Global Dynamics of Higher-Order Exponential Systems of Difference Equations

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In this paper, we study the global dynamics of three higher-order exponential systems of rational difference equations. This suggested work considerably extends and improves some existing results in the literature.

1. Introduction

Global dynamical properties of difference equations or systems of difference equations involving exponential term have been widely investigated in recent years. For instance, Ozturk et al. [1] have explored the dynamical properties of the following exponential difference equation:

\[ x_{n+1} = \alpha + \beta e^{-x_n} + \gamma + x_{n-1}, \]  

(1)

where \( \alpha, \beta, \gamma \), and \( x_p \) \((p = 0, -1)\) are positive real numbers. Cömert et al. [2] have explored the dynamical properties of the following higher-order exponential difference equation:

\[ x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{y + x_{n-1}}, \]  

(2)

where \( \alpha, \beta, \gamma \), and \( x_p \) \((p = 0, -1, \ldots, -k)\) are positive real numbers. Bozkurt [3] has explored the dynamical properties of the following exponential difference equation:

\[ x_{n+1} = \frac{\alpha e^{-x_n} + \beta e^{-x_{n-1}}}{y + \alpha x_n + \beta x_{n-1}}, \]  

(3)

where \( \alpha, \beta, \gamma \), and \( x_p \) \((p = 0, -1)\) are positive real numbers. In 2009, Ozturk et al. [4] have explored the dynamical properties of the following higher-order exponential difference equation:

\[ x_{n+1} = \frac{\alpha e^{-\gamma n + (n-k)x_{n-k}}}{\beta + nx_n + (n-k)x_{n-k}}, \]  

(4)

where \( \alpha, \beta, \gamma \), and \( x_p \) \((p = 0, -1, \ldots, -k)\) are positive real numbers. Papaschinopoulos et al. [5] have explored the dynamical properties of the following exponential systems of difference equations:

\[ x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{y + y_{n-1}}, \]  

\[ y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\zeta + y_{n-1}}, \]

\[ x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{y + x_{n-1}}, \]  

\[ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}}, \]

\[ x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{y + x_{n-1}}, \]  

\[ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + x_{n-1}}, \]  

(5)
where \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta \) and \( x_p, y_p (p = 0, -1) \) are positive real numbers. Khan and Sharif [6] have explored the dynamical properties of the following 3 \times 6 exponential systems of difference equations:

\[
\begin{align*}
x_{n+1} &= \frac{a_1 y_n + c_1 x_n e^{k_1 - d_1 x_n}}{b_1 + y_n}, \\
y_{n+1} &= \frac{a_2 y_n + c_2 y_n e^{k_2 - d_2 y_n}}{b_2 + x_n}, \\
z_{n+1} &= \frac{a_3 y_n + c_3 y_n e^{k_3 - d_3 y_n}}{b_3 + z_n}, \\
x_{n+1} &= \frac{a_4 y_n + c_4 y_n e^{k_4 - d_4 y_n}}{b_4 + y_n}, \\
y_{n+1} &= \frac{a_5 y_n + c_5 y_n e^{k_5 - d_5 y_n}}{b_5 + x_n}, \\
z_{n+1} &= \frac{a_6 y_n + c_6 y_n e^{k_6 - d_6 y_n}}{b_6 + z_n}.
\end{align*}
\]

(6)

where \( \alpha_p, \beta_p, \gamma, (p = 4, 5, \cdots, 18) \) and \( x_n, y_n, z_n (p = 0, -1) \) are positive real numbers. Khan and Qureshi [7] have explored the dynamical properties of the following exponential system of difference equations:

\[
\begin{align*}
x_{n+1} &= \frac{a_1 y_n + c_1 x_n e^{k_1 - d_1 x_n}}{b_1 + y_n}, \\
y_{n+1} &= \frac{a_2 y_n + c_2 y_n e^{k_2 - d_2 y_n}}{b_2 + x_n}, \\
z_{n+1} &= \frac{a_3 y_n + c_3 y_n e^{k_3 - d_3 y_n}}{b_3 + z_n}, \\
x_{n+1} &= \frac{a_4 y_n + c_4 y_n e^{k_4 - d_4 y_n}}{b_4 + y_n}, \\
y_{n+1} &= \frac{a_5 y_n + c_5 y_n e^{k_5 - d_5 y_n}}{b_5 + x_n}, \\
z_{n+1} &= \frac{a_6 y_n + c_6 y_n e^{k_6 - d_6 y_n}}{b_6 + z_n}.
\end{align*}
\]

where \( \alpha, \beta, \gamma, \alpha_p, \beta_p, \gamma_p (p = 1, 2, q = 1, 2, 3) \) and \( x_0, y_0 \) are positive real numbers. Mylona et al. [9] have explored the dynamical properties of the following two 3\times3 close-to-cyclic systems of exponential difference equations:

\[
\begin{align*}
x_{n+1} &= \frac{a_1 x_n + b_1 y_n e^{-k_1 x_n}}{b_1 + y_n}, \\
y_{n+1} &= \frac{a_2 y_n + b_2 y_n e^{-k_2 y_n}}{b_2 + x_n}, \\
z_{n+1} &= \frac{a_3 z_n + b_3 z_n e^{-k_3 z_n}}{b_3 + z_n}.
\end{align*}
\]

(9)

where \( a_p, b_p, c_p, d_p, k_q (p = 1, 2, q = 1, 2, 3) \) and \( x_0, y_0, z_0 \) are positive real numbers. Mylona et al. [10] have explored the dynamical
properties of the following systems of difference equations with exponential terms:

\[ x_{n+1} = a_1 x_n + a_2 y_n e^{-x_n}, \]
\[ y_{n+1} = a_3 y_n + a_4 x_n e^{-y_n}, \]

(10)

where \( a_p \) (\( p = 1, \ldots, 4 \)) and \( x_0, y_0 \) are positive real numbers, and for more other interesting results on difference equations as well as systems of difference equations, we refer the reader to [11–13] and the references cited therein. Motivated by the above systemic studies, in this paper we aim to explore the dynamical properties of the following higher-order exponential systems of difference equations, which are natural extension of the work studied by Ozturk et al. [4]:

\[ x_{n+1} = \frac{ae^{-(nx_n+(n-k)x_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}}, \]
\[ y_{n+1} = \frac{\alpha_1 e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1 + nx_n + (n-k)x_{n-k}}, \]

(11)

\[ x_{n+1} = \frac{ae^{-(mx_n+(m-k)x_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}}, \]
\[ y_{n+1} = \frac{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1 + nx_n + (n-k)x_{n-k}}, \]

(12)

\[ x_{n+1} = \frac{ae^{-(nx_n+(n-k)x_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}}, \]
\[ y_{n+1} = \frac{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1 + nx_n + (n-k)x_{n-k}}, \]

(13)

where \( \alpha, \beta, \alpha_1, \beta_1 \) and \( x_p, y_p \) (\( p = 0, -1, \ldots, -k \)) are positive real numbers.

The rest of this paper is organized as follows: Section 2 is about the study of boundedness and persistence of systems (11), (12), and (13). This also includes construction of invariant rectangle of these systems. Section 3 is about the uniqueness and existence of \(+ve\) fixed point of systems (11), (12), and (13). Section 4 deals with the study of local stability, whereas in Section 5 we study the global dynamics of the unique \(+ve\) fixed point of these systems. Section 6 deals with the study of rate of convergence of systems (11), (12), and (13). Brief conclusion is given in Section 7.

2. Boundedness, Persistence, and Construction of Invariant Rectangle of Systems (11), (12), and (13)

**Theorem 1.** If \( 2n \geq k \), then every \(+ve\) solution \( \{x_n, y_n\}_{n=-k}^{\infty} \) of (i) system (11), (ii) system (12), and (iii) system (13) is bounded and persists.
From (13) and (20) we get
\[ x_n \geq \frac{\alpha e^{-k(2n)}(\alpha/\beta)}{\beta + (k - 2n)(\alpha/\beta)} = L_5, \]
\[ y_n \geq \frac{\alpha_1 e^{-k(2n)}(\alpha/\beta)}{\beta_1 + (k - 2n)(\alpha/\beta)} = L_6, \] (21)
\[ n = 2, 3, \ldots. \]

Finally from (20) and (21) we get
\[ L_5 \leq x_n \leq U_5, \]
\[ L_6 \leq y_n \leq U_6, \] (22)
\[ n = 3, 4, \ldots. \]

Theorem 2. If \(2n \geq k\), then the following holds:
(i) If \((x_n, y_n)_{n=k}^\infty\) is a +ve solution of (11), then \([L_1, U_1] \times [L_2, U_2]\) is invariant rectangle for (11).
(ii) If \((x_n, y_n)_{n=k}^\infty\) is a +ve solution of (12), then \([L_4, U_4] \times [L_3, U_3]\) is invariant rectangle for (12).
(iii) If \((x_n, y_n)_{n=k}^\infty\) is a +ve solution of (13), then \([L_5, U_5] \times [L_6, U_6]\) is invariant rectangle for (13).

Proof. The proof follows from mathematical induction. \(\square\)

3. Existence and Uniqueness of +ve Equilibrium of Systems (11), (12), and (13)

Theorem 3. Assume that \(2n \geq k\), then the following statements hold:
(i) If \(0 < \beta < 1\) and
\[ e^{-(k + 2n)L_1} (\beta + (-k + 2n) U_1) \left((-k + 2n) U_1 + 1\right) \]
\[ \cdot \left( e^{-(k + 2n)L_2} + (1 - \beta) U_1 \right) \left((-k + 2n) U_2 + 1\right) \]
\[ \cdot L_1^2 \left( e^{-(k + 2n)L_4} - \beta U_4 \right), \] (23)
then (11) has a unique +ve equilibrium \((\overline{x}, \overline{y})\) \(\in [L_1, U_1] \times [L_3, U_3].\)
(ii) If
\[ e^{-(k + 2n)L_1} (\beta + (-k + 2n) U_1) \]
\[ \cdot \left( \beta_1 + 2 \ln \left( \frac{\alpha}{\beta + (-k + 2n) L_1} \right) \right) \]
\[ \cdot \frac{\alpha_1}{\beta_1 + (-k + 2n) L_1} \]
\[ < \alpha_1 (-k + 2n)^2 (\beta + (-k + 2n) L_1), \] (24)
then (12) has a unique +ve equilibrium \((\overline{x}, \overline{y})\) \(\in [L_3, U_3] \times [L_4, U_4].\)

(iii) If
\[ \alpha_1 (2n - k)^2 e^{-(2n-k)L_1} \left( \beta_1 + (2n - k) U_1 \right) \]
\[ + (2n-k) \alpha_1 e^{-(2n-k)L_1} U_1 \Omega_1 < (\beta_1 \]
\[ + (2n-k) L_1)^2 \Omega_2, \] (25)
then (13) has a unique +ve equilibrium \((\overline{x}, \overline{y})\) \(\in [L_5, U_5] \times [L_6, U_6]\) where \(\Omega_1\) and \(\Omega_2\) are defined in (53) and (54).

Proof. (i) Consider
\[ x = \frac{\alpha e^{-(2n-k)x}}{\beta + (2n - k) y}, \]
\[ y = \frac{\alpha_1 e^{-(2n-k)y}}{\beta_1 + (2n - k) x}. \] (26)
From (26), we have
\[ y = \frac{\alpha e^{-(k+2n)x}}{(-k + 2n) x} - \frac{\beta}{(-k + 2n)}, \]
\[ x = \frac{\alpha_1 e^{-(k+2n)y}}{(-k + 2n) y} - \frac{\beta_1}{(-k + 2n)}. \] (27)
From (27), define
\[ Q(x) = \frac{\alpha_1 e^{-(k+2n)q(x)}}{(-k + 2n) q(x)} - \frac{\beta_1}{(-k + 2n)} - x, \] (28)
where
\[ q(x) = \frac{\alpha e^{-(k+2n)x}}{(-k + 2n) x} - \frac{\beta}{(-k + 2n)}, \] (29)
and \(x \in [L_1, U_1]\). Now our claim is \(Q(x) = 0\) has a single solution \(x \in [L_1, U_1]\). From (28) and (29) one gets
\[ Q'(x) = -\frac{\alpha_1 e^{-(k+2n)q(x)}}{(-k + 2n) q(x)} \left((-k + 2n) q(x) + 1\right) \]
\[ \cdot \frac{(k+2n) q(x)}{(-k + 2n) q(x)} - 1, \] (30)
where
\[ q'(x) = -\frac{\alpha e^{-(k+2n)x}}{(-k + 2n) x} \left((-k + 2n) x + 1\right) \]
\[ \cdot \frac{(-k + 2n) x^2}{(-k + 2n) x}. \] (31)
If \(\overline{x} \in [L_1, U_1]\) is a solution of \(Q(x) = 0\), then from (28) and (29) one gets
\[ \frac{\alpha_1 e^{-(k+2n)q(\overline{x})}}{q(\overline{x})} = \beta_1 + (-k + 2n) \overline{x}, \] (32)
where
\[ q(\overline{x}) = \frac{\alpha e^{-(k+2n)x}}{(-k + 2n) \overline{x}} - \frac{\beta}{(-k + 2n)}. \] (33)
In view of (31), (32), and (33), (30) becomes

\[ Q'(x) = \frac{\alpha e^{-(k+2n)x} (\beta_1 + (-k + 2n) x) (-k + 2n) x + 1) (\alpha e^{-(k+2n)x} + (1 - \beta) x)}{(-k + 2n) x^2 (\alpha e^{-(k+2n)x} - \beta x)} - 1, \]

\[ \leq \frac{\alpha e^{-(k+2n)x_1} (\beta_1 + (-k + 2n) U_1) (-k + 2n) U_1 + 1) (\alpha e^{-(k+2n)x_1} + (1 - \beta) U_1)}{(-k + 2n) L_1^2 (\alpha e^{-(k+2n)x_1} - \beta U_1)} - 1. \]

Now assume that (23) holds, then from (34) one gets \( Q'(x) < 0. \)

(ii). From (12) one has

\[ x = \frac{\alpha e^{-(k+2n)y}}{\beta + (-k + 2n) x}, \]

\[ y = \frac{\alpha_1 e^{-(k+2n)x}}{\beta_1 + (-k + 2n) y}. \]

From (35) one gets

\[ y = \frac{1}{-k + 2n} \ln \left( \frac{\alpha}{(\beta + (-k + 2n) x) x} \right), \]

\[ x = \frac{1}{-k + 2n} \ln \left( \frac{\alpha_1}{(\beta_1 + (-k + 2n) y) y} \right). \]

From (36), define

\[ G(x) = \frac{1}{-k + 2n} \ln \left( \frac{\alpha_1}{(\beta_1 + (-k + 2n) g(x)) g(x)} \right) - x, \]

where

\[ g(x) = \frac{1}{-k + 2n} \ln \left( \frac{\alpha}{(\beta + (-k + 2n) x) x} \right), \]

and \( x \in [L_3, U_3]. \) We claim that \( G(x) = 0 \) has a single solution \( x \in [L_3, U_3]. \) From (37) and (38) one gets

\[ G'(x) = -\frac{g'(x) (\beta_1 + 2 (-k + 2n) g(x)) g(x)}{(-k + 2n) (\beta_1 + (-k + 2n) g(x)) g(x)} - 1, \]

where

\[ g'(x) = -\frac{\beta + 2 (2n - k) x}{(2n - k) (\beta + (2n - k) x) x}. \]

If \( \bar{x} \in [L_3, U_3] \) is a solution of \( G(x) = 0, \) then from (37) and (38) one gets

\[ (\beta_1 + (-k + 2n) g(\bar{x}) g(\bar{x}) = \alpha_1 e^{-(k+2n)x}, \]

where

\[ g(\bar{x}) = \frac{1}{2n - k} \ln \left( \frac{\alpha}{(\beta + (2n - k) \bar{x}) \bar{x}} \right). \]

In view of (40), (41), and (42), (39) becomes

\[ G'(x) = \frac{e^{(2n-k)x} (\beta + 2 (-k + 2n) \bar{x}) (\beta + 2 \ln (\alpha/(\beta + (-k + 2n) \bar{x}) \bar{x}))}{\alpha_1 (-k + 2n)^2 (\beta + (-k + 2n) \bar{x})} - 1, \]

\[ \leq \frac{e^{-(k+2n)x_1} (\beta + 2 (-k + 2n) U_3) (\beta + 2 \ln (\alpha/ (\beta + (-k + 2n) L_3) L_3))}{\alpha_1 (-k + 2n)^2 (\beta + (-k + 2n) L_3)} - 1. \]

Finally assume that (24) holds, then from (43) one gets \( G'(x) < 0. \)

(iii) From system (13)

\[ x = \frac{\alpha e^{-(2n-k)y}}{\beta + (2n - k) y}, \]

\[ y = \frac{\alpha_1 e^{-(2n-k)x}}{\beta_1 + (2n - k) x}. \]

From (44), define

\[ f(x) = \frac{\alpha e^{-(k+2n)x}}{\beta + (-k + 2n) x} - x, \]

where

\[ j(x) = \frac{\alpha_1 e^{-(k+2n)x}}{\beta_1 + (-k + 2n) x}. \]
and \( x \in [L_5, U_5] \). We claim that \( J(x) = 0 \) has a single solution \( x \in [L_5, U_5] \). From (45) and (46) one gets

\[
J' (x) = -\frac{\alpha (\beta + (-k + 2n) x)}{(\beta + (-k + 2n) x)^2} - 1,
\]

where

\[
J' (x) = -\frac{\alpha (2n - k) e^{(-2n-k)x} (\beta_1 + (2n - k) x) + 1}{(\beta_1 + (2n - k) x)^2}.
\]

If \( \bar{x} \in [L_5, U_5] \) is a solution of \( J(x) = 0 \), then from (45) and (46) one gets

\[
e^{-(-k+2n)j(\bar{x})} = \left( \frac{\beta + (-k + 2n) j(\bar{x})}{\alpha} \right) \bar{x},
\]

where

\[
j(\bar{x}) = \frac{\alpha_1 e^{-(-k+2n)x}}{\beta_1 + (-k + 2n) \bar{x}}.
\]

In view of (48), (49), and (50), (47) becomes

\[
J' (x) = -\frac{\alpha_1 (-k + 2n)^2 e^{(-2n-k)x}}{(\beta_1 + (-k + 2n) x)^2} - 1,
\]

where

\[
\Omega = (\beta_1 + (2n - k) x)
\]

\[
+ (-k + 2n) \alpha_1 e^{(-2n-k)x},
\]

\[
\Omega_1 = (\beta_1 + (2n - k) x) U_3
\]

\[
+ (-k + 2n) \alpha_1 e^{(-2n-k)x},
\]

\[
\Omega_2 = \beta_1 (\beta_1 + (2n - k) L_5) + (2n - k) \alpha_1 e^{(-2n-k)x}.
\]

Finally assume that (25) along with (53) and (54) holds, then from (51) one gets \( J'(x) < 0 \).

4. Local Asymptotic Stability about Equilibrium of Systems (11), (12), and (13)

Theorem 4. Assume that \( 2n \geq k \), then the following statements hold:

(i) If

\[
(-k + n) \alpha_1 e^{(-2n-k)L_1} \left( \beta + (2n - k) U_2 + 1 \right)
\]

\[
+ na\alpha_1 e^{(-2n-k)L_1} \left( \beta + (2n - k) L_2 + 1 \right)
\]

\[
< (\beta + (2n - k) L_2)^2,
\]

and

\[
(-k + n) \alpha_1 e^{(-2n-k)L_1} \left( \beta_1 + (-k + 2n) U_1 + 1 \right)
\]

\[
+ na\alpha_1 e^{(-2n-k)L_1} \left( \beta_1 + (-k + 2n) L_1 + 1 \right)
\]

\[
< (\beta_1 + (-k + 2n) L_1)^2,
\]

then \( (\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2] \) of (11) is a sink.

(ii) If

\[
(-k + n) \alpha_1 e^{(-2n-k)L_4} \left( \beta + (-k + 2n) U_3 + 1 \right)
\]

\[
+ na\alpha_1 e^{(-2n-k)L_4} \left( \beta + (-k + 2n) L_3 + 1 \right)
\]

\[
< (\beta + (-k + 2n) L_3)^2,
\]

and

\[
(-k + n) \alpha_1 e^{(-2n-k)L_4} \left( \beta_1 + (2n - k) U_4 + 1 \right)
\]

\[
+ na\alpha_1 e^{(-2n-k)L_4} \left( \beta_1 + (2n - k) L_4 + 1 \right)
\]

\[
< (\beta_1 + (2n - k) L_4)^2,
\]

then \( (\bar{x}, \bar{y}) \in [L_3, U_3] \times [L_4, U_4] \) of (12) is a sink.

(iii) If

\[
(-k + n) \alpha_1 e^{(-2n-k)L_6} \left( \beta + (-k + 2n) U_5 + 1 \right)
\]

\[
+ na\alpha_1 e^{(-2n-k)L_6} \left( \beta + (-k + 2n) L_5 + 1 \right)
\]

\[
< (\beta + (-k + 2n) L_5)^2,
\]

and

\[
(-k + n) \alpha_1 e^{(-2n-k)L_6} \left( \beta_1 + (-k + 2n) U_6 + 1 \right)
\]

\[
+ na\alpha_1 e^{(-2n-k)L_6} \left( \beta_1 + (-k + 2n) L_6 + 1 \right)
\]

\[
< (\beta_1 + (-k + 2n) L_6)^2,
\]

then \( (\bar{x}, \bar{y}) \in [L_5, U_5] \times [L_6, U_6] \) of (13) is a sink.
Moreover, linearized equation of (11) about \((\bar{x}, \bar{y})\) is

\[
\Phi_{n+1} = E\Phi_n, 
\]

where

\[
E = \begin{pmatrix}
\mu_1 & 0 & 0 & \ldots & 0 & \mu_2 & \mu_3 & 0 & \ldots & 0 & \mu_4 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\mu_5 & 0 & 0 & \mu_6 & \mu_7 & 0 & 0 & 0 & \mu_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
\end{pmatrix}.
\]

Let eigenvalues of \(E\) be \(\lambda_i, i = 1, 2, \ldots, 2k + 2\). Let \(D\) be \(\text{diag}(\tau_1, \tau_2, \ldots, \tau_{2k+2})\) be a diagonal matrix with \(\tau_i = \tau_{k+2} = 1\), \(\tau_{1+m} = \tau_{k+2+m} = 1 - me (1 \leq m \leq k)\), and

\[
0 < e < \min \left\{ \frac{1}{k} \left( 1 - \frac{(k + n)\alpha e^{-\frac{(k+2)n\tau}{\beta + (k + 2n)\bar{y}}} (\beta + (k + 2n)\bar{y})}{\beta + (k + 2n)\bar{y}} + 1 \right) \right\},
\]

\[
1 \left( 1 - \frac{(k + n)\alpha e^{-\frac{(k+2)n\tau}{\beta + (k + 2n)\bar{y}}} (\beta + (k + 2n)\bar{y})}{\beta + (k + 2n)\bar{y}} + 1 \right) \right\}.
\]

Since \(D\) is invertible, So,

\[
DED^{-1} = \begin{pmatrix}
\tau_1 \tau_1^{-1} & \mu_1 & 0 & \ldots & 0 & \tau_1 \tau_1^{-1} & \mu_2 & \tau_1 \tau_1^{-1} & \mu_3 & 0 & \ldots & 0 & \tau_1 \tau_1^{-1} & \mu_4 \\
\tau_2 \tau_1^{-1} & 0 & 0 & \ldots & 0 & \tau_1 \tau_1^{-1} & \mu_2 & \tau_1 \tau_1^{-1} & \mu_3 & 0 & \ldots & 0 & \tau_1 \tau_1^{-1} & \mu_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \tau_{k+1} \tau_1^{-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\
\tau_{k+2} \tau_1^{-1} & \mu_5 & 0 & 0 & \ldots & 0 & \tau_{k+1} \tau_1^{-1} & \mu_6 & \tau_{k+2} \tau_1^{-1} & \mu_7 & 0 & \ldots & 0 & \tau_{k+1} \tau_1^{-1} & \mu_8 \\
0 & 0 & \tau_{k+1} \tau_1^{-1} & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \tau_{k+2} \tau_{k+1}^{-1} & 0 & 0 & \ldots & 0 & \tau_{k+2} \tau_{k+1}^{-1} & 0 \\
\end{pmatrix}.
\]
The following two inequalities,

\[ \tau_1 > \tau_2 > \cdots > \tau_{k+1} > 0, \]
\[ \tau_{k+2} > \tau_{k+3} > \cdots > \tau_{2k+2} > 0, \]

imply that

\[ \tau_2 \tau_1^{-1} < 1, \]
\[ \tau_3 \tau_2^{-1} < 1, \]
\[ \vdots \]
\[ \tau_{k+1} \tau_k^{-1} < 1, \]
\[ \tau_{k+2} \tau_{k+1}^{-1} < 1, \]
\[ \tau_{k+3} \tau_{k+2}^{-1} < 1, \]
\[ \tau_{k+4} \tau_{k+3}^{-1} < 1, \]
\[ \vdots \]
\[ \tau_{2k+2} \tau_{2k+1}^{-1} < 1. \]

Furthermore,

\[ \tau_1^{-1} \mu_1 + \tau_1^{-1} \mu_2 + \tau_1^{-1} \mu_3 + \tau_1^{-1} \mu_4 = \alpha e^{-(k+2n)} \left( \beta + (k + 2n) \frac{\tau_1}{\tau_2} - 1 \right) \left( \frac{n + k + n}{1 - k \epsilon} \right) < 1. \]

It is a well-known fact that \( E \) and \( DED^{-1} \) have the same eigenvalues. Thus

\[ \max_{1 \leq m \leq 2k+2} |\lambda_m| \leq \left\| DED^{-1} \right\|_o = \max \left\{ \tau_2 \tau_1^{-1}, \cdots, \right. \]
\[ \left. \tau_{k+1} \tau_k^{-1}, \tau_{k+2} \tau_{k+1}^{-1}, \cdots, \tau_{2k+2} \tau_{2k+1}^{-1}, \right. \]
\[ \alpha e^{-(k+2n)} \left( \beta + (2n-k) \frac{\mu_1}{\mu_2} - 1 \right) \left( \frac{n + k + n}{1 - k \epsilon} \right), \]
\[ \alpha e^{-(k+2n)} \left( \beta + (2n-k) \frac{\mu_9}{\mu_10} - 1 \right) \left( \frac{n + k + n}{1 - k \epsilon} \right), \]
\[ \alpha e^{-(k+2n)} \left( \beta + (2n-k) \frac{\mu_{13}}{\mu_{14}} - 1 \right) \left( \frac{n + k + n}{1 - k \epsilon} \right), \]
\[ \alpha e^{-(k+2n)} \left( \beta + (2n-k) \frac{\mu_{15}}{\mu_{16}} - 1 \right) \left( \frac{n + k + n}{1 - k \epsilon} \right), \]
\[ \alpha e^{-(k+2n)} \left( \beta + (2n-k) \frac{\mu_{17}}{\mu_{18}} - 1 \right) \left( \frac{n + k + n}{1 - k \epsilon} \right) < 1. \]

Hence \((\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2] \) of (11) is a sink.

(ii). If \((\overline{x}, \overline{y})\) is equilibrium of (12), then

\[ \overline{x} = \frac{\alpha e^{-(k+2n)} \overline{y}}{\beta + (2n-k) \overline{y}}, \]
\[ \overline{y} = \frac{\alpha_1 e^{-(k+2n)} \overline{y}}{\beta_1 + (2n-k) \overline{y}}, \]

and linearized equation of (12) about \((\overline{x}, \overline{y})\) is

\[ \Phi_{n+1} = E \Phi_n, \]

where \( \Phi_n \) is defined in (63). Also

\[ E = \begin{pmatrix} \mu_9 & 0 & \cdots & 0 & \mu_{10} & 0 & \cdots & 0 & \mu_{12} \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \mu_{13} & 0 & \cdots & 0 & \mu_{14} & 0 & \cdots & 0 & \mu_{16} \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \]

and

\[ \mu_9 = - \frac{n \alpha e^{-(k+2n)} \overline{y}}{\beta + (k + 2n) \overline{x}} \]
\[ \mu_{10} = - \frac{(k+n) \alpha e^{-(k+2n)} \overline{y}}{\beta + (k + 2n) \overline{x}} \]
\[ \mu_{11} = - \frac{n \alpha e^{-(k+2n)} \overline{x}}{\beta + (k + 2n) \overline{x}} \]
\[ \mu_{12} = - \frac{(k+n) \alpha e^{-(k+2n)} \overline{x}}{\beta + (2n-k) \overline{x}} \]
\[ \mu_{13} = - \frac{n \alpha_1 e^{-(k+2n)} \overline{x}}{\beta_1 + (k + 2n) \overline{y}} \]
\[ \mu_{14} = - \frac{(k+n) \alpha_1 e^{-(k+2n)} \overline{x}}{\beta_1 + (k + 2n) \overline{y}} \]
\[ \mu_{15} = - \frac{n \alpha e^{-(k+2n)} \overline{y}}{\beta_1 + (k + 2n) \overline{y}} \]
\[ \mu_{16} = - \frac{(k+n) \alpha e^{-(k+2n)} \overline{y}}{\beta_1 + (k + 2n) \overline{y}} \]
Using arrangements as in the proof of (i) one gets

\[
0 < \varepsilon < \min \left\{ \frac{1}{K} \left( 1 - \frac{(-k + n)\alpha e^{(-k+2n)\tau}}{\left( \beta + (-k + 2n) X + 1 \right)^2} - n\alpha e^{(-k+2n)\tau} \left( \beta + (-k + 2n) X + 1 \right) \right) \right\},
\]

and

\[
D^{-1} E D^{-1} = \begin{pmatrix}
\tau_1 \tau_1^{-1} \mu_9 & 0 & \ldots & 0 & -\tau_1 \tau_{k+1}^{-1} \mu_{10} & -\tau_1 \tau_{k+2}^{-1} \mu_{11} & 0 & \ldots & 0 & -\tau_1 \tau_{2k+2}^{-1} \mu_{12} \\
\tau_2 \tau_1^{-1} & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \tau_{k+1} \tau_1^{-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\
-\tau_{k+2} \tau_1^{-1} \mu_{13} & 0 & \ldots & 0 & -\tau_{k+2} \tau_{k+1}^{-1} \mu_{14} & -\tau_{k+2} \tau_{k+2}^{-1} \mu_{15} & 0 & \ldots & 0 & -\tau_{k+2} \tau_{2k+2}^{-1} \mu_{16} \\
0 & 0 & \ldots & 0 & 0 & \tau_{k+3} \tau_2^{-1} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \tau_{2k+2} \tau_{2k+1}^{-1} & 0
\end{pmatrix}
\]

Assume that (68) and (69) hold, then

\[
\mu_9 + \tau_1 \tau_1^{-1} \mu_{10} + \tau_1 \tau_{k+1}^{-1} \mu_{11} + \tau_1 \tau_{2k+2}^{-1} \mu_{12} = \frac{\alpha e^{(-k+2n)\tau}}{\left( \beta + (-k + 2n) X + 1 \right)^2} \left( n + \frac{-k + n}{1 - k \varepsilon} \right) < 1.
\]

\[
\mu_{13} + \tau_{k+2} \tau_{k+1}^{-1} \mu_{14} + \mu_{15} + \tau_{k+2} \tau_{2k+2}^{-1} \mu_{16} = \frac{\alpha_1 e^{(-k+2n)\tau}}{\left( \beta_1 + (-k + 2n) \bar{Y} + 1 \right)^2} \left( n + \frac{-k + n}{1 - k \varepsilon} \right) < 1.
\]

Finally,

\[
\max_{1 \leq m \leq 2k+2} \left| \lambda_m \right| \leq \left\| D^{-1} E D^{-1} \right\|_\infty = \max \left\{ \tau_1 \tau_1^{-1}, \ldots, \tau_{k+1} \tau_k^{-1}, \tau_{k+2} \tau_{k+2}^{-1}, \ldots, \tau_{2k+2} \tau_{2k+1}^{-1} \right\},
\]

and linearized equation of (13) about \((\bar{X}, \bar{Y})\) is

\[
\Phi_{n+1} = E \Phi_n,
\]

Hence equilibrium \((\bar{X}, \bar{Y}) \in [L_3, U_3] \times [L_4, U_4] (12)\) is a sink.
where $\Phi_n$ is again the same as (63). Moreover

$$E = \begin{pmatrix}
0 & 0 & \ldots & 0 & \mu_{17} & 0 & \ldots & 0 & \mu_{18} \\
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\
\mu_{19} & 0 & \ldots & 0 & \mu_{20} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 
\end{pmatrix}, \quad (84)$$

where

$$\mu_{17} = -\frac{n \alpha e^{-\frac{n}{k} \varpi} (\beta + (2n-k) \varpi + 1)}{(\beta + (2n-k) \varpi)^2},$$

$$\mu_{18} = -\frac{(k-n) \alpha e^{-\frac{k+2n}{k} \varpi} (\beta + (2n-k) \varpi + 1)}{(\beta + (2n-k) \varpi)^2},$$

$$\mu_{19} = -\frac{n \alpha_1 e^{-\frac{k+2n}{k} \varpi} (\beta_1 + (k+2n) \varpi + 1)}{(\beta_1 + (k+2n) \varpi)^2},$$

$$\mu_{20} = -\frac{(k-n) \alpha_1 e^{-\frac{k+2n}{k} \varpi} (\beta_1 + (k+2n) \varpi + 1)}{(\beta_1 + (k+2n) \varpi)^2}.$$

Again using similar arrangements as in the proof of (i) one gets

$$0 < \epsilon < \min \left\{ \frac{1}{k} \left( 1 - \frac{(k+n) \alpha e^{-\frac{k+2n}{k} \varpi} (\beta + (2n-k) \varpi + 1)}{(\beta + (2n-k) \varpi)^2} - n \alpha e^{-\frac{k+2n}{k} \varpi} (\beta + (2n-k) \varpi + 1) \right), \right\}.$$

Moreover,

$$DED^{-1} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & -\tau_1^{-1} \mu_{17} & 0 & \ldots & 0 & -\tau_1^{-1} \mu_{18} \\
\tau_2 \tau_1^{-1} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \tau_{k+1}^{-1} \mu_{19} & 0 & 0 & \ldots & 0 & 0 \\
-\tau_{k+2}^{-1} \mu_{19} & 0 & \ldots & 0 & -\tau_{k+2}^{-1} \mu_{20} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \tau_{k+2}^{-1} \mu_{20} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \tau_{k+2}^{-1} \tau_{k+1}^{-1} 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}, \quad (87)$$

Assume that (68) and (69) hold true, then

$$\tau_{k+2}^{-1} \mu_{17} + \tau_{k+2}^{-1} \mu_{18} = \frac{\alpha e^{-\frac{k+2n}{k} \varpi} (\beta + (2n-k) \varpi + 1)}{(\beta + (2n-k) \varpi)^2} \left( n + \frac{-k+n}{1-\varpi k} \right) < 1. \quad (88)$$

$$\tau_{k+2}^{-1} \mu_{19} + \tau_{k+2}^{-1} \mu_{20} = \frac{\alpha_1 e^{-\frac{k+2n}{k} \varpi} (\beta_1 + (k+2n) \varpi + 1)}{(\beta_1 + (k+2n) \varpi)^2} \left( n + \frac{n-k}{1-\varpi k} \right) < 1. \quad (89)$$

Finally,

$$\max_{1 \leq m \leq 2k+2} |\lambda_m| \leq \|DED^{-1}\|_{\infty} = \max \left\{ \tau_{k+1}^{-1} \tau_1^{-1}, \ldots, \right\}.$$
Hence equilibrium \((\bar{x}, \bar{y}) \in [L_5, U_5] \times [L_6, U_6]\) of (13) is a sink.

Hereafter we study the global dynamics about unique +ve equilibrium of systems (11), (12), and (13). Since investigating the global stability of difference equations or systems of difference equation for higher-order is a challenging task in recent year, here we will investigate the global dynamics about unique +ve equilibrium of systems (11), (12), and (13) by utilizing discrete-time Lyapunov function motivated by the work of [14–16].

5. Global Stability about Equilibrium of Systems (11), (12), and (13)

Theorem 5. Assume that \(2n \geq k\), then the following statements hold:

(i) If
\[
\alpha e^{-(k+2n)L_1} < \bar{x} (\beta + (-k + 2n) L_2),
\]
and
\[
\alpha_1 e^{-(k+2n)L_2} < \bar{y} (\beta_1 + (-k + 2n) L_4),
\]
then \((\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]\) of (11) is globally asymptotically stable.

(ii) If
\[
\alpha e^{-(k+2n)L_3} < \bar{x} (\beta + (2n - k) L_5),
\]
and
\[
\alpha_1 e^{-(k+2n)L_4} < \bar{y} (\beta_1 + (2n - k) L_8),
\]
then \((\bar{x}, \bar{y}) \in [L_3, U_3] \times [L_4, U_4]\) of (12) is globally asymptotically stable.

(iii) If
\[
\alpha e^{-(k+2n)L_4} < \bar{x} (\beta + (-k + 2n) L_6),
\]
and
\[
\alpha_1 e^{-(k+2n)L_5} < \bar{y} (\beta_1 + (-k + 2n) L_7),
\]
then \((\bar{x}, \bar{y}) \in [L_5, U_5] \times [L_6, U_6]\) of (13) is globally asymptotically stable.

Proof. (i) Consider
\[
\Gamma_n = \bar{x} \left( -1 + \frac{x_n}{\bar{x}} - \ln \frac{x_n}{\bar{x}} \right) + \bar{y} \left( -1 + \frac{y_n}{\bar{y}} - \ln \frac{y_n}{\bar{y}} \right),
\]
where its positivity follows from the inequality below:
\[
-1 + x - \ln x \geq 0, \quad \forall x > 0.
\]

Moreover,
\[
-\ln \left( \frac{x_{n+1}}{x_n} \right) = \ln \left( 1 - \frac{x_n}{x_{n+1}} \right) + 1 \leq -\frac{x_n - x_{n+1}}{x_n},
\]
\[
-\ln \left( \frac{y_{n+1}}{y_n} \right) = \ln \left( 1 - \frac{y_n}{y_{n+1}} \right) + 1 \leq -\frac{y_n - y_{n+1}}{y_n}.
\]

Now
\[
\Gamma_{n+1} - \Gamma_n = (x_{n+1} - x_n) - \bar{x} \ln \frac{x_{n+1}}{x_n} + (y_{n+1} - y_n) - \bar{y} \ln \frac{y_{n+1}}{y_n}.
\]

In view of (99), (100) takes the form
\[
\Gamma_{n+1} - \Gamma_n \leq (x_{n+1} - x_n) \left(1 - \frac{\bar{x} \ln \frac{x_{n+1}}{x_n}}{x_n + (2n - k) \ln x_n - k} \right) + (y_{n+1} - y_n) \left(1 - \frac{\bar{y} \ln \frac{y_{n+1}}{y_n}}{y_n + (2n - k) \ln y_n - k} \right).
\]

Assuming that (91) and (92) hold, then from (101) one gets
\[
\Gamma_{n+1} - \Gamma_n \leq 0 \quad \forall n \geq 0.
\]
Hence, \(\lim_{n \to \infty} (\Gamma_{n+1} - \Gamma_n) = 0\). Therefore, \((\bar{x}, \bar{y}) \in [L_3, U_3] \times [L_4, U_4]\) of (11) is globally asymptotically stable.
(i) Using similar construction as in the proof of (i) one gets
\[ \Gamma_{n+1} - \Gamma_n \leq (x_{n+1} - x_n) \]
\[ \frac{\alpha e^{-(n, k)(k+1)} - x}{\alpha e^{-(n, k)(k+1)}} \]
\[ + (y_{n+1} - y_n) \]
\[ \frac{\alpha e^{-(n, k)(k+1)} - x}{\alpha e^{-(n, k)(k+1)}} \]
\[ \leq (U_3 - L_3) \left( \frac{\alpha e^{-(k+2n)(k+1)} L_3 - \chi (\beta + (-k + 2n) L_3)}{\alpha e^{-(k+2n)(k+1)}} \right) \]
\[ + (U_4 - L_4) \]
\[ \frac{\alpha e^{-(k+2n)(k+1)} L_3 - \chi (\beta + (-k + 2n) L_3)}{\alpha e^{-(k+2n)(k+1)}} \].

Assuming that (93) and (94) hold, then from (102) one gets \( \Gamma_{n+1} - \Gamma_n < 0 \) for all \( n \geq 0 \). Hence, \( \lim_{n \to \infty} \Gamma_{n+1} - \Gamma_n = 0 \). Therefore, \( (\chi, \chi, \chi) \in [L_3, U_3] \times [L_4, U_4] \) of (12) is globally asymptotically stable.

(iii) Again using similar arrangements, one gets
\[ \Gamma_{n+1} - \Gamma_n \leq (x_{n+1} - x_n) \]
\[ \frac{\alpha e^{-(n, k)(k+1)} - x}{\alpha e^{-(n, k)(k+1)}} \]
\[ + (y_{n+1} - y_n) \]
\[ \frac{\alpha e^{-(n, k)(k+1)} - x}{\alpha e^{-(n, k)(k+1)}} \]
\[ \leq (U_3 - L_3) \left( \frac{\alpha e^{-(k+2n)(k+1)} L_3 - \chi (\beta + (-k + 2n) L_3)}{\alpha e^{-(k+2n)(k+1)}} \right) \]
\[ + (U_4 - L_4) \]
\[ \frac{\alpha e^{-(k+2n)(k+1)} L_3 - \chi (\beta + (-k + 2n) L_3)}{\alpha e^{-(k+2n)(k+1)}} \].

Assuming that (95) and (96) hold, then from (103) one gets \( \Gamma_{n+1} - \Gamma_n < 0 \) for all \( n \geq 0 \). Hence, \( \lim_{n \to \infty} \Gamma_{n+1} - \Gamma_n = 0 \). Thus, \( (\chi, \chi, \chi) \in [L_3, U_3] \times [L_4, U_4] \) of (13) is globally asymptotically stable.

Hereafter we will study rate of convergence of systems (11), (12), and (13) motivated by the work of [15, 17, 18].

6. Rate of Convergence of Systems (11), (12), and (13)

6.1. Rate of Convergence of System (11). Let \( \{x_n, y_n\} \) be any solution of (11) such that the following hold:
\[ \lim_{n \to \infty} x_n = \chi, \]
\[ \lim_{n \to \infty} y_n = \chi. \] (104)

Now computing error terms one gets
\[ x_{n+1} - \chi = \frac{\alpha e^{-(n, k)(k+1)} - \beta + ny_n + (n-k) x_{n-k}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ - \frac{\alpha e^{-(k+2n)(k+1)}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ \frac{\alpha e^{-(n, k)(k+1)} - \beta + ny_n + (n-k) x_{n-k}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ - \frac{\alpha e^{-(k+2n)(k+1)}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ (y_{n+1} - y_n) \]
\[ - \gamma. \] (105)

After some tedious calculations, from (105) one gets
\[ x_{n+1} - \chi = \frac{\alpha e^{-(n, k)(k+1)} - \beta + ny_n + (n-k) x_{n-k}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ - \frac{\alpha e^{-(k+2n)(k+1)}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ (y_{n+1} - y_n) \]
\[ - \gamma. \] (106)

Similarly,
\[ y_{n+1} - \gamma = \frac{\alpha e^{-(n, k)(k+1)} - \beta + ny_n + (n-k) x_{n-k}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ - \frac{\alpha e^{-(k+2n)(k+1)}}{\beta + \alpha e^{-(k+2n)(k+1)}} \]
\[ (y_{n+1} - y_n) \]
\[ - \gamma + O_1 (ny_n + (n-k) x_{n-k} - \gamma) \] (107)
From (106) and (107), we have

\[ x_{n+1} - x \approx - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} (x_n - x) \]

\[ - \frac{\alpha (n-k) e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} (x_n - x) \]

\[ - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} (y_n - y) \]

\[ y_{n+1} - y \approx - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n x_n + (n-k) x_{n-k}} (y_n - y) \]

\[ (108) \]

Let

\[ \epsilon_1 = x_n - x, \]

\[ \epsilon_2 = y_n - y, \]

and then (108) becomes

\[ \epsilon_{1n+1} = A_{n_1} \epsilon_1^n + A_{n_2} \epsilon_2^n + A_{n_3} \epsilon_1^{n-k} + A_{n_4} \epsilon_2^{n-k}, \]

\[ \epsilon_{2n+1} = A_{n_5} \epsilon_1^n + A_{n_6} \epsilon_2^n + A_{n_7} \epsilon_1^{n-k} + A_{n_8} \epsilon_2^{n-k}, \]

where

\[ A_{n_1} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}}, \]

\[ A_{n_2} = - \frac{\alpha (n-k) e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}}, \]

\[ A_{n_3} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} \]

\[ A_{n_4} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} \]

\[ A_{n_5} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} \]

\[ A_{n_6} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} \]

\[ A_{n_7} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} \]

\[ A_{n_8} = - \frac{\alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + n y_n + (n-k) y_{n-k}} \]

Moreover,

\[ \lim_{n \to \infty} A_{n_1} = - \frac{n \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_2} = - \frac{(n-k) \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_3} = - \frac{n \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_4} = - \frac{(n-k) \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_5} = - \frac{n \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_6} = - \frac{(n-k) \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_7} = - \frac{n \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_{n_8} = - \frac{(n-k) \alpha e^{-(\alpha \epsilon + (n-k) x_{n-k})}}{\beta + (n-k) x_{n-k}} \]

Hence limiting system of error terms becomes

\[ E_{n+1} = EE_n, \]
where

\[
E = \begin{pmatrix}
- \frac{nae^{-(k+2n)x}}{\beta + (-k + 2n)y} & 0 & \ldots & 0 & - \frac{(k + n)ae^{-(k+2n)x}}{\beta + (-k + 2n)y} \\
- \frac{nae^{-(k+2n)x}}{\beta + (-k + 2n)y} & 0 & \ldots & 0 & - \frac{(k + n)ae^{-(k+2n)x}}{\beta + (-k + 2n)y} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 \\
\end{pmatrix}
\]

which is similar to linearized system of (11). Finally we have the following theorem by utilizing Perron’s Theorems (see [19]).

**Theorem 6.** Assume that \((x_n, y_n)\) is a positive solution of (11) such that (104) along with the following relation holds:

\[
\overline{x} \in [L_1, U_1], \\
\overline{y} \in [L_2, U_2].
\]

Then the error vector

\[
E_n = \begin{pmatrix}
\epsilon_n^1 \\
\epsilon_{n-1}^1 \\
\vdots \\
\epsilon_{n-k}^1 \\
\epsilon_n^2 \\
\epsilon_{n-1}^2 \\
\vdots \\
\epsilon_{n-k}^2
\end{pmatrix}
\]

of every solution of (11) satisfies

\[
\lim_{n \to \infty} (\|E_n\|)^{1/n} = |\lambda E|,
\]

\[
\lim_{n \to \infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda E|,
\]

where \(\lambda E\) are the characteristic roots of \(E\) about \((\overline{x}, \overline{y})\).

---

6.2. Rate of Convergence of System (12). If \((x_n, y_n)\) is any solution of (12) such that (104) holds, then

\[
x_{n+1} - \overline{x} = \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} \left( \frac{ae^{-(k+2n)y}}{\beta + (-k + 2n)\overline{y}} - \frac{\alpha (n-k)e^{-(k+2n)\overline{y}}}{\beta + nx_n + (n-k)x_{n-k}} \right)
\]

\[
\alpha (n-k)e^{-(k+2n)\overline{y}} \left( \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} - 1 \right)
\]

After calculations, from (118) one gets

\[
x_{n+1} - \overline{x} = \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} \left( \frac{ae^{-(k+2n)y}}{\beta + (-k + 2n)\overline{y}} - \frac{\alpha (n-k)e^{-(k+2n)\overline{y}}}{\beta + nx_n + (n-k)x_{n-k}} \right) (x_n - \overline{x})
\]

\[
\alpha (n-k)e^{-(k+2n)\overline{y}} \left( \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} - 1 \right)
\]

\[
\beta + nx_n + (n-k)x_{n-k}
\]

\[
\alpha (n-k)e^{-(k+2n)\overline{y}} \left( \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} - 1 \right)
\]

\[
\beta + nx_n + (n-k)x_{n-k}
\]

\[
\alpha (n-k)e^{-(k+2n)\overline{y}} \left( \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} - 1 \right)
\]

\[
\beta + nx_n + (n-k)x_{n-k}
\]

\[
\alpha (n-k)e^{-(k+2n)\overline{y}} \left( \frac{ae^{-(ny_n + (n-k)y_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} - 1 \right)
\]

\[
\beta + nx_n + (n-k)x_{n-k}
\]
From (119) and (120), we have

\[ y_{n+1} - \bar{y} = -\frac{\alpha y e^{-\alpha x_n (n-k)x_{n-k}}}{\beta + n x_n + (n-k) x_{n-k}} (x_n - \bar{x}) \]

where

\[ A_n = \frac{\alpha e^{-\alpha x_n (n-k)x_{n-k}}}{\beta + n x_n + (n-k) x_{n-k}} \]

Let (109) hold, then from (121) one gets

\[ e_{n+1} = A_{n+1} e_n + A_{n+2} e_{n+1} + A_{n+3} e_{n+2} + A_{n+4} e_{n+3} + A_{n+5} e_{n+4} \]

\[ e_{n+1} = A_{n+1} e_n + A_{n+2} e_{n+1} + A_{n+3} e_{n+2} + A_{n+4} e_{n+3} + A_{n+5} e_{n+4} \]

Moreover,

\[ \lim_{n \to \infty} A_n = \frac{n \alpha e^{-\alpha x_n (n-k)x_{n-k}}}{\beta + n x_n + (n-k) x_{n-k}} \]

\[ \lim_{n \to \infty} A_n = \frac{(-k + n) \alpha e^{-\alpha x_n (n-k)x_{n-k}}}{\beta + n x_n + (n-k) x_{n-k}} \]

Hence limiting system of error terms becomes

\[ E_{n+1} = EE_n \]
where

\[
E = \begin{pmatrix}
-\frac{nae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
-\frac{nae^{-(k+2n)y}}{\beta_1 + ((-k+2n)x)^2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]

which is the same as linearized system of (12) about \((\mathcal{X}, \mathcal{Y})\). Finally we have following theorem.

**Theorem 7.** Assume that \(\{(x_n, y_n)\}\) is a positive solution of (12) such that (104) along with the following relation holds:

\[
\mathcal{X} \in [L_3, U_3], \quad \mathcal{Y} \in [L_4, U_4].
\]

Then the error vector \(E_n\) defined in (116) satisfies

\[
\lim_{n \to \infty} \left(\|E_n\|\right)^{1/n} = |\lambda E|
\]

\[
\lim_{n \to \infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda E|
\]

where \(\lambda E\) are the characteristic roots of \(E\) about \((\mathcal{X}, \mathcal{Y})\).

6.3 Rate of Convergence of System (13). If \(\{(x_n, y_n)\}\) is any solution of (13) such that (104) holds, then

\[
x_{n+1} - \mathcal{X} = -\frac{ae^{-(ny_n - (n-k)y_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}} (y_n - \mathcal{X})
\]

\[
\approx -\frac{ae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} (y_n - \mathcal{X}) - \alpha \frac{ae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} (y_n - \mathcal{Y})
\]

\[
\approx \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_n - \mathcal{Y}) - \alpha \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_n - \mathcal{Y})
\]

\[
\approx \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_n - \mathcal{Y}) - \alpha \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_n - \mathcal{Y})
\]

From (130) and (131), we have

\[
x_{n+1} - \mathcal{X} = \frac{ae^{-(ny_n - (n-k)y_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}} (y_{n-k} - \mathcal{X})
\]

\[
\approx \frac{ae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{X}) - \alpha \frac{ae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{Y})
\]

\[
\approx \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{Y}) - \alpha \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{Y})
\]

After simplifying, one gets

\[
x_{n+1} - \mathcal{X} = \frac{ae^{-(ny_n - (n-k)y_{n-k})}}{\beta + ny_n + (n-k)y_{n-k}} (y_{n-k} - \mathcal{X})
\]

\[
\approx \frac{ae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{X}) - \alpha \frac{ae^{-(k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{Y})
\]

\[
\approx \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{Y}) - \alpha \frac{ae^{-(-k+2n)y}}{\beta + ((-k+2n)x)^2} (y_{n-k} - \mathcal{Y})
\]
Let (109) hold, then (132) can be represented as

\[
\begin{align*}
\varepsilon_{n+1}^1 & = A_{n_1} \varepsilon_{n}^1 + A_{n_2} \varepsilon_{n-k}^1, \\
\varepsilon_{n+1}^2 & = A_{n_2} \varepsilon_{n}^2 + A_{n_3} \varepsilon_{n-k}^2,
\end{align*}
\]

where

\[
A_{n_1} = -\alpha (-k+n) \left( e^{\alpha \gamma (n-k)} e^{(-k+2n \gamma)} \right),
\]

\[
A_{n_2} = -\alpha (-k+n) \left( e^{\alpha \gamma (n-k)} e^{(-k+2n \gamma)} \right).
\]

Moreover,

\[
\lim_{n \to \infty} A_{n_{17}} = - \frac{n \alpha e^{(-k+2n \gamma)} \left( \beta + (-k+2n) \gamma \right) + 1}{\left( \beta + (-k+2n) \gamma \right)^2},
\]

\[
\lim_{n \to \infty} A_{n_{18}} = - \frac{(k+n) \alpha e^{(-k+2n \gamma)} \left( \beta + (-k+2n) \gamma \right) + 1}{\left( \beta + (-k+2n) \gamma \right)^2},
\]

\[
\lim_{n \to \infty} A_{n_{19}} = - \frac{n \alpha e^{(-k+2n \gamma)} \left( \beta + (-k+2n) \gamma + 1 \right)}{\left( \beta + (-k+2n) \gamma + 1 \right)^2},
\]

\[
\lim_{n \to \infty} A_{n_{20}} = - \frac{(k+n) \alpha e^{(-k+2n \gamma)} \left( \beta + (-k+2n) \gamma + 1 \right)}{\left( \beta + (-k+2n) \gamma + 1 \right)^2}.
\]

Hence limiting system of error terms becomes

\[E_{n+1} = EE_{n},\]

where

\[
E = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(136)

\[
A_{n_{10}} = -\alpha (-k+n) \left( e^{\alpha \gamma (n-k)} e^{(-k+2n \gamma)} \right),
\]

\[
\left( \frac{\beta + (-k+2n) \gamma + 1}{\beta + (-k+2n) \gamma + 1} \right).
\]

(137)

7. Conclusion

This work is about the global dynamics of three higher-order exponential systems of difference equations. We proved that every positive solution is bounded and persistent, and further \([L_1, U_1] \times [L_2, U_2], [L_3, U_3] \times [L_4, U_4], and [L_5, U_5] \times [L_6, U_6]\), respectively, are invariant rectangle for systems (11), (12), and (13). We studied existence and uniqueness of +ve equilibrium and global stability, and conclusions are presented in Table 1. Finally rate of convergence for (11), (12), and (13) is also investigated.

Data Availability

All the data utilized are included in this article and their sources are cited accordingly.
### Table 1: Systems with Corresponding Qualitative Behavior

<table>
<thead>
<tr>
<th>Systems</th>
<th>Corresponding behavior</th>
</tr>
</thead>
</table>
| (11) Unique positive equilibrium point \((\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]\) if \(0 < \beta < 1\) and \(\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)U_1 + 1) (\alpha (e^{-(k+2n)L_1} + (1 - \beta)U_1) < (k + 2n)L_2 \alpha e^{-(k+2n)L_1} - \beta U_1); (\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2] is locally asymptotically stable if \((-k + n)\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)U_1 + 1) + n\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)L_1 + 1) < (\beta + (-k + 2n)L_2)^2,
\]
| (12) Unique positive equilibrium point \((\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]\) if \(\alpha e^{-(k+2n)L_1} (\beta + 2(-k + 2n)U_1) (\beta + (-k + 2n)U_1 + 1) < \alpha e^{-(k+2n)L_2} < \alpha e^{-(k+2n)L_1} < \beta + (-k + 2n)L_2) and \alpha e^{-(k+2n)L_2} < \beta + (-k + 2n)L_1). Unique positive equilibrium point \((\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]\) if \((-k + n)\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)U_1 + 1) + n\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)L_1 + 1) < (\beta + (-k + 2n)L_2)^2,
\]
| (13) Unique positive equilibrium point \((\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]\) if \(\alpha e^{-(k+2n)L_1} (\beta + 2(-k + 2n)U_1) (\beta + (-k + 2n)U_1 + 1) < \alpha e^{-(k+2n)L_2} < \alpha e^{-(k+2n)L_1} < \beta + (-k + 2n)L_2) and \alpha e^{-(k+2n)L_2} < \beta + (-k + 2n)L_1). Unique positive equilibrium point \((\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]\) if \((-k + n)\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)U_1 + 1) + n\alpha e^{-(k+2n)L_1} (\beta + (-k + 2n)L_1 + 1) < (\beta + (-k + 2n)L_2)^2,
\]

### Conflicts of Interest
The authors declare that they have no conflicts of interest regarding the publication of this paper.

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