of differential equation (1), using integral average method, which has the result of Wintner as a particular case. The obtained result in [8] states that the condition

$$\limsup_{t \longrightarrow \infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m q(s)\, ds = \infty, \tag{4}$$

for some integer $m > 1$, is sufficient for the oscillation of (3).

In 1989, Philos [26] improved the Kamenev type criterion to obtain the following results for (3).

**Theorem 1.** *Let $H : D \equiv \{(t,s) : t \geq s \geq t_0\} \longrightarrow \mathbb{R}$ be a continuous function which satisfies $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $t > s \geq t_0$; $H$ has a continuous and nonpositive partial derivative on $D_0 \equiv \{(t,s) : t > s \geq t_0\}$ with respect to the second variable. Moreover, let $h : D_0 \longrightarrow \mathbb{R}$ be a continuous function with*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \tag{5}$$

*for all $(t,s) \in D_0$. Then (3) is oscillatory if*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right]ds = \infty. \tag{6}$$

**Theorem 2.** *Let $H$ and $h$ be defined as in Theorem 1. Suppose that*

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\longrightarrow\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty \tag{7}$$

*and*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}h^2(t,s)\,ds < \infty. \tag{8}$$

*Moreover, there exists a continuous function $A : [t_0,\infty) \longrightarrow \mathbb{R}$ such that, for every $T \geq t_0$,*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right]ds \geq A(T), \tag{9}$$

*where $A_+(t) = \max\{A(t),0\}$. Then (3) is oscillatory provided*

$$\int_{t_0}^{\infty}A_+^2(t)\,dt = \infty. \tag{10}$$

**Theorem 3.** *Let $H$ and $h$ be defined as in Theorem 1. Suppose that (7) and*

$$\liminf_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)q(s)\,ds < \infty \tag{11}$$

*hold. Moreover, there exists a continuous function $A : [t_0,\infty) \longrightarrow \mathbb{R}$ such that*

$$\liminf_{t\longrightarrow\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right]ds \geq A(T), \tag{12}$$

*for every $T \geq t_0$. Then (10) implies the oscillation of (3).*

As a new research field, the oscillation of fractional differential equations has been widely studied by many authors (e.g., see [27–35] and the references quoted therein). Among these oscillation criteria, most of them are major in the fractional differential equations with forcing terms. The Kamenev type oscillation criteria of fractional differential equations without forcing terms are obtained [27] under the frame of Caputo derivative. However, to our best knowledge, the study of oscillatory behavior of linear conformable fractional differential equation has not been seen in literature. The purpose of this paper is to obtain new Kamenev type oscillatory criteria for (1) using integral average method based on the properties of conformable fractional derivatives and integral. By investigating some new properties of this derivative, the classical oscillatory problem of (2) can be extended to (1). These oscillation criteria improve the results mentioned above. Examples are also given to illustrate the effectiveness of our main results.

## 2. Main Results

Firstly, we give some definitions and properties of conformable fractional derivatives and integral, which are important in the proofs of the main results.

*Definition 4.* The left conformable fractional derivative starting from $t_0$ of a function $f : [t_0,\infty) \longrightarrow \mathbb{R}$ of order $\alpha$ with $0 < \alpha \leq 1$ is defined by

$$\left(\mathbf{T}_{t_0}^{\alpha}f\right)(t) = f^{(\alpha)}(t) = \lim_{\varepsilon\longrightarrow 0}\frac{f\left(t+\varepsilon(t-t_0)^{1-\alpha}\right)-f(t)}{\varepsilon}, \tag{13}$$

when $\alpha = 1$, this derivative of $f(t)$ coincides with $f'(t)$. If $(\mathbf{T}_{t_0}^{\alpha}f)(t)$ exists on $(t_0,t_1)$ then

$$\left(\mathbf{T}_{t_0}^{\alpha}f\right)(t_0) = \lim_{t\longrightarrow t_0^+}f^{(\alpha)}(t). \tag{14}$$

*Definition 5.* Let $\alpha \in (0,1]$. Then the left conformable fractional integral of order $\alpha$ starting at $t_0$ is defined by

$$\left(\mathbf{I}_{t_0}^{\alpha}f\right)(t) = \int_{t_0}^{t}(s-t_0)^{\alpha-1}f(s)\,ds := \int_{t_0}^{t}f(s)\,d_{t_0}^{\alpha}s. \tag{15}$$

If the conformable fractional integral of a given function $f$ exists, we call that $f$ is $\alpha$−integrable.

**Lemma 6** (see [4]). *If $\alpha \in (0,1]$ and $f \in C^1([t_0,\infty),\mathbb{R})$, then, for all $t > t_0$, we have*

$$\mathbf{I}_{t_0}^{\alpha}\mathbf{T}_{t_0}^{\alpha}(f)(t) = f(t)-f(t_0) \tag{16}$$

*and*

$$\mathbf{T}_{t_0}^{\alpha}\mathbf{I}_{t_0}^{\alpha}(f)(t) = f(t). \tag{17}$$

**Lemma 7** (see [3]). *(1) $\mathbf{T}_{t_0}^{\alpha}(af+bg) = a\mathbf{T}_{t_0}^{\alpha}(f)+b\mathbf{T}_{t_0}^{\alpha}(g)$ for all real constant a, b.*

*(2) $\mathbf{T}_{t_0}^{\alpha}(fg) = f\mathbf{T}_{t_0}^{\alpha}(g)+g\mathbf{T}_{t_0}^{\alpha}(f)$.*

*(3) $\mathbf{T}_{t_0}^{\alpha}(t^p) = pt^{p-\alpha}$, for all p.*

*(4) $\mathbf{T}_{t_0}^{\alpha}(f/g) = (g\mathbf{T}_{t_0}^{\alpha}(f)-f\mathbf{T}_{t_0}^{\alpha}(g))/g^2$.*

*(5) $\mathbf{T}_{t_0}^{\alpha}(c) = 0$, where c is a constant.*

**Lemma 8** (see [6]). *Let $f, g : [t_0, t_1] \longrightarrow \mathbb{R}$ be two functions such that $fg$ is differentiable. Then*

$$\int_{t_0}^{t_1} f(s)\, g^{(\alpha)}(s)\, d_{t_0}^{\alpha} s = f(s)\, g(s)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} g(s)\, f^{(\alpha)}(s)\, d_{t_0}^{\alpha} s. \tag{18}$$

**Lemma 9** (Cauchy-Schwarz inequality with conformable fractional derivative). *Let $f, g : [t_0, \infty) \longrightarrow \mathbb{R}$ be two functions such that $f^2$ and $g^2$ are $\alpha$–integrable. Then*

$$\left|\int_{t_0}^{t} f(s)\, g(s)\, d_{t_0}^{\alpha} s\right|^2 \le \int_{t_0}^{t} |f(s)|^2\, d_{t_0}^{\alpha} s \int_{t_0}^{t} |g(s)|^2\, d_{t_0}^{\alpha} s. \tag{19}$$

**Theorem 10.** *Let $H(t,s)$ and $h(t,s)$ be defined as that in Theorem 1. Then (1) is oscillatory if*

$$\limsup_{t \longrightarrow \infty} \frac{1}{H(t,t_0)} \int_{t_0+1}^{t} \left[H(t,s)\, q(s) - \frac{1}{4}(s-t_0)^{2-2\alpha}\, p(s)\, h^2(t,s)\right] d_{t_0}^{\alpha} s = \infty. \tag{20}$$

*Proof.* Suppose on the contrary that there exists a nontrivial solution $y(t)$ of (1) which is not oscillatory. Without loss of generality, we may suppose that $y(t) \neq 0$ for all $t \geq t_0$. Define $V(t) = p(t) y^{(\alpha)}(t)/y(t)$ for $t \geq t_0$; by the properties of conformable fractional derivative, we get the Riccati equation

$$V^{(\alpha)}(t) + q(t) + \frac{1}{p(t)} V^2(t) = 0, \quad t \geq t_0. \tag{21}$$

Thus, for every $t, T$ with $t \geq T \geq t_0$, we have

$$\int_{T}^{t} H(t,s)\, q(s)\, d_{t_0}^{\alpha} s = -\int_{T}^{t} H(t,s)\, V^{(\alpha)}(s)\, d_{t_0}^{\alpha} s - \int_{T}^{t} H(t,s) \frac{V^2(s)}{p(s)} d_{t_0}^{\alpha} s. \tag{22}$$

Using Lemma 8, for $t \geq T \geq t_0$, we get

$$\begin{aligned}
\int_{T}^{t} H(t,s)\, q(s)\, d_{t_0}^{\alpha} s &= -\int_{T}^{t} H(t,s)\, V^{(\alpha)}(s)\, d_{t_0}^{\alpha} s - \int_{T}^{t} H(t,s) \frac{V^2(s)}{p(s)} d_{t_0}^{\alpha} s = H(t,T)\, V(T) \\
&+ \int_{T}^{t} \sqrt{H(t,s)}\, (s-t_0)^{1-\alpha}\, h(t,s)\, V(s)\, d_{t_0}^{\alpha} s \\
&- \int_{T}^{t} H(t,s) \frac{V^2(s)}{p(s)} d_{t_0}^{\alpha} s = H(t,T)\, V(T) + \frac{1}{4} \int_{T}^{t} (s \\
&- t_0)^{2-2\alpha}\, p(s)\, h^2(t,s)\, d_{t_0}^{\alpha} s - \int_{T}^{t} \left[\sqrt{H(t,s)} \frac{V(s)}{\sqrt{p(s)}} \right. \\
&\left. + \frac{(s-t_0)^{1-\alpha}}{2} \sqrt{p(s)} h(t,s)\right]^2 d_{t_0}^{\alpha} s \le H(t,T)\, V(T) \\
&+ \frac{1}{4} \int_{T}^{t} (s-t_0)^{2-2\alpha}\, p(s)\, h^2(t,s)\, d_{t_0}^{\alpha} s.
\end{aligned} \tag{23}$$

Thus, for every $t \geq T \geq t_0$,

$$\begin{aligned}
&\int_{T}^{t} \left[H(t,s)\, q(s) - \frac{1}{4}(s-t_0)^{2-2\alpha}\, p(s)\, h^2(t,s)\right] d_{t_0}^{\alpha} s \\
&\quad \le H(t,T)\, V(T) \le H(t,T)\, |V(T)| \\
&\quad \le H(t,t_0)\, |V(T)|.
\end{aligned} \tag{24}$$

This inequality holds for $T = t_0 + 1$, which means

$$\begin{aligned}
&\int_{t_0+1}^{t} \left[H(t,s)\, q(s) - \frac{1}{4}(s-t_0)^{2-2\alpha}\, p(s)\, h^2(t,s)\right] d_{t_0}^{\alpha} s \\
&\quad \le H(t,t_0)\, |V(t_0+1)|.
\end{aligned} \tag{25}$$

Dividing both sides of (25) by $H(t,t_0)$, and taking the upper limit in both sides as $t \longrightarrow \infty$, the right hand side is always bounded, which contradicts hypothesis (20). This completes the proof of Theorem 10. □

*Remark 11.* The singularity of integral at $t_0$ has no affection on the oscillation of conformable fractional differential equations since oscillation is a qualitative property at infinite. The results are still true if the lower bound of the integration in (20) is replaced by any other point larger than $t_0$.

Let us consider the function $H$ defined by

$$H(t,s) = (t-s)^m, \quad t \geq s \geq t_0, \tag{26}$$

where $m$ is an integer with $m > 1$. Then $H(t,t) = 0$, $H(t,s) > 0$ for $t > s \geq t_0$, and by direct calculation, we have

$$h(t,s) = m(t-s)^{(m-2)/2}, \quad t > s \geq t_0. \tag{27}$$

Under a modification of the hypotheses of Theorem 10, we can obtain the following result.

**Corollary 12.** *If there exists an integer $m > 1$ such that*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^m}\int_{t_0+1}^{t}\left[(t-s)^m q(s)-\frac{m^2}{4}(t-s)^{m-2}(s-t_0)^{2-2\alpha}\right]d_{t_0}^{\alpha}s=\infty, \tag{28}$$

*then (1) is oscillatory.*

If $p(t)\equiv 1$ in (1), we can get

$$\lim_{t\longrightarrow\infty}\frac{1}{t^m}\int_{t_0+1}^{t}\frac{m^2}{4}(t-s)^{m-2}(s-t_0)^{2-2\alpha}d_{t_0}^{\alpha}s<\infty, \tag{29}$$

and then we further obtain the following.

**Corollary 13.** *If $p(t)\equiv 1$ in (1) and there exists an integer $m>1$ such that*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^m}\int_{t_0+1}^{t}(t-s)^m q(s)\,d_{t_0}^{\alpha}s=\infty, \tag{30}$$

*then (1) is oscillatory.*

**Theorem 14.** *Let $H$ and $h$ be defined as in Theorem 10. Moreover, suppose that*

$$0<\inf_{s\geq t_0}\left[\liminf_{t\longrightarrow\infty}\frac{H(t,s)}{H(t,t_0)}\right]\leq\infty \tag{31}$$

*and*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}p(s)(s-t_0)^{2(1-\alpha)}h^2(t,s)\,d_{t_0}^{\alpha}s<\infty \tag{32}$$

*hold. If there exists a function $A\in C([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s)-\frac{1}{4}p(s)(s-t_0)^{2(1-\alpha)}h^2(t,s)\right]d_{t_0}^{\alpha}s\geq A(T) \tag{33}$$

*for every $T\geq t_0$, then (1) is oscillatory provided*

$$\int_{t_0+1}^{\infty}\frac{A_+^2(t)}{p(t)}d_{t_0}^{\alpha}t=\infty. \tag{34}$$

*Proof.* Suppose that (1) possesses a nonoscillatory solution $y(t)$; without loss of generality, we may also suppose that $y(t)\neq 0$ for $t\geq t_0$. Define $V(t)=p(t)y^{(\alpha)}(t)/y(t)$; as in the proof of Theorem 10, we get

$$\begin{aligned}&\int_{T}^{t}\left[H(t,s)q(s)-\frac{1}{4}(s-t_0)^{2-2\alpha}p(s)h^2(t,s)\right]d_{t_0}^{\alpha}s\\&=H(t,T)V(T)-\int_{T}^{t}\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2 d_{t_0}^{\alpha}s\end{aligned} \tag{35}$$

for all $t$, $T$ with $t\geq T\geq t_0$. Then we have

$$\begin{aligned}&\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s)\right.\\&\left.-\frac{1}{4}(s-t_0)^{2-2\alpha}p(s)h^2(t,s)\right]d_{t_0}^{\alpha}s=V(T)\\&-\frac{1}{H(t,T)}\int_{T}^{t}\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2 d_{t_0}^{\alpha}s.\end{aligned} \tag{36}$$

Hence,

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)q(s)\right.\\&\left.-\frac{1}{4}(s-t_0)^{2-2\alpha}p(s)h^2(t,s)\right]d_{t_0}^{\alpha}s=V(T)\\&-\liminf_{t\longrightarrow\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2 d_{t_0}^{\alpha}s.\end{aligned} \tag{37}$$

Thus, by condition (33), we obtain for $T\geq t_0$

$$\begin{aligned}V(T)\geq{}&A(T)+\liminf_{t\longrightarrow\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2 d_{t_0}^{\alpha}s.\end{aligned} \tag{38}$$

This implies that

$$V(T) \geq A(T) \tag{39}$$

for every $T \geq t_0$, and

$$\liminf_{t\longrightarrow\infty} \frac{1}{H(t,T)} \int_T^t \left[ \sqrt{H(t,s)} \frac{V(s)}{\sqrt{p(s)}} + \frac{(s-t_0)^{1-\alpha}}{2} \sqrt{p(s)} h(t,s) \right]^2 d_{t_0}^{\alpha} s < \infty, \quad T \geq t_0. \tag{40}$$

Then for every $T_0 > t_0$, we get

$$\begin{aligned} \infty &> \liminf_{t\longrightarrow\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \left[ \sqrt{H(t,s)} \frac{V(s)}{\sqrt{p(s)}} + \frac{(s-t_0)^{1-\alpha}}{2} \sqrt{p(s)} h(t,s) \right]^2 d_{t_0}^{\alpha} s \\ &\geq \liminf_{t\longrightarrow\infty} \left[ \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{H(t,s)}{p(s)} V^2(s)\, d_{t_0}^{\alpha} s + \frac{1}{H(t,T_0)} \cdot \int_{T_0}^t \sqrt{H(t,s)} (s-t_0)^{1-\alpha} h(t,s) V(s)\, d_{t_0}^{\alpha} s \right], \end{aligned} \tag{41}$$

i.e., we have

$$\liminf_{t\longrightarrow\infty} [P(t) + Q(t)] < \infty, \tag{42}$$

where

$$P(t) = \frac{1}{H(t,T_0)} \int_{T_0}^t H(t,s) \frac{V^2(s)}{p(s)} d_{t_0}^{\alpha} s, \quad t \geq T_0 > t_0 \tag{43}$$

and

$$Q(t) = \frac{1}{H(t,T_0)} \cdot \int_{T_0}^t \sqrt{H(t,s)} (s-t_0)^{1-\alpha} h(t,s) V(s)\, d_{t_0}^{\alpha} s, \quad t \geq T_0 > t_0. \tag{44}$$

Now we claim that

$$\int_{T_0}^{\infty} \frac{V^2(t)}{p(t)} d_{t_0}^{\alpha} t < \infty. \tag{45}$$

By condition (31), there exists a constant $\gamma$ with

$$\inf_{s\geq t_0} \left[ \liminf_{t\longrightarrow\infty} \frac{H(t,s)}{H(t,t_0)} \right] > \gamma > 0. \tag{46}$$

If (45) is not true, then for each $\lambda > 0$, there exists a $T_1 > T_0$ such that

$$\int_{T_0}^t \frac{V^2(s)}{p(s)} d_{t_0}^{\alpha} s \geq \frac{2\lambda}{\gamma}, \quad t \geq T_1. \tag{47}$$

Then for all $t \geq T_1$, we have

$$\begin{aligned} P(t) &= \frac{1}{H(t,T_0)} \int_{T_0}^t H(t,s) \left[ \int_{T_0}^s \frac{V^2(\xi)}{p(\xi)} d_{t_0}^{\alpha} \xi \right]^{(\alpha)} d_{t_0}^{\alpha} s \\ &= -\frac{1}{H(t,T_0)} \int_{T_0}^t \left[ \int_{T_0}^s \frac{V^2(\xi)}{p(\xi)} d_{t_0}^{\alpha} \xi \right] [H(t,s)]_s^{(\alpha)} d_{t_0}^{\alpha} s \\ &\geq -\frac{1}{H(t,T_0)} \int_{T_1}^t \left[ \int_{T_0}^s \frac{V^2(\xi)}{p(\xi)} d_{t_0}^{\alpha} \xi \right] [H(t,s)]_s^{(\alpha)} d_{t_0}^{\alpha} s \\ &\geq -\frac{2\lambda}{\gamma} \frac{1}{H(t,T_0)} \int_{T_1}^t [H(t,s)]_s^{(\alpha)} d_{t_0}^{\alpha} s \\ &= \frac{2\lambda}{\gamma} \frac{H(t,T_1)}{H(t,T_0)}. \end{aligned} \tag{48}$$

However,

$$\liminf_{t\longrightarrow\infty} \frac{H(t,T_1)}{H(t,t_0)} > \gamma, \tag{49}$$

using the continuity of $H$ and noticing that $T_0 > t_0$ can be arbitrarily close to $t_0$, we can choose a $T_1' \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,T_0)} \geq \frac{\gamma}{2} \tag{50}$$

for every $t \geq T_1'$. Hence, we have $P(t) \geq \lambda$, for all $t \geq T_1'$. Since $\lambda > 0$ is arbitrary, we have

$$\lim_{t\longrightarrow\infty} P(t) = \infty. \tag{51}$$

Next, let a sequence $\{t_i\}, i = 1, 2, \cdots$, in the interval $[T_0, \infty)$, be selected such that $\lim_{i\longrightarrow\infty} t_i = \infty$, and

$$\lim_{i\longrightarrow\infty} [P(t_i) + Q(t_i)] = \liminf_{t\longrightarrow\infty} [P(t) + Q(t)]. \tag{52}$$

By (42), there exists a constant $C_0$ such that

$$P(t_i) + Q(t_i) \leq C_0, \quad i = 1, 2, \cdots. \tag{53}$$

Using (51) and (53), we have

$$\lim_{i\longrightarrow\infty} P(t_i) = \infty \tag{54}$$

and hence

$$\lim_{i\longrightarrow\infty} Q(t_i) = -\infty. \tag{55}$$

By (53) and (54), we obtain

$$1 + \frac{Q(t_i)}{P(t_i)} \leq \frac{M}{P(t_i)} < \frac{1}{2}, \tag{56}$$

provided that $i$ is sufficiently large. Therefore,

$$\frac{Q(t_i)}{P(t_i)} < -\frac{1}{2}, \tag{57}$$

for all large $i$. Then (55) ensures that

$$\lim_{i\longrightarrow\infty}\frac{Q^2(t_i)}{P(t_i)}=\infty. \tag{58}$$

Using Lemma 9 and Cauchy-Schwarz inequality, we have, for any positive integer $i$,

$$\begin{aligned}Q^2(t_i)&=\frac{1}{H^2(t_i,T_0)}\left[\int_{T_0}^{t_i}h(t_i,s)\right.\\&\quad\cdot\left.\sqrt{H(t_i,s)}(s-t_0)^{1-\alpha}V(s)\,d_{t_0}^{\alpha}s\right]^2\le\left[\frac{1}{H(t_i,T_0)}\right.\\&\quad\cdot\left.\int_{T_0}^{t_i}p(s)(s-t_0)^{2(1-\alpha)}h^2(t_i,s)\,d_{t_0}^{\alpha}s\right]\cdot\left[\frac{1}{H(t_i,T_0)}\right.\\&\quad\cdot\left.\int_{T_0}^{t_i}\frac{H(t_i,s)}{p(s)}V^2(s)\,d_{t_0}^{\alpha}s\right]\le\left[\frac{1}{H(t_i,T_0)}\int_{T_0}^{t_i}p(s)\right.\\&\quad\cdot\left.(s-t_0)^{2(1-\alpha)}h^2(t_i,s)\,d_{t_0}^{\alpha}s\right]P(t_i),\end{aligned}\tag{59}$$

and consequently

$$\begin{aligned}&\frac{Q^2(t_i)}{P(t_i)}\\&\quad\le\frac{1}{H(t_i,T_0)}\int_{T_0}^{t_i}p(s)(s-t_0)^{2(1-\alpha)}h^2(t_i,s)\,d_{t_0}^{\alpha}s\end{aligned}\tag{60}$$

for all large $i$. By (46), we get

$$\liminf_{t\longrightarrow\infty}\frac{H(t,T_0)}{H(t,t_0)}>\gamma, \tag{61}$$

and then there exists a $T_0''\ge T_0$, such that

$$\frac{H(t,T_0)}{H(t,t_0)}\ge\gamma, \tag{62}$$

for every $t\ge T_0''$. Hence,

$$\frac{H(t_i,T_0)}{H(t_i,t_0)}\ge\gamma, \tag{63}$$

for sufficiently large $i$. Therefore,

$$\begin{aligned}&\frac{Q^2(t_i)}{P(t_i)}\\&\quad\le\frac{1}{\gamma}\frac{1}{H(t_i,t_0)}\int_{T_0}^{t_i}p(s)(s-t_0)^{2(1-\alpha)}h^2(t_i,s)\,d_{t_0}^{\alpha}s\\&\quad\le\frac{1}{\gamma}\frac{1}{H(t_i,t_0)}\int_{t_0}^{t_i}p(s)(s-t_0)^{2(1-\alpha)}h^2(t_i,s)\,d_{t_0}^{\alpha}s\end{aligned}\tag{64}$$

for all large $i$. Using (58), we have

$$\begin{aligned}&\lim_{i\longrightarrow\infty}\frac{1}{H(t_i,t_0)}\int_{t_0}^{t_i}p(s)(s-t_0)^{2(1-\alpha)}h^2(t_i,s)\,d_{t_0}^{\alpha}s\\&\quad=\infty.\end{aligned}\tag{65}$$

This implies that

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}p(s)(s-t_0)^{2(1-\alpha)}h^2(t,s)\,d_{t_0}^{\alpha}s\\&\quad=\infty,\end{aligned}\tag{66}$$

which contradicts condition (32). This gives (45). Since $T_0>t_0$ can be selected arbitrarily, we have

$$\int_{t_0+1}^{\infty}\frac{V^2(s)}{p(s)}d_{t_0}^{\alpha}s<\infty. \tag{67}$$

Using (39), we have

$$\int_{t_0+1}^{\infty}\frac{A_+^2(s)}{p(s)}d_{t_0}^{\alpha}s\le\int_{t_0+1}^{\infty}\frac{V^2(s)}{p(s)}d_{t_0}^{\alpha}s<\infty, \tag{68}$$

which contradicts condition (34). This completes the proof of Theorem 14. □

**Corollary 15.** *If $H(t,s)=(t-s)^m$, $m>1$ in Theorem 14, and (32) and (33) are replaced by*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^m}\int_{t_0}^{t}p(s)(s-t_0)^{2(1-\alpha)}(t-s)^{m-2}\,d_{t_0}^{\alpha}s<\infty, \tag{69}$$

*and*

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\frac{1}{t^m}\int_{T}^{t}\left[(t-s)^m q(s)\right.\\&\quad\left.-\frac{m^2p(s)}{4}(s-t_0)^{2(1-\alpha)}(t-s)^{2(m-2)}\right]d_{t_0}^{\alpha}s\ge A(T),\end{aligned}\tag{70}$$

*then (1) is oscillatory provided (34) holds.*

**Theorem 16.** *Let $H$ and $h$ be as in Theorem 10. Moreover, suppose that (31) and*

$$\liminf_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)q(s)\,d_{t_0}^{\alpha}s<\infty \tag{71}$$

*hold. If there exists a continuous function $A$ on $[t_0,\infty)$ such that*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)q(s)-\frac{1}{4}p(s)(s-t_0)^{2(1-\alpha)}h^2(t,s)\right]d_{t_0}^{\alpha}s\ge A(T) \tag{72}$$

*for every $T\ge t_0$, then (1) is oscillatory provided (34) is satisfied.*

*Proof.* Suppose on the contrary that there exists a prepared solution $y$ of (1) which is not oscillatory. Suppose that $y(t)\neq 0$, for all $t\ge t_0$. Let $V(t)=p(t)y^{(\alpha)}(t)/y(t)$, on $[t_0,\infty)$; as in the proof of Theorem 14, we get (35) for any $t,T_0,T$ with $t\ge T\ge T_0>t_0$. Thus

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)q(s)\right.\\&\left.-\frac{1}{4}(s-t_0)^{2-2\alpha}p(s)h^2(t,s)\right]d_{t_0}^{\alpha}s=V(T)\\&-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2d_{t_0}^{\alpha}s.\end{aligned}\tag{73}$$

Using condition (72), we get

$$\begin{aligned}V(T)\ge A(T)+\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2d_{t_0}^{\alpha}s.\end{aligned}\tag{74}$$

By (39), we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2d_{t_0}^{\alpha}s<\infty,\quad T\ge T_0.\end{aligned}\tag{75}$$

Hence, we have

$$\begin{aligned}\infty&>\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\left[\sqrt{H(t,s)}\frac{V(s)}{\sqrt{p(s)}}\right.\\&\left.+\frac{(s-t_0)^{1-\alpha}}{2}\sqrt{p(s)}h(t,s)\right]^2d_{t_0}^{\alpha}s\\&=\limsup_{t\to\infty}\left[\frac{1}{H(t,T_0)}\int_{T_0}^tH(t,s)\frac{V^2(s)}{p(s)}d_{t_0}^{\alpha}s\right.\\&+\frac{1}{H(t,T_0)}\\&\left.\cdot\int_{T_0}^t(s-t_0)^{1-\alpha}h(t,s)\sqrt{H(t,s)}V(s)\,d_{t_0}^{\alpha}s\right],\end{aligned}\tag{76}$$

i.e.,

$$\limsup_{t\to\infty}[P(t)+Q(t)]<\infty,\tag{77}$$

where $P$ and $Q$ are defined by (43) and (44). Using condition (72), we obtain

$$\begin{aligned}A(t_0)&\le\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^tH(t,s)q(s)\,d_{t_0}^{\alpha}s-\frac{1}{4}\\&\cdot\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\\&\cdot\int_{t_0}^t(s-t_0)^{2(1-\alpha)}p(s)h^2(t,s)\,d_{t_0}^{\alpha}s\end{aligned}\tag{78}$$

and by condition (71), we get

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t(s-t_0)^{2(1-\alpha)}p(s)h^2(t,s)\,d_{t_0}^{\alpha}s\\&<\infty.\end{aligned}\tag{79}$$

Hence there exists a sequence $\{t_i\}$, $i=1,2,\cdots$ in the interval $(T_0,\infty)$ with $\lim_{i\to\infty}t_i=\infty$ and such that

$$\begin{aligned}&\liminf_{i\to\infty}\frac{1}{H(t_i,t_0)}\int_{t_0}^{t_i}(s-t_0)^{2(1-\alpha)}p(s)h^2(t_i,s)\,d_{t_0}^{\alpha}s\\&<\infty.\end{aligned}\tag{80}$$

If (47) holds, following the procedure of the proof of Theorem 14, we conclude that (51) is satisfied. By (77), there exists a constant $C_0$, such that (53) is fulfilled. Then, as in the proof of Theorem 14, we can arrive at (65), which contradicts (80). This proves that (45) holds. The remainder of the proof proceeds as in the proof of Theorem 14. □

**Corollary 17.** *If $H(t,s)=(t-s)^m$, $m>1$ in Theorem 16, and (71) and (72) are replaced by*

$$\liminf_{t\to\infty}\frac{1}{t^m}\int_{t_0}^t(t-s)^mq(s)\,d_{t_0}^{\alpha}s<\infty,\tag{81}$$

*and*

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t^m}\int_T^t\left[(t-s)^mq(s)\right.\\&\left.-\frac{m^2}{4}(s-t_0)^{2(1-\alpha)}(t-s)^{(m-2)}p(s)\right]d_{t_0}^{\alpha}s\ge A(T)\end{aligned}\tag{82}$$

*for every $T\ge t_0$, then (1) is oscillatory provided (34) is satisfied.*

*Remark 18.* Choosing different functions $H(t,s)$ and $h(t,s)$ in Theorems 10, 14, and 16, we can obtain various oscillation criteria for (1). For example, $H(t,s) \equiv \rho(t-s)$, where $\rho(u)$ is a continuously differentiable function on $[0,\infty)$, $\rho(0)=0$, $\rho(u)>0$, $\rho'(u)\geq 0$ for $u>0$; $H(t,s)=(\ln(t/s))^{m-1}$, $t\geq s\geq t_0$; $H(t,s)=(\int_s^t (dz/\rho(z)))^{n-1}$, $t\geq s\geq t_0$, where $n$ is an integer with $n>2$ and $\rho$ is a positive continuous function on $[t_0,\infty)$ such that $\int_{t_0}^{\infty}(1/\rho(z))dz=\infty$.

*Remark 19.* If $\alpha \equiv 1$, then Theorem 10, Theorem 14, and Theorem 16 reduce to Theorem 1.1, Theorem 1.2, and Theorem 1.3 in [26], respectively. Our results improve the results mentioned above, since they can handle the cases not covered by known results.

*Remark 20.* Theorems 10–16 are not suitable for differential equations with Caputo or Riemann-Liouville fractional derivatives in general cases.

*Example 21.* Consider the following conformable fractional differential equation:

$$\left(\gamma (t-t_0)^{\lambda} y^{(\alpha)}(t)\right)^{(\alpha)} + y(t) = 0, \quad t \geq t_0,\ 0<\alpha\leq 1, \tag{83}$$

where $\gamma$, $\lambda$ are constants. Using Corollary 12, we can verify that

$$\begin{aligned}
&\frac{1}{t^m}\int_{T_1}^{t}\left[(t-s)^m - \frac{m^2\gamma}{4}(t-s)^{m-2}(s-t_0)^{2-2\alpha}(s-t_0)^{\lambda}\right] d_{t_0}^{\alpha} s = \frac{1}{t^m} \\
&\cdot \int_{T_1}^{t}(t-s)^m (s-t_0)^{\alpha-1} ds - \frac{1}{t^m}\int_{T_1}^{t}\frac{m^2\gamma}{4}(t \\
&- s)^{m-2}(s-t_0)^{\lambda-\alpha+1} ds \geq \frac{1}{t^m}\int_{T_1}^{t}(t-s)^m (s \\
&- T_1)^{\alpha-1} ds - \frac{m^2\gamma}{4t^m}\int_{t_0}^{t}(t-s)^{m-2}(s \\
&- t_0)^{\lambda-\alpha+1} ds = \frac{1}{t^m}(t-T_1)^{m+\alpha} B(m+1,\alpha) \\
&- \frac{m^2\gamma}{4t^m}(t-t_0)^{m+\lambda-\alpha} B(m-1,\lambda-\alpha+2),
\end{aligned} \tag{84}$$

where $B(\cdot,\cdot)$ is the classical Beta function defined by

$$B(m,n) = \int_0^1 (1-s)^{m-1} s^{n-1} ds. \tag{85}$$

By

$$\lim_{t\to\infty}\frac{1}{t^m}(t-T_1)^{m+\alpha} B(m+1,\alpha) = \infty \tag{86}$$

and

$$\lim_{t\to\infty}\frac{m^2\gamma}{4t^m}(t-t_0)^{m+\lambda-\alpha} B(m-1,\lambda-\alpha+2) = \begin{cases} \dfrac{m^2\gamma}{4}B(m-1,2), & \lambda=\alpha, \\ 0, & \lambda<\alpha. \end{cases} \tag{87}$$

we get

$$\begin{aligned}
\infty &\geq \limsup_{t\to\infty}\frac{1}{t^m}\int_{T_1}^{t}\left[(t-s)^m - \frac{m^2\gamma}{4}(t-s)^{m-2}(s-t_0)^{2-2\alpha}(s-t_0)^{\lambda}\right] d_{t_0}^{\alpha} s \\
&\geq \limsup_{t\to\infty}\frac{1}{t^m}(t-T_1)^{m+\alpha} B(m+1,\alpha) \\
&- \limsup_{t\to\infty}\frac{m^2\gamma}{4t^m}(t-t_0)^{m+\lambda-\alpha} B(m-1,\lambda-\alpha+2) = \infty.
\end{aligned} \tag{88}$$

Hence by Corollary 12, we obtain that the conformable fractional differential equation (83) is oscillatory.

*Example 22.* Consider the following conformable fractional differential equation:

$$\left(y^{(\alpha)}(t)\right)^{(\alpha)} + t^{\gamma} y(t) = 0, \quad t\geq 0,\ 0<\alpha\leq 1, \tag{89}$$

where $\gamma$ is constant. Since $p(t)\equiv 1$, using Corollary 13, we can verify that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^m}\int_1^t s^{\gamma}(t-s)^m d_{t_0}^{\alpha} s \\
&= \lim_{t\to\infty}\frac{1}{t^m}\int_0^1 (t-s)^m s^{\gamma+\alpha-1} ds \\
&+ \lim_{t\to\infty}\frac{1}{t^m}\int_1^t (t-s)^m s^{\gamma+\alpha-1} ds \\
&= \lim_{t\to\infty}\frac{1}{t^m}B(m+1,\alpha+\gamma)\, t^{m+\alpha+\gamma} = +\infty \\
&\qquad \text{for } \gamma > -\alpha.
\end{aligned} \tag{90}$$

Hence by Corollary 13, we conclude that the conformable fractional differential equation (89) is oscillatory for $\gamma > -\alpha$.

*Example 23.* Consider the following conformable fractional differential equation:

$$\left(t^{\lambda} y^{(\alpha)}(t)\right)^{(\alpha)} + t^{\gamma}\cos\left(\frac{t^{\alpha}}{\alpha}\right) y(t) = 0, \quad t\geq 0,\ 0<\alpha\leq 1, \tag{91}$$

where $\gamma, \lambda$ are constants. We conclude that (91) is oscillatory for $\lambda\leq\alpha$ and $2\gamma-\lambda+\alpha\geq 0$. In fact, taking $H(t,s)=(t-s)^2$, then direct calculation implies $h(t,s)=2$. Using Corollary 15, we can verify that

$$\limsup_{t\longrightarrow\infty}\frac{1}{t^2}\int_0^t s^{\lambda}s^{2(1-\alpha)}d_{t_0}^{\alpha}s=\limsup_{t\longrightarrow\infty}\frac{1}{t^2}\int_0^t s^{\lambda+1-\alpha}ds$$
$$=\lim_{t\longrightarrow\infty}\frac{1}{(\lambda+2-\alpha)t^2}t^{\lambda+2-\alpha}<+\infty\quad\text{for }\lambda\le\alpha.$$

$$\begin{aligned}&\frac{1}{t^m}\int_T^t\left[(t-s)^m q(s)\right.\\&\left.-\frac{m^2p(s)}{4}(s-t_0)^{2(1-\alpha)}(t-s)^{2(m-2)}\right]d_{t_0}^{\alpha}s=\frac{1}{t^2}\\&\cdot\int_T^t\left[(t-s)^2s^{\gamma}\cos\left(\frac{t^{\alpha}}{\alpha}\right)-s^{\lambda+2(1-\alpha)}\right]d_{t_0}^{\alpha}s\ge-T^{\gamma}\\&\cdot\sin\left(\frac{T^{\alpha}}{\alpha}\right)-k,\end{aligned}\tag{92}$$

where $k$ is a positive constant. Set $A(s)=-s^{\gamma}\sin(s^{\alpha}/\alpha)-k$; there is an integer $N>0$ such that for $n>N$ and $(2n+1)\pi+\pi/4\le s\le 2(n+1)\pi-\pi/4$, $A(s)=-s^{\gamma}\sin(s^{\alpha}/\alpha)-k>\varepsilon s^{\gamma}$, where $\varepsilon$ is a small constant. Noticing $2\gamma-\lambda+\alpha\ge 0$, we have

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_1^t\frac{A_+^2(s)}{p(s)}d_{t_0}^{\alpha}s&\ge\sum_{n=N}^{\infty}\int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4}\frac{A_+^2(s)}{p(s)}d_{t_0}^{\alpha}s\\&\ge\varepsilon^2\sum_{n=N}^{\infty}\int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4}s^{2\gamma-\lambda}d_{t_0}^{\alpha}s\\&=\varepsilon^2\sum_{n=N}^{\infty}\int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4}s^{2\gamma-\lambda+\alpha-1}ds\\&\ge\varepsilon^2\sum_{n=N}^{\infty}\int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4}s^{-1}ds=\infty.\end{aligned}\tag{93}$$

Hence by Corollary 15, we get that the conformable fractional differential equation (91) is oscillatory for $\lambda\le\alpha$ and $2\gamma-\lambda+\alpha\ge 0$.

## Data Availability

The data used to support the findings of this study are available within the paper.

## Conflicts of Interest

The authors declare no conflicts of interest.

## Authors' Contributions

The authors contributed equally to the writing of this paper.

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