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Research Article

Dynamic Behaviors of a Competitive System with Beddington-DeAngelis Functional Response

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This article studies a competitive system with Beddington-DeAngelis functional response and establishes sufficient conditions on permanence, partial extinction, and the existence of a unique almost periodic solution for the system. The results supplement and generalize the main conclusions in recent literature. Numerical simulations have been presented to validate the analytical results.

1. Introduction

For a continuous bounded function $f(t)$, we define

$$f^l = \inf_{t\in R} f(t),\qquad f^u = \sup_{t\in R} f(t). \tag{1}$$

In the paper, we investigate the dynamic behaviors of the following competitive system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(r_1(t) - a_1(t)x_1(t) - \frac{b_1(t)x_2(t)}{\alpha_1(t)+\beta_1(t)x_1(t)+\gamma_1(t)x_2(t)}\right),\\ \dot{x}_2(t) &= x_2(t)\left(r_2(t) - a_2(t)x_2(t) - \frac{b_2(t)x_1(t)}{\alpha_2(t)+\beta_2(t)x_1(t)+\gamma_2(t)x_2(t)}\right),\end{aligned} \tag{2}$$

where $x_1(t)$ and $x_2(t)$ are the biomass of species x_1 and x_2 at time t, respectively. For $i = 1,2$, $r_i(t)$ are the intrinsic growth rates of species x_i; $a_i(t)$ are the rates of intraspecific competition of the first and second species; the interspecific competition between two species takes the Beddington-DeAngelis functional response type $b_1(t)x_2(t)/(\alpha_1(t) + \beta_1(t)x_1(t) + \gamma_1(t)x_2(t))$ and $b_2(t)x_1(t)/(\alpha_2(t) + \beta_2(t)x_1(t) + \gamma_2(t)x_2(t))$, respectively; $r_i(t), a_i(t), b_i(t), \alpha_i(t), \beta_i(t), \gamma_i(t)$ are all continuous and bounded functions with upper and lower positive bounds. For the biological meaning, we will consider system (2) with the following initial conditions:

$$x(0) > 0,\qquad y(0) > 0. \tag{3}$$

It is not difficult to obtain that the corresponding solution $(x_1(t), x_2(t))^T$ satisfies $x_1(t) > 0, x_2(t) > 0$ for all $t \geq 0$.

Motivated by Gopalsamy [1], Wang, Liu and Li [2] introduced the following Lotka-Volterra competitive system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(r_1(t) - a_1(t)x_1(t) - \frac{b_1(t)x_2(t)}{1+x_2(t)}\right),\\ \dot{x}_2(t) &= x_2(t)\left(r_2(t) - a_2(t)x_2(t) - \frac{b_2(t)x_1(t)}{1+x_1(t)}\right),\end{aligned} \tag{4}$$

which is a special case of system (2) with $\alpha_i(t) = \gamma_i(t) = 1$ and $\beta_i(t) = 0$ $(i = 1,2)$. Wang et al. [2] showed the existence and stability of positive almost periodic solutions of system (4). We [3] obtained partial extinction of system (4) by constructing some suitable Lyapunov type extinction functions. Liu and Wang [4] incorporated the impulsive perturbations to the system (4) and investigated the uniqueness of positive almost periodic solutions. Liu, Wu and Cheke [5] also investigated dynamic behaviour of (4) with delay, impulsive harvesting and stocking controls. Xie et al. [6] further considered the partial extinction of system (4) with one toxin producing species.

Qin et al.[7] and Wang et al.[8] both studied the discrete time version of system (4)

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{r_1(n) - a_1(n)x_1(n) - \frac{b_1(n)x_2(n)}{1+x_2(n)}\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{r_2(n) - a_2(n)x_2(n) - \frac{b_2(n)x_1(n)}{1+x_1(n)}\right\} \end{aligned} \tag{5}$$

and obtained the permanence, stability, and almost periodic solutions of the system. Yue [9] considered the partial extinction of system (5) with one toxin producing species.

Considering the interference of unpredictable forces for ecosystems in nature, Wang et al. [10] further incorporated feedback controls to system (5)

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{r_1(n) - a_1(n)x_1(n) - \frac{b_1(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n)\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{r_2(n) - a_2(n)x_2(n) - \frac{b_2(n)x_1(n)}{1+x_1(n)} - e_2(n)u_2(n)\right\}, \\ \Delta u_1(n) &= -f_1(n)u_1(n) + d_1(n)x_1(n), \\ \Delta u_2(n) &= -f_2(n)u_2(n) + d_2(n)x_2(n) \end{aligned} \tag{6}$$

and established some results on almost periodic solutions of the system. We [11, 12] investigated the effect of feedback control variables on permanence and extinction of (6).

Motivated by Gopalsamy [1], Ma, Gao, and Xie [13] investigated the following discrete two-species competitive system:

$$\begin{aligned} x_1(n+1) &= x_1(n) \cdot \exp\left\{r_1(n) - a_1(n)x_1(n) - \frac{b_1(n)x_2(n)}{1+\beta_1(n)x_1(n)}\right\}, \\ x_2(n+1) &= x_2(n) \cdot \exp\left\{r_2(n) - a_2(n)x_2(n) - \frac{b_2(n)x_1(n)}{1+\beta_2(n)x_2(n)}\right\} \end{aligned} \tag{7}$$

and obtained the almost periodic solutions of the system. However, to the best of our knowledge, there are no researches on the dynamic behaviors of continuous analogue of system (7) which is a special case of system (2) under $\gamma_i(t) = 0$ and $\alpha_i(t) = 1$ $(i = 1, 2)$.

Based on the above papers, Chen, Chen and Huang [14] proposed system (2) with the effect of toxic substances and obtained the partial extinction of system. However, authors in [14] did not study some important topics such as permanence, stability, and almost periodic solutions of the system. Hence, the goal of this paper is to obtain results on permanence, partial extinction, and the existence of a unique almost periodic solution of system (2) and (3). Our results supplement the main results of [13, 14] and generalize [2, 3]. Many important results concerned this direction; one could see [15–18] and so on.

This paper is distributed as follows: Section 2 is devoted to the results on permanence and extinction for system (2). In Section 3, we discuss the global attractivity of the system (2) and of one species under the other one is extinct. In Section 4, the uniqueness of positive almost periodic solutions of system (2) is obtained. Numerical simulations are presented to validate the analytical results in Section 5. Finally, we conclude in Section 6.

2. Permanence and Extinction

First, let us introduce the following lemma which is useful for our main result.

Lemma 1 (see [19]). *If $a > 0$, $b > 0$, and $\dot{x} \geq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t\longrightarrow+\infty} x(t) \geq \frac{b}{a}. \tag{8}$$

If $a > 0$, $b > 0$, and $\dot{x} \leq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have

$$\limsup_{t\longrightarrow+\infty} x(t) \leq \frac{b}{a}. \tag{9}$$

Theorem 2. *Suppose*

$$\begin{aligned} r_1^l\alpha_1^l &> \left(b_1^u - r_1^l\gamma_1^l\right)M_2, \\ r_2^l\alpha_2^l &> \left(b_2^u - r_2^l\gamma_2^l\right)M_1 \end{aligned} \tag{H_1}$$

holds, where $M_i = r_i^u/a_i^l$, $(i = 1, 2)$, then system (2) with initial condition (3) is permanent. That is, each positive solution $(x_1(t), x_2(t))^T$ of system (2) satisfies

$$\begin{aligned} 0 < m_1 \leq \liminf_{t\longrightarrow+\infty} x_1(t) \leq \limsup_{t\longrightarrow+\infty} x_1(t) \leq M_1, \\ 0 < m_2 \leq \liminf_{t\longrightarrow+\infty} x_2(t) \leq \limsup_{t\longrightarrow+\infty} x_2(t) \leq M_2, \end{aligned} \tag{10}$$

where $m_1 = (r_1^l\alpha_1^l - (b_1^u - r_1^l\gamma_1^l)M_2)/a_1^u(\alpha_1^l + \gamma_1^l M_2)$, $m_2 = (r_2^l\alpha_2^l - (b_2^u - r_1^l\gamma_2^l)M_1)/a_2^u(\alpha_2^l + \gamma_2^l M_1)$.

Proof. For any small enough $\varepsilon > 0$, it follows from condition (H_1) that

$$r_1^l\alpha_1^l > \left(b_1^u - r_1^l\gamma_1^l\right)\left(M_2 + \varepsilon\right),$$
$$r_2^l\alpha_2^l > \left(b_2^u - r_2^l\gamma_2^l\right)\left(M_1 + \varepsilon\right). \tag{11}$$

The first equation of system (2) yields

$$\dot{x}_1(t) \le x_1(t)\left(r_1^u - a_1^l x_1(t)\right). \tag{12}$$

By applying Lemma 1, we obtain

$$\limsup_{t\longrightarrow+\infty} x_1(t) \le \frac{r_1^u}{a_1^l} \triangleq M_1. \tag{13}$$

Analogously,

$$\limsup_{t\longrightarrow+\infty} x_2(t) \le \frac{r_2^u}{a_2^l} \triangleq M_2. \tag{14}$$

Above two inequalities imply that there exists a $T_1 > 0$ such that, for $t \ge T_1$,

$$x_1(t) \le M_1 + \varepsilon \triangleq M_{1\varepsilon},$$
$$x_2(t) \le M_2 + \varepsilon \triangleq M_{2\varepsilon}. \tag{15}$$

Hence, (15) and the second equation of (2) show that, for $t \ge T_1$,

$$\dot{x}_1(t) \ge x_1(t)\left(r_1^l - a_1^u x_1(t) - \frac{b_1^u M_{2\varepsilon}}{\alpha_1^l + \gamma_1^l M_{2\varepsilon}}\right). \tag{16}$$

From (16), according to (11) and Lemma 1, one can get

$$\liminf_{t\longrightarrow+\infty} x_1(t) \ge \frac{r_1^l\alpha_1^l - \left(b_1^u - r_1^l\gamma_1^l\right)M_{2\varepsilon}}{a_1^u\left(\alpha_1^l + \gamma_1^l M_{2\varepsilon}\right)}. \tag{17}$$

Similarly, for $t \ge T_1$, we can easily obtain

$$\liminf_{t\longrightarrow+\infty} x_2(t) \ge \frac{r_2^l\alpha_2^l - \left(b_2^u - r_1^l\gamma_2^l\right)M_{1\varepsilon}}{a_2^u\left(\alpha_2^l + \gamma_2^l M_{1\varepsilon}\right)}. \tag{18}$$

Setting $\varepsilon \longrightarrow 0$, one can get

$$\liminf_{t\longrightarrow+\infty} x_1(t) \ge \frac{r_1^l\alpha_1^l - \left(b_1^u - r_1^l\gamma_1^l\right)M_2}{a_1^u\left(\alpha_1^l + \gamma_1^l M_2\right)} \triangleq m_1,$$
$$\liminf_{t\longrightarrow+\infty} x_2(t) \ge \frac{r_2^l\alpha_2^l - \left(b_2^u - r_1^l\gamma_2^l\right)M_1}{a_2^u\left(\alpha_2^l + \gamma_2^l M_1\right)} \triangleq m_2. \tag{19}$$

Equations (12), (13), and (19) show that system (2) is permanent. □

Theorem 3. *Assume*

$$\frac{r_2^u}{r_1^l} < \min\left\{\frac{b_2^l}{a_1^u\left[\alpha_2^u + \beta_2^u M_1 + \gamma_2^u M_2\right]}, \frac{a_2^l\alpha_1^l}{b_1^u}\right\}, \tag{H_2}$$

holds, where M_i is defined in Theorem 2, then for each positive solution $(x_1(t), x_2(t))^T$ of system (2), $\lim_{t\longrightarrow+\infty} x_2(t) = 0$.

Proof. (H_2) implies that we can choose $\varepsilon_1 > 0$ small enough and two positive constants p, q such that

$$\frac{r_2^u}{r_1^l} < \min\left\{\frac{b_2^l}{a_1^u\left[\alpha_2^u + \beta_2^u\left(M_1 + \varepsilon_1\right) + \gamma_2^u\left(M_2 + \varepsilon_1\right)\right]}, \frac{a_2^l\alpha_1^l}{b_1^u}\right\}, \tag{20}$$

and

$$\frac{r_2^u}{r_1^l} < \frac{p}{q} < \min\left\{\frac{b_2^l}{a_1^u\left[\alpha_2^u + \beta_2^u\left(M_1 + \varepsilon_1\right) + \gamma_2^u\left(M_2 + \varepsilon_1\right)\right]}, \frac{a_2^l\alpha_1^l}{b_1^u}\right\}. \tag{21}$$

Thus,

$$-pr_1^l + qr_2^u \overset{\text{def}}{=} -d_1 < 0,$$
$$pa_1^u - \frac{qb_2^l}{\alpha_2^u + \beta_2^u\left(M_1 + \varepsilon_1\right) + \gamma_2^u\left(M_2 + \varepsilon_1\right)} < 0, \tag{22}$$
$$\frac{pb_1^u}{\alpha_1^l} - qa_2^l < 0.$$

For above ε_1, Theorem 2 shows there exists $T_2 > 0$ and for $t \ge T_2$,

$$x_i(t) < M_i + \varepsilon_1, \quad i = 1, 2. \tag{23}$$

It follows from system (2) that

$$\frac{\dot{x}_1(t)}{x_1(t)} = r_1(t) - a_1(t)x_1(t) - \frac{b_1(t)x_2(t)}{\alpha_1(t) + \beta_1(t)x_1(t) + \gamma_1(t)x_2(t)},$$
$$\frac{\dot{x}_2(t)}{x_2(t)} = r_2(t) - a_2(t)x_2(t) - \frac{b_2(t)x_1(t)}{\alpha_2(t) + \beta_2(t)x_1(t) + \gamma_2(t)x_2(t)}. \tag{24}$$

Consider the following Lyapunov type extinction function

$$V(t) = x_1^{-p}(t)\,x_2^{q}(t). \tag{25}$$

For $t \geq T_2$, from (22)-(24), we have

$$\begin{aligned}\dot{V}(t) &= V(t)\Bigg\{\left[-pr_1(t)+qr_2(t)\right] \\ &+\left[pa_1(t)-\frac{qb_2(t)}{\alpha_2(t)+\beta_2(t)x_1(t)+\gamma_2(t)x_2(t)}\right] \\ &\cdot x_1(t) \\ &+\left[\frac{pb_1(t)}{\alpha_1(t)+\beta_1(t)x_1(t)+\gamma_1(t)x_2(t)}-qa_2(t)\right] \\ &\cdot x_2(t)\Bigg\} \leq V(t)\Bigg\{\left[-pr_1^l+qr_2^u\right] \\ &+\left[pa_1^u-\frac{qb_2^l}{\alpha_2^u+\beta_2^u(M_1+\varepsilon_1)+\gamma_2^u(M_2+\varepsilon_1)}\right] \\ &\cdot x_1(t)+\left[\frac{pb_1^u}{\alpha_1^l}-qa_2^l\right]x_2(t)\Bigg\} \leq -d_1V(t)<0.\end{aligned} \tag{26}$$

Integrating the above inequality from T_2 to t ($t \geq T_2$), one has

$$V(t) \leq V(T_2)\exp\left(-d_1(t-T_2)\right). \tag{27}$$

Choose $M = 2\max\{M_1, M_2\}$, then (23) shows that $x_i(t) < M$, for $t \geq T_2$. Hence, (27) implies that

$$\begin{aligned}&x_2(t) \\ &< M^{p/q}(x_1(T_2))^{-p/q}x_2(T_2)\exp\left(-\frac{d_1}{q}(t-T_2)\right),\end{aligned} \tag{28}$$

and consequently, $\lim_{t\longrightarrow+\infty}x_2(t)=0$. □

Theorem 4. *Assume*

$$\frac{r_2^l}{r_1^u} > \max\left\{\frac{b_2^u}{\alpha_2^l a_1^l}, \frac{a_2^u\left[\alpha_1^u+\beta_1^u M_1+\gamma_1^u M_2\right]}{b_1^l}\right\} \tag{H_3}$$

holds, where M_i is defined in Theorem 2, then for each positive solution $(x_1(t), x_2(t))^T$ of system (2), $\lim_{t\longrightarrow+\infty}x_1(t)=0$.

Proof. Due to (H_3), there exist three positive constants ε_2, p, and q such that

$$\begin{aligned}\frac{r_2^l}{r_1^u} &> \max\Bigg\{\frac{b_2^u}{\alpha_2^l a_1^l}, \\ &\frac{a_2^u\left[\alpha_1^u+\beta_1^u(M_1+\varepsilon_2)+\gamma_1^u(M_2+\varepsilon_2)\right]}{b_1^l}\Bigg\}.\end{aligned} \tag{29}$$

and

$$\begin{aligned}\frac{r_2^l}{r_1^u} > \frac{p}{q} &> \max\Bigg\{\frac{b_2^u}{\alpha_2^l a_1^l}, \\ &\frac{a_2^u\left[\alpha_1^u+\beta_1^u(M_1+\varepsilon_2)+\gamma_1^u(M_2+\varepsilon_2)\right]}{b_1^l}\Bigg\}.\end{aligned} \tag{30}$$

Thus,

$$\begin{aligned}&pr_1^u-qr_2^l \overset{\text{def}}{=} -d_2<0, \\ &-pa_1^l+\frac{qb_2^u}{\alpha_2^l}<0, \\ &-\frac{pb_1^l}{\alpha_1^u+\beta_1^u(M_1+\varepsilon_2)+\gamma_1^u(M_2+\varepsilon_2)}+qa_2^u<0.\end{aligned} \tag{31}$$

Consider the following Lyapunov type extinction function:

$$V(t) = x_1^p(t)x_2^{-q}(t). \tag{32}$$

For $t \geq T_2$, it follows from (23)-(24) and (31) that

$$\begin{aligned}\dot{V}(t) &= V(t)\Bigg\{\left[pr_1(t)-qr_2(t)\right] \\ &+\left[-pa_1(t)+\frac{qb_2(t)}{\alpha_2(t)+\beta_2(t)x_1(t)+\gamma_2(t)x_2(t)}\right] \\ &\cdot x_1(t) \\ &+\left[-\frac{pb_1(t)}{\alpha_1(t)+\beta_1(t)x_1(t)+\gamma_1(t)x_2(t)}+\beta a_2(t)\right] \\ &\cdot x_2(t)\Bigg\} \leq V(t)\Bigg\{\left[pr_1^u-\beta r_2^l\right]+\left[-pa_1^l+\frac{qb_2^u}{\alpha_2^l}\right] \\ &\cdot x_1(t) \\ &+\left[-\frac{pb_1^l}{\alpha_1^u+\beta_1^u(M_1+\varepsilon_2)+\gamma_1^u(M_2+\varepsilon_2)}+qa_2^u\right] \\ &\cdot x_2(t)\Bigg\} \leq -d_2V(t)<0.\end{aligned} \tag{33}$$

Similarly to the analysis in Theorem 3, one can get $\lim_{t\longrightarrow+\infty}x_1(t)=0$. □

3. Stability

We will derive the global attractivity of the system and of one species under the other one is extinct in this section. Firstly, we introduce some useful lemmas firstly.

Lemma 5 (see [20], fluctuation lemma). *Let $x(t)$ be a bounded differentiable function on $[\alpha, \infty)$. Then there exist sequences $\tau_n \longrightarrow \infty$ and $\sigma_n \longrightarrow \infty$ such that*

(i) *$\dot{x}(\tau_n) \longrightarrow 0$ and $x(\tau_n) \longrightarrow \limsup_{t\longrightarrow\infty}x(t) = \overline{x}$ as $n \longrightarrow \infty$;*

(ii) *$\dot{x}(\sigma_n) \longrightarrow 0$ and $x(\sigma_n) \longrightarrow \liminf_{t\longrightarrow\infty}x(t) = \underline{x}$ as $n \longrightarrow \infty$.*

According to Lemma 2.1 in Zhao et al.[21], one can get

Lemma 6. *Suppose that $r_1(t)$ and $a_1(t)$ are continuous functions bounded above and below by positive constants, then any positive solutions of the following equation*

$$\dot{x}(t) = x(t)\left(r_1(t) - a_1(t)x(t)\right) \tag{34}$$

are defined on $[0, +\infty)$, *bounded above and below by positive constants and globally attractive.*

Theorem 7. *Assume* (H_1) *and*

$$\begin{gathered}\left[a_1(t) - \frac{b_1(t)\beta_1(t)M_2}{\Delta_1^2(t,m_1,m_2)} - \frac{b_2(t)\alpha_2(t)}{\Delta_2^2(t,m_1,m_2)}\right.\\ \left.-\frac{b_2(t)\gamma_2(t)M_2}{\Delta_2^2(t,m_1,m_2)}\right]^l > 0,\end{gathered} \tag{H_4}$$

$$\begin{gathered}\left[a_2(t) - \frac{b_1(t)\beta_1(t)M_1}{\Delta_1^2(t,m_1,m_2)} - \frac{b_1(t)\alpha_1(t)}{\Delta_1^2(t,m_1,m_2)}\right.\\ \left.-\frac{b_2(t)\gamma_2(t)M_1)}{\Delta_2^2(t,m_1,m_2)}\right]^l > 0\end{gathered} \tag{H_5}$$

hold, where m_i *and* M_i $(i = 1, 2)$ *are defined in Theorem 2 and*

$$\Delta_i(t, x(t), y(t)) = \alpha_i(t) + \beta_i(t)x(t) + \gamma_i(t)y(t), \quad i = 1, 2. \tag{35}$$

Let $z_1(t) = (x_1(t), x_2(t))^T$, $z_2(t) = (y_1(t), y_2(t))^T$ *be any two positive solutions of system (2), then*

$$\begin{gathered}\lim_{t\to+\infty}\left|x_1(t) - y_1(t)\right| = 0,\\ \lim_{t\to+\infty}\left|x_2(t) - y_2(t)\right| = 0.\end{gathered} \tag{36}$$

Proof. According to (H_4), (H_5), and Theorem 2 that there exist $\varepsilon_3 > 0$ and $T_3 > T_1$, when $t \geq T_3$,

$$\begin{gathered}\left[a_1(t) - \frac{b_1(t)\beta_1(t)(M_2+\varepsilon_3)}{\Delta_1^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\right.\\ -\frac{b_2(t)\alpha_2(t)}{\Delta_2^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\\ \left.-\frac{b_2(t)\gamma_2(t)(M_2+\varepsilon_3)}{\Delta_2^2(t,m_1-\varepsilon,m_2-\varepsilon_3)}\right]^l > \varepsilon_3,\\ \left[a_2(t) - \frac{b_1(t)\beta_1(t)(M_1+\varepsilon_3)}{\Delta_1^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\right.\\ -\frac{b_1(t)\alpha_1(t)}{\Delta_1^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\\ \left.-\frac{b_2(t)\gamma_2(t)(M_1+\varepsilon_3)}{\Delta_2^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\right]^l > \varepsilon_3\end{gathered} \tag{37}$$

and

$$\begin{gathered}m_1 - \varepsilon_3 \leq x_1(t) \leq M_1 + \varepsilon_3,\\ m_2 - \varepsilon_3 \leq x_2(t) \leq M_2 + \varepsilon_3,\\ m_1 - \varepsilon_3 \leq y_1(t) \leq M_1 + \varepsilon_3,\\ m_2 - \varepsilon_3 \leq y_2(t) \leq M_2 + \varepsilon_3.\end{gathered} \tag{38}$$

Set $V(t) = V_1(t) + V_2(t)$, where $V_i(t) = |\ln x_i(t) - \ln y_i(t)|$, $i = 1, 2$. Calculation of the upper right derivatives for $V_1(t)$ and $V_2(t)$ along the solution of system (2) yields

$$\begin{aligned}D^+V_1(t) &= \operatorname{sgn}\left(x_1(t) - y_1(t)\right)\left[-a_1(t)\left(x_1(t) - y_1(t)\right) - b_1(t)\left(\frac{x_2(t)}{\Delta_1(t,x_1(t),x_2(t))} - \frac{y_2(t)}{\Delta_1(t,y_1(t),y_2(t))}\right)\right]\\ &= -a_1(t)\left|x_1(t) - y_1(t)\right| - \operatorname{sgn}\left(x_1(t) - y_1(t)\right)b_1(t)\left[\frac{\alpha_1(t)\left(x_2(t) - y_2(t)\right) + \beta_1(t)y_1(t)x_2(t) - \beta_1(t)x_1(t)y_2(t)}{\Delta_1(t,x_1(t),x_2(t))\cdot\Delta_1(t,y_1(t),y_2(t))}\right]\\ &= -a_1(t)\left|x_1(t) - y_1(t)\right| - \operatorname{sgn}\left(x_1(t) - y_1(t)\right)b_1(t)\\ &\quad\cdot\left[\frac{\alpha_1(t)\left(x_2(t) - y_2(t)\right)}{\Delta_1(t,x_1(t),x_2(t))\cdot\Delta_1(t,y_1(t),y_2(t))} + \frac{\beta_1(t)y_1(t)\left(x_2(t) - y_2(t)\right) + \beta_1(t)y_2(t)\left(y_1(t) - x_1(t)\right)}{\Delta_1(t,x_1(t),x_2(t))\cdot\Delta_1(t,y_1(t),y_2(t))}\right] \leq -a_1(t)\\ &\quad\cdot\left|x_1(t) - y_1(t)\right| + b_1(t)\\ &\quad\cdot\left[\frac{\alpha_1(t)\left|x_2(t) - y_2(t)\right|}{\Delta_1(t,x_1(t),x_2(t))\cdot\Delta_1(t,y_1(t),y_2(t))} + \frac{\beta_1(t)y_1(t)\left|x_2(t) - y_2(t)\right| + \beta_1(t)y_2(t)\left|x_1(t) - y_1(t)\right|}{\Delta_1(t,y_1(t),y_2(t))\cdot\Delta_1(t,x_1(t),x_2(t))}\right],\end{aligned} \tag{39}$$

and

$$\begin{aligned}D^+V_2(t) &\leq -a_2(t)\left|x_2(t) - y_2(t)\right| + b_2(t)\\ &\quad\cdot\left[\frac{\alpha_2(t)\left|x_1(t) - y_1(t)\right|}{\Delta_2(t,x_1(t),x_2(t))\cdot\Delta_2(t,y_1(t),y_2(t))}\right.\\ &\quad\left.+\frac{\gamma_2(t)y_2(t)\left|x_1(t) - y_1(t)\right| + \gamma_2(t)y_1(t)\left|x_2(t) - y_2(t)\right|}{\Delta_2(t,y_1(t),y_2(t))\cdot\Delta_2(t,x_1(t),x_2(t))}\right].\end{aligned} \tag{40}$$

Thus, by (37), one can obtain that, for $t \geq T_3$,

$$D^+V(t) \leq -\left[a_1(t) - \frac{b_1(t)\beta_1(t)y_2(t)}{\Delta_1(t,y_1(t),y_2(t))\cdot\Delta_1(t,x_1(t),x_2(t))}\right.$$

$$
\begin{aligned}
&-\frac{b_2(t)\alpha_2(t)}{\Delta_2(t,x_1(t),x_2(t))\cdot\Delta_2(t,y_1(t),y_2(t))} \\
&-\left.\frac{b_2(t)\gamma_2(t)y_2(t)}{\Delta_2(t,y_1(t),y_2(t))\cdot\Delta_2(t,x_1(t),x_2(t))}\right]|x_1(t) \\
&-y_1(t)|-\left[a_2(t)\right. \\
&-\frac{b_1(t)\beta_1(t)y_1(t)}{\Delta_1(t,y_1(t),y_2(t))\cdot\Delta_1(t,x_1(t),x_2(t))} \\
&-\frac{b_1(t)\alpha_1(t)}{\Delta_1(t,y_1(t),y_2(t))\cdot\Delta_1(t,x_1(t),x_2(t))} \\
&-\left.\frac{b_2(t)\gamma_2(t)y_1(t)}{\Delta_2(t,y_1(t),y_2(t))\cdot\Delta_2(t,x_1(t),x_2(t))}\right] \\
&\cdot|x_2(t)-y_2(t)|\le-\left[a_1(t)\right. \\
&-\frac{b_1(t)\beta_1(t)(M_2+\varepsilon_3)}{\Delta_1^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)} \\
&-\frac{b_2(t)\alpha_2(t)}{\Delta_2^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)} \\
&-\left.\frac{b_2(t)\gamma_2(t)(M_2+\varepsilon_3)}{\Delta_2^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\right]|x_1(t)-y_1(t)| \\
&-\left[a_2(t)-\frac{b_1(t)\beta_1(t)(M_1+\varepsilon_3)}{\Delta_1^2(t,m_1-\varepsilon,m_2-\varepsilon_3)}\right. \\
&-\frac{b_1(t)\alpha_1(t)}{\Delta_1^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)} \\
&-\left.\frac{b_2(t)\gamma_2(t)(M_1+\varepsilon_3)}{\Delta_2^2(t,m_1-\varepsilon_3,m_2-\varepsilon_3)}\right]|x_2(t)-y_2(t)|. \\
&\le-\varepsilon_3|x_1(t)-y_1(t)|-\varepsilon_3|x_2(t)-y_2(t)|.
\end{aligned}
\tag{41}
$$

Consequently, $V(t)$ is nonincreasing on $[T_3,+\infty)$. Integrating inequality (41) from T_3 to t, we get

$$
\begin{aligned}
&V(t)+\varepsilon_3\int_{T_3}^{t}|x_1(s)-y_1(s)|\,ds \\
&\quad+\varepsilon_3\int_{T_3}^{t}|x_2(s)-y_2(s)|\,ds<V(T_3)<+\infty, \\
&\qquad t\ge T_3.
\end{aligned}
\tag{42}
$$

Then

$$
\begin{aligned}
&\int_{T_3}^{t}|x_1(s)-y_1(s)|\,ds<\frac{V(T_3)}{\varepsilon_3}<+\infty, \\
&\int_{T_3}^{t}|x_2(s)-y_2(s)|\,ds<\frac{V(T_3)}{\varepsilon}<+\infty,
\end{aligned}
\tag{43}
$$

and so $|x_1(t)-y_1(t)|, |x_2(t)-y_2(t)| \in L^1[T_3,+\infty)$. Moreover, Theorem 2 and system (2) show that both $x_1(t), y_1(t), x_2(t), y_2(t)$ and their derivatives are bounded on $[T_3,+\infty)$, and therefor $|x_1(t)-y_1(t)|$ and $|x_2(t)-y_2(t)|$ are uniformly continuous on $[T_3,+\infty)$. Using Barbălat's Lemma [22], one has

$$
\begin{aligned}
&\lim_{t\longrightarrow+\infty}|x_1(t)-y_1(t)|=0, \\
&\lim_{t\longrightarrow+\infty}|x_2(t)-y_2(t)|=0.
\end{aligned}
\tag{44}
$$

□

Theorem 8. *Assume* (H_2) *holds, then*

$$
\begin{aligned}
&\lim_{t\longrightarrow+\infty}x_2(t)=0 \\
&\text{and }\lim_{t\longrightarrow+\infty}\left(x_1(t)-x_1^*(t)\right)=0,
\end{aligned}
\tag{45}
$$

where $(x_1(t),x_2(t))^T$ *and* $x_1^*(t)$ *are any two positive solutions of system (2) and (34), respectively.*

Proof. We first show that if (H_2) holds, then

$$
A_1\le\liminf_{t\longrightarrow+\infty}x_1(t)\le\limsup_{t\longrightarrow+\infty}x_1(t)\le M_1,\tag{46}
$$

where $M_1=r_1^u/a_1^l$ and $A_1=r_1^l/a_1^u$.

By (13) and Theorem 3, one can obtain

$$
\begin{aligned}
&\limsup_{t\longrightarrow+\infty}x_1(t)\le M_1, \\
&\lim_{t\longrightarrow+\infty}x_2(t)=0.
\end{aligned}
\tag{47}
$$

Since $r_1^l>0$, one can choose a small enough $\varepsilon_4>0$ such that

$$
A_\varepsilon\overset{\text{def}}{=}r_1^l-\frac{b_1^u}{\alpha_1^l}\varepsilon_4>0.\tag{48}
$$

For above ε_4, from (47), there exists a $T_4>T_3$ such that for $t\ge T_4$,

$$
\begin{aligned}
&x_1(t)\le M_1+\varepsilon_4, \\
&x_2(t)\le\varepsilon_4,
\end{aligned}
\tag{49}
$$

and so

$$
\dot{x}_1(t)\ge x_1(t)\left\{r_1^l-a_1^u x_1(t)-\frac{b_1^u}{\alpha_1^l}\varepsilon_4\right\}.\tag{50}
$$

In view of (48) and (50), it derives from Lemma 1 that $\liminf_{t\longrightarrow+\infty}x_1(t)\ge A_\varepsilon/a_1^u$. Setting $\varepsilon_4\longrightarrow0$ in the above inequality, we have $\liminf_{t\longrightarrow+\infty} x_1(t) \geq r_1^l/a_1^u$ and (46) holds. It follows from (46) and Lemma 6 that there exist three positive constants $T_5 > T_4$, η_1, and η_2 such that $A_1/2 < x_1(t) < 3M_1/2$ and $\eta_1 < x_1^*(t) < \eta_2$, for $t \geq T_5$. Set $v(t) = 1/x_1(t)$, $v^*(t) = 1/x_1^*(t)$, and $z(t) = v(t) - v^*(t)$, then $2/3M_1 < v(t) < 2/A_1$, $1/\eta_2 < v^*(t) < 1/\eta_1$, and

$$\begin{aligned}\dot{v}(t) &= -r_1(t)v(t) + a_1(t) \\ &\quad + \frac{b_1(t)x_2(t)v^2(t)}{\alpha_1(t)v(t) + \beta_1(t) + \gamma_1(t)x_2(t)v(t)}, \\ \dot{v}^*(t) &= -r_1(t)v^*(t) + a_1(t).\end{aligned} \tag{51}$$

Hence,

$$\begin{aligned}\dot{z}(t) &= -r_1(t)z(t) \\ &\quad + \frac{b_1(t)x_2(t)v^2(t)}{\alpha_1(t)v(t) + \beta_1(t) + \gamma_1(t)x_2(t)v(t)},\end{aligned} \tag{52}$$

or

$$\begin{aligned}z(t) &= -\frac{\dot{z}(t)}{r_1(t)} \\ &\quad + \frac{b_1(t)x_2(t)v^2(t)}{r_1(t)\left[\alpha_1(t)v(t) + \beta_1(t) + \gamma_1(t)x_2(t)v(t)\right]}.\end{aligned} \tag{53}$$

So $z(t)$ is bounded and differentiable. By Lemma 5, there exist sequences $\tau_n \longrightarrow \infty$ and $\sigma_n \longrightarrow \infty$ satisfying $\dot{z}(\tau_n) \longrightarrow 0$, $z(\tau_n) \longrightarrow \overline{z}$; $\dot{z}(\sigma_n) \longrightarrow 0$, $z(\sigma_n) \longrightarrow \underline{z}$ as $n \longrightarrow \infty$. Noting

$$\begin{aligned}0 &< \frac{b_1(t)x_2(t)w^2(t)}{r_1(t)\left[\alpha_1(t)w(t) + \beta_1(t) + \gamma_1(t)x_2(t)w(t)\right]} \\ &< \frac{4b_1^u x_2(t)}{r_1^l\left[2\alpha_1^l A_1 + \beta_1^l A_1^2 + 2\gamma_1^l A_1 x_2(t)\right]},\end{aligned} \tag{54}$$

and

$$0 < \frac{1}{r_1(t)} < \frac{1}{r_1^l}, \tag{55}$$

according to $\lim_{t\longrightarrow+\infty} x_2(t) = 0$ and (53)

$$\begin{aligned}\lim_{n\longrightarrow\infty} z(\tau_n) &= -\lim_{n\longrightarrow\infty}\frac{\dot{z}(\tau_n)}{r_1(\tau_n)} + \lim_{n\longrightarrow\infty} \\ &\quad \cdot \frac{b_1(\tau_n)x_2(\tau_n)w^2(\tau_n)}{r_1(\tau_n)\left[\alpha_1(\tau_n)w(\tau_n) + \beta_1(\tau_n) + \gamma_1(\tau_n)x_2(\tau_n)w(\tau_n)\right]}; \\ \lim_{n\longrightarrow\infty} z(\sigma_n) &= -\lim_{n\longrightarrow\infty}\frac{\dot{z}(\sigma_n)}{r_1(\sigma_n)} + \lim_{n\longrightarrow\infty} \\ &\quad \cdot \frac{b_1(\sigma_n)x_2(\sigma_n)w^2(\sigma_n)}{r_1(\sigma_n)\left[\alpha_1(\sigma_n)w(\sigma_n) + \beta_1(\sigma_n) + \gamma_1(\sigma_n)x_2(\sigma_n)w(\sigma_n)\right]}.\end{aligned} \tag{56}$$

Hence $\overline{z} = \underline{z} = 0$ that is $\lim_{t\longrightarrow+\infty} z(t) = 0$. Moreover, since $x_1(t), x_1^*(t)$ are bounded functions and $|x_1(t) - x_1^*(t)| = |v(t) - v^*(t)|x_1(t)x_1^*(t) = |z(t)|x_1(t)x_1^*(t)$, we have $\lim_{t\longrightarrow+\infty}(x_1(t) - x_1^*(t)) = 0$. □

Theorem 9. *Suppose (H_3) holds, then*

$$\begin{aligned}&\lim_{t\longrightarrow+\infty} x_1(t) = 0 \\ &\text{and} \lim_{t\longrightarrow+\infty}\left(x_2(t) - x_2^*(t)\right) = 0,\end{aligned} \tag{57}$$

where $(x_1(t), x_2(t))^T$ and $x_1^*(t)$ are any two positive solutions of system (2) and equation $\dot{x}(t) = (r_2(t) - a_2(t)x(t))$, respectively.

Proof. Similar arguments as the proof of Theorem 8 can show Theorem 9. We omit the details here. □

4. Existence of a Unique Almost Periodic Solution

Now, we are in a position to show the existence of a unique almost periodic solution of system (2) under the assumption that $r_i(t), a_i(t), b_i(t), \alpha_i(t), \beta_i(t), \gamma_i(t)$ $(i = 1, 2)$ are all continuous almost periodic functions with upper and lower positive bounds. For information on almost periodic functions, one can refer to [23, 24].

Let $S(E)$ be the set of all solutions $(x_1(t), x_2(t))^T$ of system (2) with $m_1 \leq x_1(t) \leq M_1$, $m_2 \leq x_2(t) \leq M_2$ for $t \in R$.

Lemma 10. *$S(E) \neq \emptyset$.*

Proof. As $r_i(t), a_i(t), b_i(t), \alpha_i(t), \beta_i(t), \gamma_i(t)$ $(i = 1, 2)$ are almost periodic functions, there exists a sequence $\{t_n\}$ satisfying $t_n \longrightarrow \infty$ and

$$\begin{aligned}r_i(t + t_n) &\longrightarrow r_i(t), \\ a_i(t + t_n) &\longrightarrow a_i(t), \\ b_i(t + t_n) &\longrightarrow b_i(t), \\ \alpha_i(t + t_n) &\longrightarrow \alpha_i(t), \\ \beta_i(t + t_n) &\longrightarrow \beta_i(t), \\ \gamma_i(t + t_n) &\longrightarrow \gamma_i(t) \\ &\qquad (i = 1, 2),\end{aligned} \tag{58}$$

uniformly on R as $n \longrightarrow \infty$. Set $(x_1(t), x_2(t))^T$ to be a solution of system (2) with $m_1 \leq x_1(t) \leq M_1$, $m_2 \leq x_2(t) \leq M_2$ for $t > T$. Then $x_1(t+t_n)$ and $x_2(t+t_n)$ are clearly uniformly bounded and equicontinuous on each bounded subset of R. According to Ascoli-Arzela theorem, one can get a subsequence of $\{t_n\}$ (still denote as $\{t_n\}$), such that $x_1(t + t_n) \longrightarrow p_1(t)$, $x_2(t + t_n) \longrightarrow p_2(t)$ uniformly on each bounded subset of R as $n \longrightarrow \infty$. Choose any $T_6 \in R$ such that $t_n + T_6 \geq T$ for all n; hence for $t \geq 0$, we have

$$x_1(t+t_n+T_6)-x_1(t_n+T_6) = \int_{T_6}^{t+T_6} x_1(s+t_n)\left(r_1(s+t_n)-a_1(s+t_n)x_1(s+t_n)-\frac{b_1(s+t_n)x_2(s+t_n)}{\alpha_1(s+t_n)+\beta_1(s+t_n)x_1(s+t_n)+\gamma_1(s+t_n)x_2(s+t_n)}\right)ds \tag{59}$$

and

$$x_2(t+t_n+T_6)-x_2(t_n+T_6) = \int_{T_6}^{t+T_6} x_2(s+t_n)\left(r_2(s+t_n)-a_2(s+t_n)x_1(s+t_n)-\frac{b_2(s+t_n)x_1(s+t_n)}{\alpha_2(s+t_n)+\beta_2(s+t_n)x_1(s+t_n)+\gamma_2(s+t_n)x_2(s+t_n)}\right)ds \tag{60}$$

Employing Lebesgue's dominated convergence theorem and taking limits in above equations lead to

$$\begin{aligned} p_1(t+T_6)-p_1(T_6) &= \int_{T_6}^{t+T_6} p_1(s)\Bigg(r_1(s) - a_1(s)x_1(s) - \frac{b_1(s)x_2(s)}{\alpha_1(s)+\beta_1(s)x_1(s)+\gamma_1(s)x_2(s)}\Bigg)ds, \\ p_2(t+T_6)-p_2(T_6) &= \int_{T_6}^{t+T_6} p_2(s)\Bigg(r_2(s) - a_2(s)x_2(s) - \frac{b_2(s)x_1(s)}{\alpha_2(s)+\beta_2(s)x_1(s)+\gamma_2(s)x_2(s)}\Bigg)ds. \end{aligned} \tag{61}$$

Due to the arbitrariness of T_1, $(p_1(t), p_2(t))^T$ is a solution of (2) on R. Obviously, $m_1 \le p_1(t) \le M_1$, $m_2 \le p_2(t) \le M_2$ for $t \in R$, so $(p_1(t), p_2(t))^T \in S(E)$ and Lemma 10 is tenable. □

Theorem 11. *Assume all conditions in Theorem 7 hold, then system (2) with (3) has a unique positive almost periodic solution.*

Proof. By Lemma 10, system (2) with (3) has a bounded positive solution $w(t) = (w_1(t), w_2(t))^T$ and so one can get a sequence $\{t'_k\}$ satisfying $\{t'_k\} \longrightarrow \infty$ as $k \longrightarrow \infty$ and $(w_1(t+t'_k), w_2(t+t'_k))^T$ be a solution of the following system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Bigg(r_1(t+t'_k)-a_1(t+t'_k)x_1(t) - \frac{b_1(t+t'_k)x_2(t)}{\alpha_1(t+t'_k)+\beta_1(t+t'_k)x_1(t)+\gamma_1(t+t'_k)x_2(t)}\Bigg), \\ \dot{x}_2(t) &= x_2(t)\Bigg(r_2(t+t'_k)-a_2(t+t'_k)x_2(t) - \frac{b_2(t+t'_k)x_1(t)}{\alpha_2(t+t'_k)+\beta_2(t+t'_k)x_1(t)+\gamma_2(t+t'_k)x_2(t)}\Bigg). \end{aligned} \tag{62}$$

Theorem 2 with the fact that all coefficients in system (2) are continuous, positive almost periodic functions leads to that $\{w_i(t+t'_k)\}$ and $\{\dot{u}(t+t'_k)\}$ $(i=1,2)$ are uniformly bounded; therefor $\{w_i(t+t'_k)\}$ $(i=1,2)$ are uniformly bounded and equicontinuous. Using Ascoli-Arzela theorem, one can get a uniformly convergent subsequence $\{w_i(t+t_k)\} \subseteq \{w_i(t+t'_k)\}$ with the property that, for any $\varepsilon_5 > 0$, there exists a $K(\varepsilon_5) > 0$ satisfying that when $m, k \ge K(\varepsilon_5)$,

$$|w_i(t+t_m)-w_i(t+t_k)| < \varepsilon_5, \quad i=1,2. \tag{63}$$

That is to say $w_i(t)$ $(i = 1,2)$ are asymptotically almost periodic functions, so one can find almost periodic functions $u_i(t+t_k)$ and continuous functions $v_i(t+t_k)$ $(i = 1,2)$ satisfying

$$w_i(t+t_k) = u_i(t+t_k)+v_i(t+t_k), \quad i=1,2, \tag{64}$$

where

$$\lim_{k\longrightarrow+\infty} u_i(t+t_k) = u_i(t), \qquad \lim_{k\longrightarrow+\infty} v_i(t+t_k) = 0, \qquad i=1,2, \tag{65}$$

and $u_i(t)$ $(i=1,2)$ are also almost periodic functions.

Thus,

$$\lim_{k\longrightarrow+\infty} w_i(t+t_k) = u_i(t), \quad i=1,2. \tag{66}$$

Moreover,

$$\begin{aligned}
&\lim_{k\to+\infty}\dot{w}_i(t+t_k)\\
&=\lim_{k\to+\infty}\lim_{h\to0}\frac{w_i(t+t_k+h)-w_i(t+t_k)}{h}\\
&=\lim_{h\to0}\lim_{k\to+\infty}\frac{w_i(t+t_k+h)-w_i(t+t_k)}{h}\\
&=\lim_{h\to0}\frac{u_i(t+h)-u_i(t)}{h},\quad i=1,2.
\end{aligned}\tag{67}$$

So $\dot{u}_i(t)$ $(i=1,2)$ exists.

By definition of almost periodic function, one can get a sequence $\{t_n\}$ with $\{t_n\}\longrightarrow\infty$ and

$$\begin{aligned}
&r_i(t+t_n)\longrightarrow r_i(t),\\
&a_i(t+t_n)\longrightarrow a_i(t),\\
&b_i(t+t_n)\longrightarrow b_i(t),\\
&\alpha_i(t+t_n)\longrightarrow \alpha_i(t),\\
&\beta_i(t+t_n)\longrightarrow \beta_i(t),\\
&\gamma_i(t+t_n)\longrightarrow \gamma_i(t)\\
&\qquad (i=1,2),
\end{aligned}\tag{68}$$

uniformly on R as $n\longrightarrow\infty$.

Obviously, $w_i(t+t_n)\longrightarrow u_i(t)$ $(i=1,2)$ as $n\longrightarrow\infty$, hence

$$\begin{aligned}
\dot{u}_1(t)&=\lim_{n\to+\infty}\dot{w}_1(t+t_n)=\lim_{n\to+\infty}w_1(t+t_n)\Big(r_1(t+t_n)\\
&\quad-a_1(t+t_n)w_1(t+t_n)\\
&\quad-\frac{b_1(t+t_n)w_2(t+t_n)}{\alpha_1(t+t_n)+\beta_1(t+t_n)w_1(t+t_n)+\gamma_1(t+t_n)w_2(t+t_n)}\Big)\\
&=u_1(t)\Big(r_1(t)-a_1(t)u_1(t)\\
&\quad-\frac{b_1(t)u_2(t)}{\alpha_1(t)+\beta_1(t)u_1(t)+\gamma_1(t)u_2(t)}\Big)
\end{aligned}\tag{69}$$

and

$$\begin{aligned}
\dot{u}_2(t)&=\lim_{n\to+\infty}\dot{w}_2(t+t_n)=\lim_{n\to+\infty}w_2(t+t_n)\Big(r_2(t+t_n)\\
&\quad-a_2(t+t_n)w_2(t+t_n)\\
&\quad-\frac{b_2(t+t_n)w_2(t+t_n)}{\alpha_2(t+t_n)+\beta_2(t+t_n)u_1(t+t_n)+\gamma_2(t+t_n)u_2(t+t_n)}\Big)\\
&=u_2(t)\Big(r_2(t)-a_2(t)u_2(t)\\
&\quad-\frac{b_2(t)u_2(t)}{\alpha_2(t)+\beta_2(t)u_1(t)+\gamma_2(t)u_2(t)}\Big).
\end{aligned}\tag{70}$$

These show that $(u_1(t),u_2(t))^T$ satisfies system (2) and is a positive almost periodic solution of (2). Combining with Theorem 7, we further obtain that there exists a unique almost periodic solution of system (2). □

5. Numeric Simulations

Example 1. Let us choose the following system:

$$\begin{aligned}
\dot{x}_1(t)&=x_1(t)\Bigg[10+2\cos\sqrt{7}t-\left(5+\sin\sqrt{5}t\right)x_1(t)\\
&\quad-\frac{\left(0.5+0.3\sin\sqrt{3}t\right)x_2(t)}{0.5+\left(8+\sin\sqrt{11}t\right)x_1(t)+0.6x_2(t)}\Bigg],\\
\dot{x}_2(t)&=x_2(t)\Bigg[17+3\sin\sqrt{5}t-\left(5+\cos\sqrt{2}t\right)x_2(t)\\
&\quad-\frac{\left(0.7+0.3\cos\sqrt{7}t\right)x_2(t)}{0.4+\left(10+\cos\sqrt{11}t\right)x_1(t)+0.5x_2(t)}\Bigg].
\end{aligned}\tag{71}$$

In this case, according to Theorem 2, we have

$$\begin{aligned}
M_1&=3,\\
M_2&=5,\\
m_1&\approx 1.1429,\\
m_2&\approx 1.2807.
\end{aligned}\tag{72}$$

Considering (H_1), (H_4), and (H_5), we can get

$$r_1^l\alpha_1^l-\left(b_1^u-r_1^l\gamma_1^l\right)M_2=24>0,\tag{73}$$

$$r_2^l\alpha_2^l-\left(b_2^u-r_2^l\gamma_2^l\right)M_1=23.6>0,\tag{74}$$

$$\begin{aligned}
&\left[a_1(t)-\frac{b_1(t)\beta_1(t)M_2}{\Delta_1^2(t,m_1,m_2)}-\frac{b_2(t)\alpha_2(t)}{\Delta_2^2(t,m_1,m_2)}\right.\\
&\quad\left.-\frac{b_2(t)\gamma_2(t)M_2}{\Delta_2^2(t,m_1,m_2)}\right]^l\geq a_1^l-\frac{b_1^u\beta_1^uM_2}{\left(\alpha_1^l+\beta_1^lm_1+\gamma_1^lm_2\right)^2}\\
&\quad-\frac{b_2^u\alpha_2^u}{\left(\alpha_2^l+\beta_2^lm_1+\gamma_2^lm_2\right)^2}-\frac{b_2^u\gamma_2^uM_2}{\left(\alpha_2^l+\beta_2^lm_1+\gamma_2^lm_2\right)^2}\\
&\quad\approx 3.5583>0,
\end{aligned}\tag{75}$$

$$\begin{aligned}
&\left[a_2(t)-\frac{b_1(t)\beta_1(t)M_1}{\Delta_1^2(t,m_1,m_2)}-\frac{b_1(t)\alpha_1(t)}{\Delta_1^2(t,m_1,m_2)}\right.\\
&\quad\left.+\frac{b_2(t)\gamma_2(t)M_1)}{\Delta_2^2(t,m_1,m_2)}\right]^l\geq a_2^l-\frac{b_1^u\beta_1^uM_1}{\left(\alpha_1^l+\beta_1^lm_1+\gamma_1^lm_2\right)^2}\\
&\quad-\frac{b_1^u\alpha_1^u}{\left(\alpha_1^l+\beta_1^lm_1+\gamma_1^lm_2\right)^2}-\frac{b_2^u\gamma_2^uM_1}{\left(\alpha_2^l+\beta_2^lm_1+\gamma_2^lm_2\right)^2}\\
&\quad\approx 3.7322>0,
\end{aligned}\tag{76}$$

Equations (73)-(76) mean that (H_1), (H_4), and (H_5) hold, so system (71) has a unique positive almost periodic solution by Theorem 11 which is also verified in Figure 1.

Example 2. Let us choose the following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\Bigg[6+\cos\sqrt{7}t - \left(0.5+0.3\sin\sqrt{5}t\right)x_1(t) - \frac{\left(0.5+0.3\sin\sqrt{3}t\right)x_2(t)}{8+\left(8+\sin\sqrt{11}t\right)x_1(t)+0.6x_2(t)}\Bigg],\\ \dot{x}_2(t) &= x_2(t)\Bigg[0.2+0.1\sin\sqrt{5}t-3x_2(t) - \frac{\left(7+\cos\sqrt{7}t\right)x_2(t)}{0.4+\left(0.6+0.1\cos\sqrt{11}t\right)x_1(t)+0.1x_2(t)}\Bigg].\end{aligned} \tag{77}$$

In this case, according to Theorem 3, we can get $M_1 = 35$, $M_2 = 0.1$, and

$$\frac{r_2^u}{r_1^l} - \frac{b_2^l}{a_1^u\left[\alpha_2^u+\beta_2^u M_1+\gamma_2^u M_2\right]} \approx -0.2411 < 0, \tag{78}$$

$$\frac{r_2^u}{r_1^l} - \frac{a_2^l\alpha_1^l}{b_1^u} = -29.94 < 0, \tag{79}$$

Equations (78)-(79) show that (H_2) holds and according to Theorem 8, for system (77), species x_2 is driven to extinction while species x_1 is asymptotic to any positive solution of

$$\dot{x}(t) = x(t)\left[6+\cos\sqrt{7}t-\left(0.5+0.3\sin\sqrt{5}t\right)x(t)\right]. \tag{80}$$

Figure 2 also supports this conclusion.

Example 3. Let us choose the following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\Bigg[0.6+0.1\cos\sqrt{7}t - \left(6+\sin\sqrt{5}t\right)x_1(t) - \frac{\left(9+3\sin\sqrt{3}t\right)x_2(t)}{8+\left(8+\sin\sqrt{11}t\right)x_1(t)+0.6x_2(t)}\Bigg],\\ \dot{x}_2(t) &= x_2(t)\Bigg[5+\sin\sqrt{5}t-0.3x_2(t) - \frac{\left(0.5+0.3\sin\sqrt{3}t\right)x_2(t)}{4+\left(0.6+0.1\cos\sqrt{11}t\right)x_1(t)+0.1x_2(t)}\Bigg].\end{aligned} \tag{81}$$

In this case, according to Theorem 4, we can get $M_1 = 0.14$, $M_2 = 20$, and

$$\frac{r_2^l}{r_1^u} - \frac{a_2^u\left[\alpha_1^u+\beta_1^u M_1+\gamma_1^u M_2\right]}{b_1^l} \approx 4.6513 > 0, \tag{82}$$

$$\frac{r_2^l}{r_1^u} - \frac{b_2^u}{\alpha_2^l a_1^l} = 5.6743 > 0, \tag{83}$$

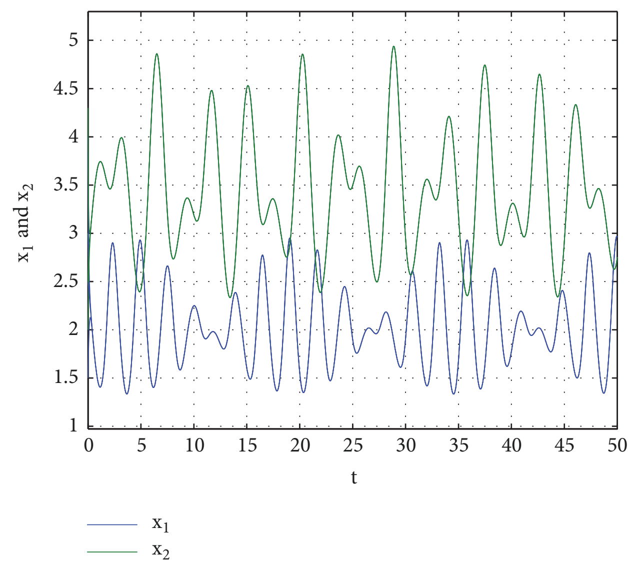

FIGURE 1: Numeric simulations of system (71) with initial values $(x(0), y(0)) = (3,3)^T, (4,2)^T, (2.5,4.3)^T$, and $(1.5,2.5)^T$, respectively.

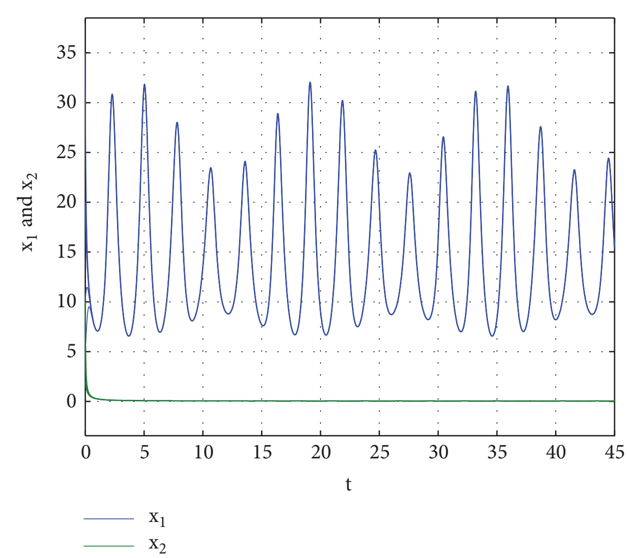

FIGURE 2: Numeric simulations of system (77) with initial values $(x(0), y(0)) = (20,10)^T, (5,5)^T, (35,2)^T$, and $(10,7)^T$, respectively.

Equations (82)-(83) show that (H_3) holds and according to Theorem 9, for system (81), species x_1 is driven to extinction while species x_2 is asymptotic to any positive solution of

$$\dot{x}(t) = x(t)\left[5+\sin\sqrt{5}t-0.3x(t)\right]. \tag{84}$$

Figure 3 also supports this conclusion.

6. Conclusion

We study a competitive system with Beddington-DeAngelis functional response in this paper and obtain sufficient conditions on permanence, partial extinction, and the existence of a unique almost periodic solution for the system. When

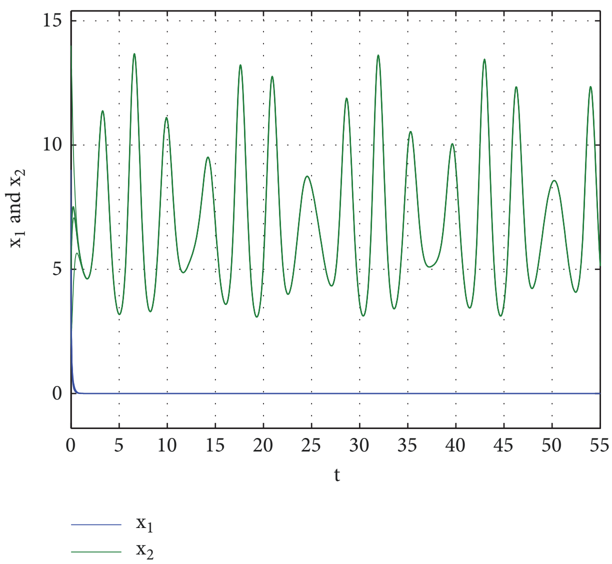

FIGURE 3: Numeric simulations of system (81) with initial values $(x(0), y(0)) = (3,14)^T$, $(8,5)^T$, $(9,2)^T$, and $(5,6)^T$, respectively.

$\beta_i(t) = 0$ and $\alpha_i(t) = \gamma_i(t) = 1$ $(i = 1, 2)$, (2) we discussed becomes (4) which was investigated by Wang et al. [2] and We [3]. Our main results Theorems 2 and 11 are in accordance with Lemma 2.2 and Theorem 3.1 in [2]; Theorems 3, 4, 8, and 9 are also consistent with Theorem 1-4 in [3], so our results generalize [2, 3]. Moreover, when $\alpha_i(t) = 1$ and $\gamma_i(t) = 0$ $(i = 1, 2)$, system (2) reduces to continuous analogue of system (7) investigated by Ma, Gao, and Xie [13]. Since we consider the permanence, global attractivity, and existence of unique almost periodic solution for the system which Ma et al. [13] and Chen et al. [14] have not studied, our results also supplement the main results of [13, 14]. Numeric simulations validate our analytical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

We declare that there are no conflicts of interest to this paper.

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