Research Article

Positive Solutions to a Coupled Fractional Differential System with p -Laplacian Operator

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In this paper, we consider a fractional differential system with multistrip and multipoint mixed boundary conditions involving p -Laplacian operator and fractional derivatives. The existence result of positive solutions is established by the Leggett-Williams fixed point theorem. Also, an example is presented to illustrate our main result.

1. Introduction

Fractional calculus is a significant branch of mathematical analysis which concerns derivatives and integrals. Fractional differential equations are caused by the intensive development of the theory of fractional calculus and have gained great interest currently, because of their crucial applications in various fields of physics, biology, and engineering such as control, porous media, electromagnetic, and other fields [1–4].

Moreover, many mathematical formulations of physical and biological phenomena lead to fractional differential equations with p -Laplacian operator, such as memory, fluid dynamics, predator harvesting, and hereditary properties of various materials, which are difficult to achieve in classical integer-order models. Hence, in recent years, more and more researchers have considered the fractional differential equations with p -Laplacian operator, by applying various mathematical theory and tools; refer to [5–9].

For example, authors in [10] discussed the following differential system with p -Laplacian operator

$$D^{\beta_1} \phi_p((D^{\alpha_1} \mu(x))) + \omega_1(x, v(x)) = 0, \quad x \in (0, 1),$$

$$D^{\beta_2} \phi_p((D^{\alpha_2} v(x))) + \omega_2(x, \mu(x)) = 0, \quad x \in (0, 1),$$

$$D^{\alpha_1} \mu(x)|_{x=0} = 0 = \mu(x)|_{x=0} = \mu''(x)|_{x=0},$$

$$\mu'(x)|_{x=1} = \eta_1 I_P^{\gamma_1} \psi_1(\mu)$$

$$\begin{aligned} &= \frac{\eta_1}{\Gamma(\gamma_1)} \int_0^P (P-\tau)^{\gamma_1-1} \psi_1(\mu(\tau)) d\tau, \\ D^{\alpha_2} v(x)|_{x=0} &= 0 = v(x)|_{x=0} = v''(x)|_{x=0}, \\ v'(x)|_{x=1} &= \eta_2 I_P^{\gamma_2} \psi_2(v) \\ &= \frac{\eta_2}{\Gamma(\gamma_2)} \int_0^P (P-\tau)^{\gamma_2-1} \psi_2(v(\tau)) d\tau, \end{aligned}$$

(1)

where $2 < \alpha_i < 3$, $0 < \beta_i < 1$, $P, \eta_i, \gamma_i > 0$, $\psi_1, \psi_2 \in L[0, 1]$, $\phi_p(s)$ is a p -Laplacian operator: $\phi_p(s) = |s|^{p-2}s$, $p > 1$ and $\omega_i : [0, 1] \times [0, \infty) \rightarrow (0, \infty)$. Authors established the conclusion of the existence and uniqueness of solution to the above problem by using topological degree theory.

By reading the existing literatures [11–24], we note that the fractional differential system involving p -Laplacian operator and multistrip and multipoint boundary conditions has not been studied yet. Thus, in this paper, we first pay close attention to the following fractional differential system, involving p -Laplacian operator and lower fractional derivatives:

$$\begin{aligned} &D_{0+}^{\beta} (\phi_p(D_{0+}^{\alpha} u(t))) \\ &= f_1(t, u(t), v(t), D_{0+}^{\alpha} u(t), D_{0+}^{\alpha} v(t)), \\ &D_{0+}^{\beta} (\phi_p(D_{0+}^{\alpha} v(t))) \\ &= f_2(t, u(t), v(t), D_{0+}^{\alpha} u(t), D_{0+}^{\alpha} v(t)), \end{aligned} \quad (2)$$

with multistrip and multipoint boundary conditions:

$$\begin{aligned}
 u(0) &= D_{0+}^{\alpha} u(0) = 0 \\
 u(1) &= \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t), \\
 \varphi_p(D_{0+}^{\alpha} u(1)) &= \sum_{i=1}^m a_i \varphi_p(D_{0+}^{\alpha} u(\xi_i)), \\
 v(0) &= D_{0+}^{\alpha} v(0) = 0 \\
 v(1) &= \sum_{i=1}^m d_i \int_0^{\eta_i} v(t) dB(t), \\
 \varphi_p(D_{0+}^{\alpha} v(1)) &= \sum_{i=1}^m c_i \varphi_p(D_{0+}^{\alpha} v(\eta_i)),
 \end{aligned} \tag{3}$$

where D_{0+}^{α} and D_{0+}^{β} are the Riemann-Liouville fractional derivatives of order α and β , $f: [0, 1] \times \mathbb{R}^{+2} \times \mathbb{R}^2 \rightarrow \mathbb{R}^{+}$ is a continuous function. For $i = 1, 2, \dots, m$, $1 < \alpha \leq 2$, $1 < \beta \leq 2$, $0 < \xi_i \leq 1$, $0 < \eta_i \leq 1$, a_i, b_i, c_i, d_i are nonnegative constants. $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $1/p + 1/q = 1$. And $A(t)$, $B(t)$ are nondecreasing functions of bounded variation in $[0, 1]$.

We emphasize that, despite the complication of the discrete and integral boundary conditions (3) and (4), operator T can be presented in a concise form based on the piecewise form of Green's function. And we can accurately estimate the upper and lower bounds of its value, which is fully prepared for the establishment of the main theorem.

Accordingly, the conclusion we reached is an extensive result and a meaningful supplement to the theory of the fractional differential systems with p -Laplacian operator.

Theorem 1 (the Leggett-Williams fixed point theorem). *Let P be a cone in a real Banach space E , $P_c = \{x \in P \mid \|x\| \leq c\}$, Ψ be a nonnegative continuous concave function on P such that $\Psi(x) \leq \|x\|$ for all $x \in \overline{P}_c$, and $P(\Psi, b, d) = \{x \in P \mid b \leq \Psi(x), \|x\| \leq d\}$. Suppose that $T: \overline{P}_c \rightarrow \overline{P}_c$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that*

- (S₁) $\{x \in P(\Psi, b, d) \mid \Psi(x) > b\} \neq \emptyset$, and $\Psi(Tx) > b$ for $x \in P(\Psi, b, d)$;
- (S₂) $\|Tx\| < a$ for $\|x\| \leq a$;
- (S₃) $\Psi(Tx) > b$ for $x \in P(\Psi, b, c)$, with $\|Tx\| > d$.

Then T has at least three fixed points x_1 , x_2 , and x_3 which satisfy

$$\begin{aligned}
 \|x_1\| &< a, \\
 b &< \Psi(x_2), \\
 a &< \|x_3\| \\
 \text{with } \Psi(x_1) &< b.
 \end{aligned} \tag{5}$$

2. Preliminaries

In this section, we will put forward some crucial definitions and theorems, preparing for establishing the existence of solutions.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f: (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{6}$$

provided the right-hand side is pointwise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$, for $\alpha > 0$.

Definition 3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f: (0, \infty) \rightarrow \mathbb{R}$ is given by $[0, \infty)$,

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \tag{7}$$

where $n = [\alpha] + 1$, $[\alpha]$ stands for the largest integer not greater than α .

According to the definitions of Riemann-Liouville's derivative, the following lemmas can be achieved.

Lemma 4. For $\alpha > 0$, if we assume that $u \in C[0, \infty) \cap L^1(0, 1)$, then we have

$$\begin{aligned}
 I_{0+}^{\alpha} (D_{0+}^{\alpha} u(t)) &= u(t) + m_1 t^{\alpha-1} + m_2 t^{\alpha-2} + \dots \\
 &\quad + m_n t^{\alpha-n},
 \end{aligned} \tag{8}$$

for some $m_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, while n is the smallest integer greater than or equal to α .

Remark 5. The following properties are useful for our discussion:

- (i) As a basic example, we quote for $\lambda > -1$,

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}; \tag{9}$$

- (ii) $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$, for $u(t) \in L^1(0, 1)$.

For convenience, we denote

$$\begin{aligned}
 N_1 &= \sum_{i=1}^m b_i \int_0^{\xi_i} t^{\alpha-1} dA(t), \\
 N_2 &= \sum_{i=1}^m d_i \int_0^{\eta_i} t^{\alpha-1} dB(t), \\
 M_k &= \sum_{i=1}^k a_i \xi_i^{\beta-1}, \\
 E_k &= \sum_{i=1}^k c_i \eta_i^{\beta-1},
 \end{aligned} \tag{10}$$

for $k = 1, 2, \dots, m$.

Besides, it should be noted that $M_0 = E_0 = 0$.

Lemma 6. For $h_1, h_2 \in C(0, 1)$, the equations

$$\begin{aligned} D_{0+}^{\beta} (\varphi_p (D_{0+}^{\alpha} u(t))) &= h_1(t), \\ D_{0+}^{\beta} (\varphi_p (D_{0+}^{\alpha} v(t))) &= h_2(t), \end{aligned} \quad (11)$$

with boundary conditions (3) and (4) have a unique solution as follows:

$$\begin{aligned} u(t) &= \int_0^1 K_1(t, s) \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds, \\ v(t) &= \int_0^1 K_2(t, s) \varphi_q \left(\int_0^1 \hat{H}(s, \tau) h_2(\tau) d\tau \right) ds, \end{aligned} \quad (12)$$

where

$$K_1(t, s) = G_{\alpha}(t, s) + \frac{t^{\alpha-1}}{1 - N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} G_{\alpha}(t, s) dA(t),$$

$$K_2(t, s) = G_{\alpha}(t, s) + \frac{t^{\alpha-1}}{1 - N_2} \sum_{i=1}^m d_i \int_0^{\eta_i} G_{\alpha}(t, s) dB(t), \quad (13)$$

$$\begin{aligned} G_{\alpha}(t, s) &= \begin{cases} \frac{1}{\Gamma(\alpha)} [t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}], & 0 \leq s \leq t \leq 1, \\ \frac{1}{\Gamma(\alpha)} t^{\alpha-1} (1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_{\beta}(t, s) &= \begin{cases} \frac{1}{\Gamma(\beta)} [t^{\beta-1} (1-s)^{\beta-1} - (t-s)^{\beta-1}], & 0 \leq s \leq t \leq 1, \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1, \end{cases} \end{aligned} \quad (14)$$

and $\bar{H}(t, s)$ is a piecewise function; the expression of the k th ($k = 0, 1, \dots, m$) piece is as follows

$$\begin{aligned} \bar{H}_k(t, s) &= G_{\beta}(t, s) + \frac{t^{\beta-1}}{1 - M_m} \sum_{i=k}^m a_i G_{\beta}(\xi_i, s) + \frac{M_k}{(1 - M_m) \Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} \\ &= \begin{cases} \frac{1}{\Gamma(\beta)} [t^{\beta-1} (1-s)^{\beta-1} - (t-s)^{\beta-1}] + \frac{t^{\beta-1}}{(1 - M_m) \Gamma(\beta)} \sum_{i=k+1}^m a_i [\xi_i^{\beta-1} (1-s)^{\beta-1} - (\xi_i - s)^{\beta-1}] \\ \quad + \frac{M_k}{(1 - M_m) \Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1}, & \text{for } \xi_k \leq s \leq t \leq \xi_{k+1}, \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} + \frac{t^{\beta-1}}{(1 - M_m) \Gamma(\beta)} \sum_{i=k+1}^m a_i \xi_i^{\beta-1} (1-s)^{\beta-1} \\ \quad + \frac{M_k}{(1 - M_m) \Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1}, & \text{for } \xi_k \leq t \leq s \leq \xi_{k+1}. \end{cases} \end{aligned} \quad (15)$$

Meanwhile, $\hat{H}(t, s)$ is also a piecewise function; the expression of k th ($k = 0, 1, \dots, m$) piece is as follows

$$\begin{aligned} \hat{H}_k(t, s) &= G_{\beta}(t, s) + \frac{t^{\beta-1}}{1 - E_m} \sum_{i=k}^m c_i G_{\beta}(\eta_i, s) + \frac{E_k}{(1 - E_m) \Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} \\ &= \begin{cases} \frac{1}{\Gamma(\beta)} [t^{\beta-1} (1-s)^{\beta-1} - (t-s)^{\beta-1}] + \frac{t^{\beta-1}}{(1 - E_m) \Gamma(\beta)} \sum_{i=k+1}^m c_i [\eta_i^{\beta-1} (1-s)^{\beta-1} - (\eta_i - s)^{\beta-1}] \\ \quad + \frac{E_k}{(1 - E_m) \Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1}, & \text{for } \eta_k \leq s \leq t \leq \eta_{k+1}, \\ \frac{1}{\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} + \frac{t^{\beta-1}}{(1 - E_m) \Gamma(\beta)} \sum_{i=k+1}^m c_i \eta_i^{\beta-1} (1-s)^{\beta-1} \\ \quad + \frac{E_k}{(1 - E_m) \Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1}, & \text{for } \eta_k \leq t \leq s \leq \eta_{k+1}. \end{cases} \end{aligned} \quad (16)$$

Proof. We shall reduce BVP (11), (3), and (4) to two fractional differential problems. To this goal, first, by means of transformation, we convert problem (11) into

$$\begin{aligned} D_{0+}^{\beta}(y_1(t)) &= h_1(t), \\ D_{0+}^{\beta}(y_2(t)) &= h_2(t), \\ y_1(0) &= 0, \\ y_1(1) &= \sum_{i=1}^m a_i y_1(\xi_i), \end{aligned} \quad (17)$$

$$\begin{aligned} y_2(0) &= 0, \\ y_2(1) &= \sum_{i=1}^m c_i y_2(\eta_i), \\ \varphi_p(D_{0+}^{\alpha} u(t)) &= y_1(t), \\ \varphi_p(D_{0+}^{\alpha} v(t)) &= y_2(t), \\ u(0) &= 0, \\ u(1) &= \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t), \\ v(0) &= 0, \\ v(1) &= \sum_{i=1}^m d_i \int_0^{\eta_i} v(t) dB(t). \end{aligned} \quad (18)$$

In view of Lemma 4, the solution of problem (17) can be written as

$$\begin{aligned} y_1(t) &= I_{0+}^{\beta} h_1(t) \\ &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h_1(s) ds + c_1 t^{\beta-1} + c_2 t^{\beta-2}, \end{aligned} \quad (19)$$

where c_1, c_2 are arbitrary real constants.

By $y_1(0) = 0$, we find $c_2 = 0$. Setting $t = 1$ in (19), together with the boundary condition $y_1(1) = \sum_{i=1}^m a_i y_1(\xi_i)$, we have

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h_1(s) ds \\ &\quad - \frac{1}{\Gamma(\beta)} \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} h_1(s) ds \\ &= c_1 \left(\sum_{i=1}^m a_i \xi_i^{\beta-1} - 1 \right). \end{aligned} \quad (20)$$

Accordingly, we can easily find out

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\beta)(1-M_m)} \left[\sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} h_1(s) ds \right. \\ &\quad \left. - \int_0^1 (1-s)^{\beta-1} h_1(s) ds \right]. \end{aligned} \quad (21)$$

Combining (21) with (19), we can obtain

$$\begin{aligned} y_1(t) &= \frac{1}{\Gamma(\beta)} \left\{ \int_0^t (t-s)^{\beta-1} h_1(s) ds \right. \\ &\quad + \frac{t^{\beta-1}}{1-M_m} \left[\sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} h_1(s) ds \right. \\ &\quad \left. \left. - \int_0^1 (1-s)^{\beta-1} h_1(s) ds \right] \right\}. \end{aligned} \quad (22)$$

Next, considering the multipoint boundary conditions, we divide the domain of integration into $m+1$ pieces. For $\xi_k \leq t \leq \xi_{k+1}$, the unique solution of (17) can be expressed as

$$\begin{aligned} y_1(t) &= \frac{1}{\Gamma(\beta)} \left\{ \int_0^t (t-s)^{\beta-1} h_1(s) ds \right. \\ &\quad + \frac{t^{\beta-1}}{1-M_m} \left[\sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\beta-1} h_1(s) \right. \\ &\quad \left. - \int_0^1 (1-s)^{\beta-1} h_1(s) ds \right] \Bigg\} = \frac{1}{\Gamma(\beta)} \int_0^{\xi_1} \left[(t-s)^{\beta-1} \right. \\ &\quad + \frac{t^{\beta-1}}{1-M_m} \sum_{i=1}^m a_i (\xi_i - s)^{\beta-1} - \frac{t^{\beta-1}(1-s)^{\beta-1}}{1-M_m} \Bigg] \\ &\quad \cdot h_1(s) ds + \frac{1}{\Gamma(\beta)} \int_{\xi_1}^{\xi_2} \left[(t-s)^{\beta-1} \right. \\ &\quad + \frac{t^{\beta-1}}{1-M_m} \sum_{i=2}^m a_i (\xi_i - s)^{\beta-1} - \frac{t^{\beta-1}(1-s)^{\beta-1}}{1-M_m} \Bigg] \\ &\quad \cdot h_1(s) ds + \cdots + \frac{1}{\Gamma(\beta)} \int_{\xi_k}^t \left[(t-s)^{\beta-1} \right. \\ &\quad + \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i (\xi_i - s)^{\beta-1} - \frac{t^{\beta-1}(1-s)^{\beta-1}}{1-M_m} \Bigg] \\ &\quad \cdot h_1(s) ds + \frac{1}{\Gamma(\beta)} \\ &\quad \cdot \int_t^{\xi_{k+1}} \left[\frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i (\xi_i - s)^{\beta-1} \right. \\ &\quad \left. - \frac{t^{\beta-1}(1-s)^{\beta-1}}{1-M_m} \right] h_1(s) ds + \cdots + \frac{1}{\Gamma(\beta)} \\ &\quad \cdot \int_{\xi_m}^1 \left[-\frac{t^{\beta-1}(1-s)^{\beta-1}}{1-M_m} \right] h_1(s) ds. \end{aligned} \quad (23)$$

When $\xi_k \leq s \leq t$, we observe that the k th integrand in (23) can be taken as follows:

$$\frac{1}{\Gamma(\beta)} \left[(t-s)^{\beta-1} + \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i (\xi_i - s)^{\beta-1} \right]$$

$$\begin{aligned}
& - \frac{t^{\beta-1} (1-s)^{\beta-1}}{1-M_m} \Bigg] = \frac{1}{\Gamma(\beta)} \left\{ (t-s)^{\beta-1} \right. \\
& - t^{\beta-1} (1-s)^{\beta-1} \\
& + \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i \left[(\xi_i - s)^{\beta-1} - \xi_i^{\beta-1} (1-s)^{\beta-1} \right] \\
& - \frac{t^{\beta-1} (1-s)^{\beta-1}}{1-M_m} + t^{\beta-1} (1-s)^{\beta-1} \\
& + \left. \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i \xi_i^{\beta-1} (1-s)^{\beta-1} \right\} \\
& = \frac{1}{\Gamma(\beta)} \left\{ (t-s)^{\beta-1} - t^{\beta-1} (1-s)^{\beta-1} \right. \\
& + \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i \left[(\xi_i - s)^{\beta-1} - \xi_i^{\beta-1} (1-s)^{\beta-1} \right] \\
& - \frac{M_k}{1-M_m} t^{\beta-1} (1-s)^{\beta-1} \Bigg\} = -G_\beta(t, s) - \frac{t^{\beta-1}}{1-M_m} \\
& \cdot \sum_{i=k+1}^m a_i G_\beta(\xi_i, s) - \frac{M_k}{(1-M_m)\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} \\
& = -\bar{H}_k(t, s),
\end{aligned} \tag{24}$$

where $\sum_{i=m+1}^m a_i G_\beta(\xi_i, s)$ is also noted by 0.

As a result, the Green function $\bar{H}_k(t, s)$ is always valid and can exactly express each integrand in the above formula, for $k = 0, 1, \dots, m$. Thus, we can take (23) as a unified form

$$y_1(t) = - \int_0^1 \bar{H}(t, s) h_1(s) ds. \tag{25}$$

$\bar{H}(t, s)$ is a piecewise function such that the form of the k th piece is given by (15).

We take (25) into (18); then

$$\varphi_p(D_0^\alpha u(t)) = - \int_0^1 \bar{H}(t, s) h_1(s) ds. \tag{26}$$

From Lemma 4, we have

$$\begin{aligned}
u(t) &= - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds \\
&+ c_1 t^{\alpha-1} + c_2 t^{\alpha-2},
\end{aligned} \tag{27}$$

where c_1, c_2 are arbitrary real constants.

By $u(0) = 0$, we easily find $c_2 = 0$. Setting $t=1$ in (27), together with the boundary condition $u(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t)$, we have

$$\begin{aligned}
& \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t) \\
&= - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds \\
&+ c_1.
\end{aligned} \tag{28}$$

Set c_1 and c_2 into (27); we find out

$$\begin{aligned}
u(t) &= \int_0^1 G_\alpha(t, s) \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds \\
&+ t^{\alpha-1} \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t).
\end{aligned} \tag{29}$$

Furthermore, by integrating and summing on both sides of (29), it is easy to see that

$$\begin{aligned}
& \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t) = \sum_{i=1}^m b_i \int_0^{\xi_i} \left[\int_0^1 G_\alpha(t, s) \right. \\
& \cdot \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds \Bigg] dA(t) \\
&+ N_1 \left(\sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t) \right).
\end{aligned} \tag{30}$$

Then, we get

$$\begin{aligned}
& \sum_{i=1}^m b_i \int_0^{\xi_i} u(t) dA(t) = \frac{1}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} \left[\int_0^1 G_\alpha(t, s) \right. \\
& \cdot \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds \Bigg] dA(t).
\end{aligned} \tag{31}$$

Take (31) into (29); we conclude that the unique solution $u(t)$ of BVP (11), (3), and (4) is

$$\begin{aligned}
u(t) &= \int_0^1 \left[G_\alpha(t, s) + \frac{t^{\alpha-1}}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} G_\alpha(t, s) dA(t) \right] \\
&\cdot \varphi_q \left(\int_0^1 \bar{H}(s, \tau) h_1(\tau) d\tau \right) ds.
\end{aligned} \tag{32}$$

Analogically, the unique solution $v(t)$ of BVP (11), (3), and (4) is

$$\begin{aligned}
v(t) &= \int_0^1 \left[G_\alpha(t, s) + \frac{t^{\alpha-1}}{1-N_2} \sum_{i=1}^m d_i \int_0^{\eta_i} G_\alpha(t, s) dB(t) \right] \\
&\cdot \varphi_q \left(\int_0^1 \hat{H}(s, \tau) h_2(\tau) d\tau \right) ds,
\end{aligned} \tag{33}$$

where $\hat{H}(t, s)$ is also a piecewise function, the k th piece $\hat{H}_k(t, s)$ is defined by (16), and the calculation process is similar.

This completes the proof. $\square$

Denote

$$\begin{aligned}\Phi_\alpha(t, s) &= \begin{cases} t^{\alpha-1} (1-s)^{\alpha-1} - (t-t^{\alpha-1}s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1} (1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \\ \Phi_\beta(t, s) &= \frac{1}{\Gamma(\beta)} \\ &\cdot \begin{cases} t^{\beta-1} (1-s)^{\beta-1} - (t-t^{\beta-1}s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1} (1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases}\end{aligned}\quad (34)$$

Lemma 7. For $t, s \in [0, 1]$ and $1 < \gamma \leq 2$, it holds that

$$\begin{aligned}t^{\gamma-1} (1-s)^{\gamma-1} - (t-s)^{\gamma-1} \\ \geq t^{\gamma-1} (1-s)^{\gamma-1} - (t-t^{\gamma-1}s)^{\gamma-1} \geq 0.\end{aligned}\quad (35)$$

Proof. For $1 < \gamma \leq 2$ and $0 \leq t \leq 1$, we have $t \leq t^{\gamma-1} \leq 1$, while $t^{\gamma-1}$ is a monotone increasing function. Therefore, we can infer that $t-s \leq t-t^{\gamma-1}s \leq t-ts$. For $(t-s)^{\gamma-1} \leq (t-t^{\gamma-1}s)^{\gamma-1} \leq (t-ts)^{\gamma-1}$, we finally find out $(t-ts)^{\gamma-1} - (t-s)^{\gamma-1} \geq (t-ts)^{\gamma-1} - (t-t^{\gamma-1}s)^{\gamma-1} \geq (t-ts)^{\gamma-1} - (t-ts)^{\gamma-1} = 0$, which implies $\Phi_\alpha(t, s) \geq 0$ and $\Phi_\beta(t, s) \geq 0$, for $1 < \alpha, \beta \leq 2$, $0 \leq t, s \leq 1$.

This completes the proof. $\square$

Lemma 8. For $t, s \in [0, 1]$, the functions $G_\alpha(t, s), G_\beta(t, s)$ defined by (14) have the following properties:

- (1) $G_\alpha(t, s) > 0, G_\beta(t, s) > 0$, for $t, s \in (0, 1)$;
- (2) $\Phi_\alpha(t, s)(1-s)^{\alpha-1} \leq \Gamma(\alpha)G_\alpha(t, s) \leq (1-s)^{\alpha-1}$, for $t, s \in (0, 1)$;
- $\Phi_\beta(t, s)(1-s)^{\beta-1} \leq G_\beta(t, s) \leq (1-s)^{\beta-1}/\Gamma(\beta)$, for $t, s \in (0, 1)$.

Proof. (1) The positive of $G_\alpha(t, s)$ and $G_\beta(t, s)$ is easily checked.

(2) For $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned}\Gamma(\alpha)G_\alpha(t, s) &= t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1} \\ &\leq t^{\alpha-1} (1-s)^{\alpha-1} \leq (1-s)^{\alpha-1}, \\ \Gamma(\alpha)G_\alpha(t, s) &= t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1} \\ &\geq t^{\alpha-1} (1-s)^{\alpha-1} - (t-t^{\alpha-1}s)^{\alpha-1} \\ &\geq \left[t^{\alpha-1} (1-s)^{\alpha-1} - (t-t^{\alpha-1}s)^{\alpha-1} \right] (1-s)^{\alpha-1} \\ &= \Phi_\alpha(t, s) (1-s)^{\alpha-1}.\end{aligned}\quad (36)$$

For $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned}\Gamma(\alpha)G_\alpha(t, s) &= t^{\alpha-1} (1-s)^{\alpha-1} \leq (1-s)^{\alpha-1}, \\ \Gamma(\alpha)G_\alpha(t, s) &= t^{\alpha-1} (1-s)^{\alpha-1} \\ &\geq t^{\alpha-1} (1-s)^{\alpha-1} (1-s)^{\alpha-1} \\ &= \Phi_\alpha(t, s) (1-s)^{\alpha-1}.\end{aligned}\quad (37)$$

The other inequality is achieved similarly. All these complete the proof of the lemma. $\square$

For $s \in [0, 1]$, denote

$$\begin{aligned}\varrho_1(s) &= \frac{1}{(1-N_1)\Gamma(\alpha)} \sum_{i=1}^m b_i \int_0^{\xi_i} \Phi_\alpha(t, s) dA(t), \\ \varrho_2(s) &= \frac{1}{(1-N_2)\Gamma(\alpha)} \sum_{i=1}^m d_i \int_0^{\eta_i} \Phi_\alpha(t, s) dB(t), \\ \rho_1 &= \frac{1}{\Gamma(\alpha)} \left[1 + \frac{1}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} dA(t) \right], \\ \rho_2 &= \frac{1}{\Gamma(\beta)} \left[\frac{1}{1-M_m} \sum_{i=1}^m a_i + \frac{1}{1-M_m} \right], \\ \rho_3 &= \frac{1}{\Gamma(\alpha)} \left[1 + \frac{1}{1-N_2} \sum_{i=1}^m d_i \int_0^{\eta_i} dB(t) \right], \\ \rho_4 &= \frac{1}{\Gamma(\beta)} \left[\frac{1}{1-E_m} \sum_{i=1}^m c_i + \frac{1}{1-E_m} \right],\end{aligned}\quad (38)$$

and the constant ρ is given by the maximum value of ρ_1, ρ_2, ρ_3 , and ρ_4 .

Lemma 9. For $(t, s) \in [0, 1] \times [0, 1]$, the functions $K_k(t, s)$ ($k = 1, 2$), $\bar{H}(t, s)$, and $\hat{H}(t, s)$ defined by Lemma 6 satisfy the following results:

- (1) $K_k(t, s), \bar{H}(t, s), \hat{H}(t, s)$ are continuous, $K_k(t, s) \geq 0, \bar{H}(t, s) \geq 0, \hat{H}(t, s) \geq 0$;
- (2) $\varrho_1(s)t^{\alpha-1}(1-s)^{\alpha-1} \leq K_1(t, s) \leq \rho(1-s)^{\alpha-1}$,
 $\varrho_2(s)t^{\alpha-1}(1-s)^{\alpha-1} \leq K_2(t, s) \leq \rho(1-s)^{\alpha-1}$;
- (3) $\Phi_\beta(t, s)t^{\beta-1}(1-s)^{\beta-1} \leq \bar{H}(t, s) \leq \rho(1-s)^{\beta-1}$,
 $\Phi_\beta(t, s)t^{\beta-1}(1-s)^{\beta-1} \leq \hat{H}(t, s) \leq \rho(1-s)^{\beta-1}$.

Proof. The nonnegativeness of $K_k(t, s), \bar{H}(t, s)$, and $\hat{H}(t, s)$ is easily checked.

(2) According to the property (2) in Lemma 8, we have

$$\begin{aligned}K_1(t, s) &= G_\alpha(t, s) + \frac{t^{\alpha-1}}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} G_\alpha(t, s) dA(t) \\ &\geq \frac{\Phi_\alpha(t, s) (1-s)^{\alpha-1}}{\Gamma(\alpha)}\end{aligned}$$

$$\begin{aligned}
& + \frac{t^{\alpha-1}}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} \frac{\Phi_\alpha(t, s) (1-s)^{\alpha-1}}{\Gamma(\alpha)} dA(t) \\
& \geq \frac{t^{\alpha-1}}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} \frac{\Phi_\alpha(t, s) (1-s)^{\alpha-1}}{\Gamma(\alpha)} dA(t) \\
& = \varrho_1(s) t^{\alpha-1} (1-s)^{\alpha-1}, \\
K_1(t, s) & = G_\alpha(t, s) + \frac{t^{\alpha-1}}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} G_\alpha(t, s) dA(t) \\
& \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} dA(t) \\
& \leq \frac{1}{\Gamma(\alpha)} \left(1 + \frac{1}{1-N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} dA(t) \right) (1-s)^{\alpha-1} \\
& = \rho_1(1-s)^{\alpha-1} \leq \rho(1-s)^{\alpha-1}.
\end{aligned} \tag{39}$$

For $k = 0, 1 \dots m$, we have

$$\begin{aligned}
H_k(t, s) & = G_\beta(t, s) + \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i G_\beta(\xi_i, s) \\
& + \frac{M_m}{(1-M_m)\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} \\
& \leq \frac{1}{\Gamma(\beta)} \left(\frac{1}{1-M_m} \sum_{i=k+1}^m a_i + \frac{1}{1-M_m} \right) (1-s)^{\beta-1} \\
& \leq \frac{1}{\Gamma(\beta)} \left(\frac{1}{1-M_m} \sum_{i=1}^m a_i + \frac{1}{1-M_m} \right) (1-s)^{\beta-1} \\
& = \rho_2(1-s)^{\beta-1} \leq \rho(1-s)^{\beta-1}. \\
H_k(t, s) & = G_\beta(t, s) + \frac{t^{\beta-1}}{1-M_m} \sum_{i=k+1}^m a_i G_\beta(\xi_i, s) \\
& + \frac{M_m}{(1-M_m)\Gamma(\beta)} t^{\beta-1} (1-s)^{\beta-1} \geq t^{\beta-1} (1-s)^{\beta-1} \\
& \cdot \left(\Phi_\beta(t, s) + \frac{\sum_{i=k+1}^m a_i \Phi_\beta(\xi_i, s)}{(1-M_m)} \right. \\
& \left. + \frac{M_m}{(1-M_m)\Gamma(\beta)} \right) \geq t^{\beta-1} (1-s)^{\beta-1} \left(\Phi_\beta(t, s) \right. \\
& \left. + \frac{\sum_{i=k+1}^m a_i \Phi_\beta(\xi_i, s)}{(1-M_m)} \right) \geq t^{\beta-1} (1-s)^{\beta-1} \Phi_\beta(t, s).
\end{aligned} \tag{40}$$

Because of the randomness of k , we have

$$\Phi_\beta(t, s) t^{\beta-1} (1-s)^{\beta-1} \leq \bar{H}(t, s) \leq \rho(1-s)^{\beta-1} \tag{41}$$

According to the above, from the property (2) of Lemma 8, we easily get

$$\begin{aligned}
\varrho_2(s) t^{\alpha-1} (1-s)^{\alpha-1} & \leq K_2(t, s) \leq \rho_3(1-s)^{\alpha-1}, \\
\Phi_\beta(t, s) t^{\beta-1} (1-s)^{\beta-1} & \leq \bar{H}(t, s) \leq \rho_4(1-s)^{\beta-1}.
\end{aligned} \tag{42}$$

This completes the proof of the lemma. $\square$

Define the space $X = \{u(t) \mid u(t) \in C[0, 1] \text{ and } D_{0+}^\alpha u(t) \in C[0, 1]\}$ is endowed with the norm $\|u\| = \max\{\|u\|_0, \|D_{0+}^\alpha u\|_0\}$, where $\|u\|_0 = \max_{t \in [0, 1]} |u(t)|$, $\|D_{0+}^\alpha u\|_0 = \max_{t \in [0, 1]} |D_{0+}^\alpha u(t)|$.

Then we introduce the product space $(X \times X, \|(u, v)\|)$ endowed with the norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$ and define a partial order $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \geq \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ where $u_1(t) \geq u_2(t)$, $v_1(t) \geq v_2(t)$, for $t \in [0, 1]$.

Lemma 10. $(X \times X, \|(u, v)\|)$ is a Banach space.

Proof. Let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in the space $(X, \|\cdot\|)$. Then clearly $\{u_n\}_{n=1}^\infty$ and $\{D_{0+}^\alpha u(t)\}_{n=1}^\infty$ are Cauchy sequences in the space $C[0, 1]$. Therefore, $\{u_n\}_{n=1}^\infty$ and $\{D_{0+}^\alpha u(t)\}_{n=1}^\infty$ converge to some v and w on $[0, 1]$ uniformly and $v, w \in C[0, 1]$. Then, we need to prove $w = D_{0+}^\alpha v$.

Note that

$$\begin{aligned}
& |I_{0+}^\alpha D_{0+}^\alpha u_n(t) - I_{0+}^\alpha w(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |D_{0+}^\alpha u_n(s) - w(s)| ds \\
& \leq \frac{1}{\Gamma(\alpha+1)} \max_{s \in [0, 1]} |D_{0+}^\alpha u_n(s) - w(s)|.
\end{aligned} \tag{43}$$

By the convergence of $\{D_{0+}^\alpha u(t)\}_{n=1}^\infty$, we get

$$\lim_{n \rightarrow \infty} I_{0+}^\alpha D_{0+}^\alpha u_n(t) = I_{0+}^\alpha w(t) \tag{44}$$

uniformly for $t \in [0, 1]$. On the other hand, by Remark 5, one has $I_{0+}^\alpha D_{0+}^\alpha u_n(t) = u_n(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$, for $t \in [0, 1]$ and some $c_1 \in \mathbb{R}$. Furthermore, we can get

$$\begin{aligned}
\lim_{n \rightarrow \infty} I_{0+}^\alpha D_{0+}^\alpha u_n(t) & = \lim_{n \rightarrow \infty} u_n(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \\
& = v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.
\end{aligned} \tag{45}$$

From (44) and (45), we obtain

$$I_{0+}^\alpha w(t) = v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad \text{for } t \in [0, 1]. \tag{46}$$

Taking the α -order derivative on both sides of yields, we have

$$I_{0+}^\alpha D_{0+}^\alpha w(t) = D_{0+}^\alpha [v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}], \tag{47}$$

for $t \in [0, 1]$.

In view of Remark 5 and Lemma 4, we find that

$$w(t) = D_{0+}^\alpha v(t), \quad \text{for } t \in [0, 1]. \tag{48}$$

Above all, we can achieve that $(X \times X, \|(u, v)\|)$ is a Banach space. This completes the proof. $\square$

Define the cone $P \subset X \times X$ by

$$P = \{(u, v) \in X \times X \mid u(t) \geq 0, v(t) \geq 0\}. \quad (49)$$

Let the nonnegative continuous concave functional Ψ on the cone P be defined by

$$\Psi((u, v)) = \min \left\{ \min_{\theta \leq t \leq 1-\theta} |u(t)|, \min_{\theta \leq t \leq 1-\theta} |v(t)| \right\}, \quad (50)$$

where θ is a positive constant, satisfying $0 < \theta < 1 - \theta < 1$.

For convenience, for $0 \leq s \leq 1$, we denote

$$c(s) = \frac{1}{1 - N_1} \sum_{i=1}^m b_i \int_0^{\xi_i} G_\alpha(t, s) dA(t), \quad (51)$$

$$e_k(s) = \frac{1}{1 - M_m} \sum_{i=k}^m a_i G_\beta(\xi_i, s) + \frac{M_k(1-s)^{\beta-1}}{(1 - M_m) \Gamma(\beta)},$$

where $k = 0, 1, \dots, m$.

For any $(u, v) \in X \times X$, we introduce the integral operator T by

$$T(u, v)(t) = \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix}, \quad (52)$$

where

$$T_1(u, v)(t) = \int_0^1 K_1(t, s) \varphi_q \left(\int_0^1 \bar{H}(s, \tau) \cdot f_1(\tau, u(\tau), v(\tau), D_{0+}^\alpha u(\tau), D_{0+}^\alpha v(\tau)) d\tau \right) ds, \quad (53)$$

$$T_2(u, v)(t) = \int_0^1 K_2(t, s) \varphi_q \left(\int_0^1 \hat{H}(s, \tau) \cdot f_2(\tau, u(\tau), v(\tau), D_{0+}^\alpha u(\tau), D_{0+}^\alpha v(\tau)) d\tau \right) ds.$$

Consequently, by calculation

$$D_{0+}^\alpha T(u, v)(t) = \begin{pmatrix} D_{0+}^\alpha T_1(u, v)(t) \\ D_{0+}^\alpha T_2(u, v)(t) \end{pmatrix}, \quad (54)$$

where

$$D_{0+}^\alpha T_1(u, v)(t) = -\varphi_q \left(\int_0^1 \bar{H}(t, s) \cdot f_1(s, u(s), v(s), D_{0+}^\alpha u(s), D_{0+}^\alpha v(s)) ds \right), \quad (55)$$

$$D_{0+}^\alpha T_2(u, v)(t) = -\varphi_q \left(\int_0^1 \hat{H}(t, s) \cdot f_2(s, u(s), v(s), D_{0+}^\alpha u(s), D_{0+}^\alpha v(s)) ds \right).$$

Lemma 11. $T : P \longrightarrow P$ is completely continuous.

Proof. Firstly, we prove that $T_1 : P \longrightarrow P$. In view of the nonnegativeness and continuity of $K_1(t, s)$, $\bar{H}(t, s)$ and $f_1(t, u(t), v(t), D_{0+}^\alpha u(t), D_{0+}^\alpha v(t))$, T_1 is continuous and $T_1(u, v)(t) \geq 0$ for $(u, v) \in P$. Hence, $T_1 P \subset P$.

Next, we show T_1 is uniformly bounded. Let $\Omega \subset P$ be bounded; that is, there exists a positive constant $L > 0$ such that $0 \leq u \leq L$, $0 \leq v \leq L$, $-L \leq D_{0+}^\alpha u(t) \leq L$, $-L \leq D_{0+}^\alpha v(t) \leq L$ for all $(u, v) \in \Omega$. Let $A_1 = \max_{t \in [0, 1]} |f_1(t, u(t), v(t), D_{0+}^\alpha u(t), D_{0+}^\alpha v(t))|$. Then for $(u, v) \in \Omega$, from the Lemma 9, we have

$$\begin{aligned} & |T_1(u, v)(t)| \\ &= \left| \int_0^1 K_1(t, s) \varphi_q \left(\int_0^1 \bar{H}(s, \tau) \cdot f_1(\tau, u(\tau), v(\tau), D_{0+}^\alpha u(\tau), D_{0+}^\alpha v(\tau)) d\tau \right) ds \right| \\ &\leq A_1^{q-1} \left| \int_0^1 K_1(t, s) \varphi_q \left(\int_0^1 \bar{H}(s, \tau) d\tau \right) ds \right| \\ &\leq A_1^{q-1} \left| \int_0^1 \rho(1 - s)^{\alpha-1} \varphi_q \left(\int_0^1 \rho(1 - \tau)^{\beta-1} d\tau \right) ds \right| \\ &= A_1^{q-1} \left(\frac{\rho}{\beta} \right)^{q-1} \frac{\rho}{\alpha}, \end{aligned} \quad (56)$$

$$\begin{aligned} & |D_{0+}^\alpha T_1(u, v)(t)| \\ &= \left| -\varphi_q \left(\int_0^1 \bar{H}(t, s) \cdot f_1(s, u(s), v(s), D_{0+}^\alpha u(s), D_{0+}^\alpha v(s)) ds \right) \right| \\ &\leq A_1^{q-1} \varphi_q \left(\int_0^1 \bar{H}(t, s) ds \right) \\ &\leq A_1^{q-1} \varphi_q \left(\int_0^1 \rho(1 - s)^{\beta-1} ds \right) = A_1^{q-1} \left(\frac{\rho}{\beta} \right)^{q-1}. \end{aligned}$$

Hence, $T_1(\Omega)$ is uniformly bounded. Then, we show T_1 is equicontinuous. Indeed, for any $(u, v) \in \Omega$, $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$\begin{aligned} & |T_1(u(t_2), v(t_2)) - T_1(u(t_1), v(t_1))| \\ &\leq \int_0^1 |K_1(t_2, s) - K_1(t_1, s)| \varphi_q \left(\int_0^1 \bar{H}(s, \tau) \right) \end{aligned}$$

$$\begin{aligned}
& \cdot f_1(\tau, u(\tau), v(\tau), D_{0^+}^\alpha u(\tau), D_{0^+}^\alpha v(\tau)) d\tau \Big) ds \\
& \leq A_1^{q-1} \left(\frac{\rho}{\beta} \right)^{q-1} \int_0^1 |K_1(t_2, s) - K_1(t_1, s)| ds \\
& \leq A_1^{q-1} \left(\frac{\rho}{\beta} \right)^{q-1} \int_0^1 (|G_\alpha(t_2, s) - G_\alpha(t_1, s)| + c(s) \\
& \cdot |t_2^{\alpha-1} - t_1^{\alpha-1}|) ds.
\end{aligned} \tag{57}$$

Moreover,

$$\begin{aligned}
& |D_{0^+}^\alpha T_1(u(t_2), v(t_2)) - D_{0^+}^\alpha T_1(u(t_1), v(t_1))| \\
& \leq \varphi_q \left(\int_0^1 |\overline{H}(t_2, s) - \overline{H}(t_1, s)| \right. \\
& \cdot f_1(s, u(s), v(s), D_{0^+}^\alpha u(s), D_{0^+}^\alpha v(s)) ds \Big) \\
& \leq A_1^{q-1} \varphi_q \left(\int_0^1 |\overline{H}(t_2, s) - \overline{H}(t_1, s)| ds \right) \\
& \leq A_1^{q-1} \varphi_q \left(\int_0^1 (|G_\beta(t_2, s) - G_\beta(t_1, s)| \right. \\
& \left. + e(s) |t_2^{\beta-1} - t_1^{\beta-1}|) ds \right).
\end{aligned} \tag{58}$$

Therefore, the above process implies that $T_1 : P \rightarrow P$ is equicontinuous for all $(u, v) \in \Omega$.

By means of the Arzela-Ascoli theorem, $T_1 : P \rightarrow P$ is completely continuous. Likewise, the proof of $T_2 : P \rightarrow P$ follows the same method as T_1 . Thus, $T : P \rightarrow P$ is completely continuous. $\square$

3. Existence Result

For convenience, we denote

$$\begin{aligned}
Q_1 &= \int_0^1 \varrho_1(s) (1-s)^{\alpha-1} \\
& \cdot \varphi_q \left(\int_0^1 \Phi_\beta(s, \tau) s^{\beta-1} (1-\tau)^{\beta-1} d\tau \right) ds, \\
Q_2 &= \int_0^1 \varrho_2(s) (1-s)^{\alpha-1} \\
& \cdot \varphi_q \left(\int_0^1 \Phi_\beta(s, \tau) s^{\beta-1} (1-\tau)^{\beta-1} d\tau \right) ds, \\
M &= \min \left\{ \left(\frac{\alpha}{\rho} \right)^{q-1} \left(\frac{\beta}{\rho} \right), \frac{\beta}{\rho} \right\}, \\
N &= \max_{k=1,2} \{ Q_k \theta^{\alpha-1} \}^{-1/(q-1)}.
\end{aligned} \tag{59}$$

Theorem 12. *There exist nonnegative numbers $0 \leq a \leq b \leq (\theta^2/\Gamma(\alpha-1))c$, and $f_k(t, u, v, w, z)$ ($k = 1, 2$) satisfies the following conditions:*

$$(H_1) \quad f_k(t, u, v, w, z) \leq M c^{1/(q-1)}, \text{ for } (t, u, v, w, z) \in [0, 1] \times [0, c] \times [0, c] \times [-c, c] \times [-c, c];$$

$$(H_2) \quad f_k(t, u, v, w, z) < M a^{1/(q-1)}, \text{ for } (t, u, v, w, z) \in [0, 1] \times [0, a] \times [0, a] \times [-a, a] \times [-a, a];$$

$$(H_3) \quad f_k(t, u, v, w, z) > N b^{1/(q-1)}, \text{ for } (t, u, v, w, z) \in [\theta, 1 - \theta] \times [b, c] \times [b, c] \times [-c, c] \times [-c, c].$$

Then BVP (2)-(4) has at least three positive solutions (u_1, v_1) , (u_2, v_2) , and (u_3, v_3) such that

$$\begin{aligned}
& \|(u_1, v_1)\| < a, \\
& b < \Psi(u_2, v_2) < \|(u_2, v_2)\| \leq c, \\
& a < \|(u_3, v_3)\| \leq c
\end{aligned} \tag{60}$$

with $\Psi(u_3, v_3) < b$.

Proof. We will verify that conditions $(S_1) - (S_3)$ of Theorem 1 are satisfied.

Let $\overline{P}_c = \{(u, v) \in X \times X \mid u(t) \geq 0, v(t) \geq 0 \text{ and } \|(u, v)\| \leq c\}$. We first prove that $T : \overline{P}_c \rightarrow \overline{P}_c$. For any $(u, v) \in \overline{P}_c$, $t \in [0, 1]$, we have $0 \leq u(t) \leq c$, $0 \leq v(t) \leq c$, $-c \leq D_{0^+}^\alpha u(t) \leq c$, $-c \leq D_{0^+}^\alpha v(t) \leq c$. Consequently, combining with properties (2) and (3) of Lemma 9,

$$\begin{aligned}
& |T_1(u, v)(t)| \\
& = \left| \int_0^1 K_1(t, s) \varphi_q \left(\int_0^1 \overline{H}(s, \tau) \right. \right. \\
& \cdot f_1(\tau, u(\tau), v(\tau), D_{0^+}^\alpha u(\tau), D_{0^+}^\alpha v(\tau)) d\tau \Big) ds \Big| \\
& \leq \int_0^1 \rho(1 \\
& - s)^{\alpha-1} \varphi_q \left(\int_0^1 \rho(1-\tau)^{\alpha-1} M c^{1/(q-1)} d\tau \right) ds \\
& = \int_0^1 \rho(1 \\
& - s)^{\alpha-1} M^{q-1} c \varphi_q \left(\int_0^1 \rho(1-\tau)^{\alpha-1} d\tau \right) ds \\
& = M^{q-1} c \left(\frac{\rho}{\beta} \right)^{q-1} \left(\frac{\rho}{\alpha} \right) \leq c, \\
& |D_{0^+}^\alpha T_1(u, v)(t)| \\
& = \left| -\varphi_q \int_0^1 \overline{H}(t, s) \right. \\
& \cdot f_1(s, u(s), v(s), D_{0^+}^\alpha u(s), D_{0^+}^\alpha v(s)) ds \Big| \\
& \leq \varphi_q \left(\int_0^1 \rho(1
\end{aligned}$$

$$\begin{aligned}
& -s)^{\beta-1} M c^{1/(q-1)} ds) = M^{q-1} c \varphi_q \left(\frac{\rho}{\beta} \right) \\
& = M^{q-1} c \left(\frac{\rho}{\beta} \right)^{q-1} \leq c.
\end{aligned} \tag{61}$$

Thus, $\|T_1(u, v)\| \leq c$ and further to get $T_1 \bar{P}_c \subset \bar{P}_c$. Therefore $T_1 : \bar{P}_c \rightarrow \bar{P}_c$ is completely continuous. The proof of T_2 is the same as T_1 . Thus, $T : \bar{P}_c \rightarrow \bar{P}_c$ is completely continuous.

By the above, condition (S_2) of Theorem 1 can be satisfied, using assumption (H_2) .

Finally, in order to verify $\{(u, v) \in P(\Psi, b, c) \mid \Psi(u, v) > b\} \neq \emptyset$, we set $(u, v) = (bt^\alpha/\theta^2, bt^\alpha/\theta^2)$, for $0 \leq t \leq 1$. It is easy to find that

$$\Psi((u, v)) = \min_{\theta \leq t \leq 1-\theta} \left\{ \min \left| \frac{bt^\alpha}{\theta^2} \right|, \min \left| \frac{bt^\alpha}{\theta^2} \right| \right\}$$

$$\Psi(T(u, v)(t))$$

$$\begin{aligned}
& = \min \left\{ \min_{\theta \leq t \leq 1-\theta} |T_1(u, v)(t)|, \min_{\theta \leq t \leq 1-\theta} |T_2(u, v)(t)| \right\} \\
& > \min \left\{ N^{q-1} b \int_0^1 K_1(t, s) \right. \\
& \quad \cdot \varphi_q \left(\int_0^1 \Phi_\beta(s, \tau) s^{\beta-1} (1-\tau)^{\beta-1} d\tau \right) ds, N^{q-1} b \int_0^1 K_2(t, s) \varphi_q \left(\int_0^1 \Phi_\beta(s, \tau) s^{\beta-1} (1-\tau)^{\beta-1} d\tau \right) ds \Big\} \\
& \geq \min \left\{ N^{q-1} b \theta^{\alpha-1} \int_0^1 \varrho_1(s) (1-s)^{\alpha-1} \right. \\
& \quad \cdot \varphi_q \left(\int_0^1 \Phi_\beta(s, \tau) s^{\beta-1} (1-\tau)^{\beta-1} d\tau \right) ds, N^{q-1} b \theta^{\alpha-1} \int_0^1 \varrho_2(s) (1-s)^{\alpha-1} \varphi_q \left(\int_0^1 \Phi_\beta(s, \tau) s^{\beta-1} (1-\tau)^{\beta-1} d\tau \right) ds \Big\} \\
& = \min_{k=1,2} \left\{ N^{q-1} b \theta^{\alpha-1} Q_k \right\} = b.
\end{aligned} \tag{64}$$

That is, $\Psi(T(u, v)(t)) > b$ for all $(u, v) \in P(\Psi, b, c)$. This shows that condition (S_1) of Theorem 1 is also satisfied.

From the above, BVP (2)-(4) has at least three positive solutions (u_1, v_1) , (u_2, v_2) , and (u_3, v_3) such that

$$\begin{aligned}
& \|(u_1, v_1)\| < a, \\
& b < \Psi(u_2, v_2) < \|(u_2, v_2)\| \leq c, \\
& a < \|(u_3, v_3)\| \leq c
\end{aligned} \tag{65}$$

$$\text{with } \Psi(u_3, v_3) < b.$$

The proof is completed. $\square$

Example 13. To verify the feasibility of our conclusion, here we consider the boundary value problem of nonlinear fractional differential system as follows:

$$\begin{aligned}
& = \min_{\theta \leq t \leq 1-\theta} \left| \frac{bt^\alpha}{\theta^2} \right| \geq \frac{b\theta^\alpha}{\theta^2} = b\theta^{\alpha-2} > b.
\end{aligned} \tag{62}$$

Hence,

$$\begin{aligned}
& \|(u, v)\| = \max \left\{ \max_{0 \leq t \leq 1} |u(t)|, \max_{0 \leq t \leq 1} |v(t)| \right\}, \\
& \max_{0 \leq t \leq 1} |D_{0^+}^\alpha u(t)|, \max_{0 \leq t \leq 1} |D_{0^+}^\alpha v(t)| \Big\} \\
& = \max \left\{ \frac{b}{\theta^2}, \frac{b}{\theta^2}, \frac{b}{\theta^2} \Gamma(\alpha+1), \frac{b}{\theta^2} \Gamma(\alpha+1) \right\} = \frac{b}{\theta^2} \\
& \cdot \Gamma(\alpha+1) < c.
\end{aligned} \tag{63}$$

If $(u, v) \in P(\Psi, b, c)$, we have $b \leq u(t) \leq c$, $b \leq v(t) \leq c$, $-c \leq D_{0^+}^\alpha u(t) \leq c$, $-c \leq D_{0^+}^\alpha v(t) \leq c$, for all $t \in [\theta, 1-\theta]$. Then

$$\begin{aligned}
& D_{0^+}^{1.7} (\varphi_p(D_{0^+}^{1.2} u(t))) \\
& = f_1(t, u(t), v(t), D_{0^+}^{1.2} u(t), D_{0^+}^{1.2} v(t)), \\
& D_{0^+}^{1.7} (\varphi_p(D_{0^+}^{1.2} v(t))) \\
& = f_2(t, u(t), v(t), D_{0^+}^{1.2} u(t), D_{0^+}^{1.2} v(t)),
\end{aligned} \tag{66}$$

with boundary conditions

$$u(0) = D_{0^+}^{1.2} u(0) = 0$$

$$u(1) = \sum_{i=1}^2 b_i \int_0^{\xi_i} u(t) dA(t),$$

$$\varphi_p(D_{0^+}^{1.2} u(1)) = \sum_{i=1}^2 a_i \varphi_p(D_{0^+}^{1.2} u(\xi_i)),$$

$$\begin{aligned}
v(0) &= D_{0^+}^{1,2} v(0) = 0 & c_1 &= \frac{2}{5}, \\
v(1) &= \sum_{i=1}^2 d_i \int_0^{\eta_i} v(t) dB(t), & c_2 &= \frac{4}{5}, \\
\varphi_p(D_{0^+}^{1,2} v(1)) &= \sum_{i=1}^2 c_i \varphi_p(D_{0^+}^{1,2} v(\eta_i)), & d_1 &= \frac{1}{3}, \\
& & d_2 &= \frac{1}{2}, \\
& & \eta_1 &= \frac{1}{2}, \\
& & \eta_2 &= \frac{3}{5}, \\
& & \xi_1 &= \frac{1}{3}, \\
& & \xi_2 &= \frac{2}{3}.
\end{aligned} \tag{67}$$

In this model, we set $p = 2$, and

$$\begin{aligned}
a_1 &= \frac{1}{9}, & \eta_1 &= \frac{1}{2}, \\
a_2 &= \frac{4}{9}, & \eta_2 &= \frac{3}{5}, \\
b_1 &= \frac{1}{4}, & \xi_1 &= \frac{1}{3}, \\
b_2 &= \frac{3}{4}, & \xi_2 &= \frac{2}{3}.
\end{aligned} \tag{68}$$

Let $A(t) = 0.5t$, $B(t) = 0.8t$ for $t \in [0, 1]$,

$$f_k(t, u, v, w, z) = \begin{cases} \frac{t}{200} + \frac{|u| + |v| + |w| + |z|}{350}, & 0 \leq u \leq 5, 0 \leq v \leq 5, w \in \mathbb{R}, z \in \mathbb{R}, \\ 43.007(|u| + |v|) - 430.010, & 5 < u \leq 10, 5 < v \leq 10, w \in \mathbb{R}, z \in \mathbb{R}, \\ 400 \cos(t) + \frac{2|v|}{4000} + 100, & \text{else.} \end{cases} \tag{69}$$

It is obvious that $f_k(t, u, v, w, z) : [0, 1] \times \mathbb{R}^{+2} \times \mathbb{R}^2 \rightarrow [0, +\infty)$ ($k = 1, 2$) is continuous. Take $\theta = 0.4$; by calculation, we obtain

$$\begin{aligned}
q &= 2, \\
\rho &= 12.463, \\
N_1 &= 0.220, \\
N_2 &= 0.277, \\
Q_1 &= 0.034, \\
Q_2 &= 0.044, \\
M &= 0.013, \\
N &= 35.327.
\end{aligned} \tag{70}$$

For $k = 1, 2$, set $a = 5$, $b = 10$, and $c = 40000$, which satisfy $0 \leq a \leq b \leq (\theta^2/\Gamma(\alpha - 1))c$, such that

$$\begin{aligned}
(F_1) \quad & f_k(t, u, v, w, z) \leq Mc, \text{ for } (t, u, v, w, z) \in [0, 1] \times [0, 40000]^2 \times [-40000, 40000]^2; \\
(F_2) \quad & f_k(t, u, v, w, z) < Ma, \text{ for } (t, u, v, w, z) \in [0, 1] \times [0, 5]^2 \times [-5, 5]^2; \\
(F_3) \quad & f_k(t, u, v, w, z) \geq Nb, \text{ for } (t, u, v, w, z) \in [0.4, 0.6] \times [10, 40000]^2 \times [-40000, 40000]^2,
\end{aligned}$$

Thus, all the conditions in Theorem 12 are achieved. Consequently, BVP (66) has at least three positive solutions (u_1, v_1) , (u_2, v_2) , and (u_3, v_3) such that $\|(u_1, v_1)\| < 5$, $10 < \min_{t \in [0.4, 0.6]} \|(u_2, v_2)(t)\| \leq 40000$, and $5 < \|(u_3, v_3)\| \leq 40000$ with $\min_{t \in [0.4, 0.6]} \|(u_3, v_3)(t)\| < 10$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the manuscript, read and approved the final draft.

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