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Research Article

Estimating the Gerber-Shiu Function in a Compound Poisson Risk Model with Stochastic Premium Income

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In this paper, we consider the compound Poisson risk model with stochastic premium income. We propose a new estimation of Gerber-Shiu function by an efficient method: Fourier-cosine series expansion. We show that the estimator is easily computed and has a fast convergence rate. Some simulation examples are illustrated to show that the estimation has a good performance when the sample size is finite.

1. Introduction

In this paper, we consider the following compound Poisson risk model with stochastic premium income:

$$U(t) = u + \sum_{i=1}^{M(t)} Y_i - \sum_{j=1}^{N(t)} X_j, \quad t \geq 0, \tag{1}$$

where $u \geq 0$ is the initial surplus and $U(t)$ denotes the surplus level of an insurance company at time t. The premium number process $M(t)$ and the claim number process $N(t)$ are homogenous Poisson processes with intensity $\mu > 0$ and $\lambda > 0$, respectively. The individual claim sizes, $X_1, X_2, \ldots$, are positive independent and identically distributed (i.i.d.) continuous random variables given by generic random variable X with density function f. The premium sizes, $Y_1, Y_2, \ldots$, are positive i.i.d. random variables given by generic random variable Y with exponential distribution function $g(y) = \beta e^{-\beta y}$, $y, \beta > 0$. Throughout this paper, we assume that $\{M(t)\}_{t\geq 0}$, $\{N(t)\}_{t\geq 0}$, $\{X_i\}_{i\geq 1}$ and $\{Y_j\}_{j\geq 1}$ are mutually independent.

Whenever the surplus process becomes negative, we say that ruin occurs. Defining the ruin time by

$$\tau = \inf\{t \geq 0 : U(t) < 0\}, \tag{2}$$

where $\tau = \infty$ if for all $t \geq 0$, $U(t) > 0$. To avoid that ruin is a certain event, suppose that the following condition holds throughout this paper.

Condition 1 (net profit condition).

$$\mu \mathbb{E}[Y] > \lambda \mathbb{E}[X]. \tag{3}$$

The above condition guarantees that the expectation of the surplus process will always be positive at any time $t > 0$; that is to say, the limit of surplus process is almost sure positive. Our research is more meaningful under this assumption.

Let $\delta > 0$ be the interest force and define the expected discounted penalty function by

$$\Phi(u) = \mathbb{E}\left[e^{-\delta\tau} \mathrm{w}(U(\tau-), |U(\tau)|)\mathbf{1}_{(\tau<\infty)} \mid U(0) = u\right], \quad u \geq 0, \tag{4}$$

where $\mathrm{w} : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is a measurable penalty function of the surplus before ruin and the deficit at ruin and $\mathbf{1}_A$ is an indicator function of the event A. This function was first proposed by Gerber and Shiu [1] to study the time to ruin; the surplus before ruin and the deficit at ruin in the classical risk model. It is also called Gerber-Shiu function in the literature. For recent research progress on the Gerber-Shiu function, we can refer to work by Zhao and Yin [2], Yin and Wang [3], Shen et al. [4], Yin and Yuen [5], Zhao and Yao [6], Li et al. [7, 8], Dong et al. [9], among others. Recently, it has become a standard risk measure in ruin theory since many important risk problems can be studied via it by taking different forms of penalty function w and different δ. Several typical examples are presented as follows:

(1) If $w(x, y) \equiv 1$, Φ is the ruin probability when $\delta = 0$, and it is the Laplace transform of ruin time when $\delta > 0$.

(2) If $w(x, y) = x + y$, Φ is the expected discounted claim size causing ruin.

(3) If $w(x, y) = \mathbf{1}_{(x \le x_1)}$ or $w(x, y) = \mathbf{1}_{(y \le y_1)}$ for $x_1, y_1 > 0$, Φ is the defective distribution of the deficit at ruin when $\delta = 0$, and it is the discounted defective distribution of the deficit at ruin when $\delta > 0$.

(4) If $w(x, y) = y^k$, Φ is the kth moment of the deficit at ruin when $\delta = 0$, and it is the discounted kth moment of the deficit at ruin when $\delta > 0$.

The classical compound Poisson risk model assumes that the premiums are received by insurance company at a constant rate over time [10–15]. However, the premiums income in the real life, especially for small insurance companies, is volatile. So it is natural to extend the classical risk model by replacing the constant premium income with a compound Poisson process, which was firstly suggested in Boucherie et al. [16]. Since then, this risk model has been widely studied by many scholars. Boikov [17] derived integral equations and exponential bounds for nonruin probability. Temnov [18] derived a representation for ruin probability. Assuming that the income rate is 1, defective renewal equations satisfied by the discounted penalty function were, respectively, obtained by Bao [19] and Yang and Zhang [20] under different distributions of interclaim times. Supposing premiums and claims follow general compound Poisson processes, a defective renewal equation satisfied by the Gerber-Shiu function was established in Labbé and Sendova [21]. Zhang and Yang [2] extended their model by assuming that a specific dependence structure exists among the claim sizes, interclaim times, and premium sizes; then the Laplace transforms and defective renewal equations for the Gerber-Shiu function were obtained when the individual premium sizes are exponentially distributed. Besides, the risk model with stochastic income has also been studied in Zhao and Yin [22], Yu [23, 24], Xie and Zou [25], Mishura and Ragulina [26], Cheng and Wang [27], Yang et al. [28], Zeng et al. [29], Deng et al. [30], Wang and Zhang [31], and so on.

In all of the above papers, it is assumed that the probability characteristics of the surplus process, such as the densities of claim size and premium size, the Poisson intensities of claim process, and premium process, are known. In reality, these characteristics for an insurance company are usually unknown, but some data information on surplus levels, claim and premium numbers, and claim and premium sizes can be obtained. Thus in recent years, many scholars began to study the estimation of ruin probability and Gerber-Shiu function based on those data information. For the classical compound Poisson risk model with unknown claim size distribution and Poisson intensity, Politis [32] and Masiello [33], respectively, proposed the semiparametric estimators of nonruin and ruin probabilities, respectively. Zhang et al. [34] presented a nonparametric estimator for ruin probability. Zhang [35] proposed a nonparametric estimation of the finite time ruin probability; Zhang and Su [36] proposed a new nonparametric estimation of Gerber-Shiu function by Laguerre series expansion. For the risk model perturbed by a Brownian motion, Shimizu [37, 38] estimated the adjustment coefficient and the Gerber-Shiu function by Laplace transform, respectively. Zhang [39] estimated the Gerber-Shiu function by Fourier-Sinc series expansion. Su et al. [40] proposed an estimator for Gerber-Shiu function by Laguerre series expansion. For the estimation of ruin probability and Gerber-Shiu function in the Lévy risk model, we refer the interested readers to Wang and Yin [41], Zhang and Yang [42, 43], Wang et al. [44], Shimizu and Zhang [45], Peng and Wang [46], and Zhang and Su [47].

In addition, Fang and Oosterlee [48] proposed a novel method for pricing European options by Fourier-cosine series expansion. This method is also called the COS method in the literature, and it can be easily used to approximate an integrable function as long as the corresponding Fourier transform has closed-form expression. Now the COS method has been widely used for pricing options and other financial derivatives. See, e.g., Fang and Oosterlee [49], Ruijter and Oosterlee [50], and Zhang and Oosterlee [51] to name a few. Recently, the COS method has also been used in risk theory to compute and estimate some risk measures by some actuarial researchers. For example, Chau et al. [52, 53] used the COS method to compute the ruin probability and Gerber-Shiu function in the Lévy risk models. Zhang [54] applied the COS method to compute the density of the time to ruin in the classical risk model. Yang et al. [55] used a two-dimensional COS method to estimate the discounted density function of the deficit at ruin in the classical risk model.

Inspired by the work Yang et al. [55], in this paper we shall use the COS method to estimate the Gerber-Shiu function in the compound Poisson risk model with stochastic income. We note that this problem has also been considered by [56] later on, where the Laguerre series expansion method is used to estimate the Gerber-Shiu function. In particular, it can be found that the convergence rate obtained in Theorem 6 in this paper is also used by [56] to make error analysis of their estimator. We would like to remark that the convergence rate in Theorem 6 is one of the main contributions of our paper, since it plays an important role in studying the consistency property of the estimator.

The remainder of this paper is organised as follows. In Section 2, we first briefly introduce the Fourier-cosine series expansion method and then derive the Fourier transform of the Gerber-Shiu function. In Section 3, an estimator of the Gerber-Shiu function is proposed by the observed sample of the surplus process. The consistent property is studied in Section 4 under large sample size setting. Finally, in Section 5 we present some simulation results to show that the estimator behaves well under finite sample size setting.

2. Preliminaries

2.1. The Fourier Transform of Gerber-Shiu Function. In this subsection, we derive the Fourier transform of the Gerber-Shiu function. Let $L^1(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+)$ denote the class of integrable and square integrable functions on the positive axis, respectively. Let $\mathscr{F}h$ and $\mathscr{L}h$ denote the Fourier transform and Laplace transform of a $h \in L^1(\mathbb{R}_+)$. For any complex number z, we denote its real part and imaginary part by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

For convenience, we introduce the Dickson-Hipp operator T_s (see, e.g., Dickson and Hipp [57] and Li and Garrido [58]), which for any integrable function h on $(0,\infty)$ and any complex number s with $\mathrm{Re}(s) \geq 0$ is defined as

$$T_s h(y) = \int_y^\infty e^{-s(x-y)} h(x)\,dx = \int_0^\infty e^{-sx} h(x+y)\,dx, \quad y \geq 0. \tag{5}$$

The Dickson-Hipp operator has been widely used in ruin theory to simplify the expression of ruin related functions. For properties on this operator, we refer the interested readers to Li and Garrido [58].

By Theorem 5.8 in Labbé and Sendova [21], we know that when $\mathbb{E}[X^2] < \infty$, the Gerber-Shiu function Φ satisfies the following renewal equation:

$$\Phi(u) = \int_0^u \Phi(u-x) f_\delta(x)\,dx + H(u), \tag{6}$$

where

$$\begin{aligned} f_\delta(x) &= \frac{\lambda}{\lambda+\mu+\delta}\left[f(x) + (\beta-\rho) T_\rho f(x)\right], \\ H(u) &= \frac{\lambda}{\lambda+\mu+\delta}\left[\omega(u) + (\beta-\rho) T_\rho \omega(u)\right], \\ \omega(u) &= \int_u^\infty \mathrm{w}(u, x-u) f(x)\,dx. \end{aligned} \tag{7}$$

It follows from Lemma 5.3 in Labbé and Sendova [21] that ρ appearing in the above formulae is the unique nonnegative root of the following equation (w.r.t. s), known as the Lundberg's fundamental equation:

$$\left[\lambda+\mu+\delta-\lambda\mathscr{L}f(s)\right](\beta-s) - \mu\beta = 0. \tag{8}$$

Remark 1. Let

$$\chi(s) = \left[\lambda+\mu+\delta-\lambda\mathscr{L}f(s)\right](s-\beta) + \mu\beta, \quad s \geq 0. \tag{9}$$

It is clear that $\chi(s)$ is a continuous function such that $\chi(0) = -\delta\beta \leq 0$, $\chi(\beta) = \mu\beta > 0$ and if $\delta > 0$, $\chi(0) < 0$. In addition, since $\mu\mathbb{E}[Y] > \lambda\mathbb{E}[X]$ and $\mathbb{E}[Y] = 1/\beta$, we have

$$\begin{aligned} \chi'(s) &= \lambda+\mu+\delta-\lambda\mathscr{L}f(s) - \lambda s\mathscr{L}f'(s) \\ &\quad + \lambda\beta\mathscr{L}f'(s) \\ &= \lambda+\mu+\delta-\lambda\int_0^\infty e^{-sx} f(x)\,dx \\ &\quad + \lambda\int_0^\infty sxe^{-sx} f(x)\,dx - \lambda\beta\int_0^\infty xe^{-sx}dx \\ &\geq \mu+\delta-\lambda\beta\mathbb{E}[X] > 0, \end{aligned} \tag{10}$$

which shows that $\chi(s)$ is an increasing function. Thus we conclude that ρ is the unique nonnegative root of the equation $\chi(s) = 0$, and it is located in the interval $[0,\beta)$. In particular, we have $\rho = 0$ when $\delta = 0$.

We assume the following conditions hold true in our literature, which could be satisfied by penalty functions.

Condition 2 (the integrability of w). For the penalty function w, it satisfies

$$\int_0^\infty \int_0^\infty (1+x)\,\mathrm{w}(x,y) f(x+y)\,dydx < \infty. \tag{11}$$

Condition 3. For the penalty function w, there exist some integers α_1, α_2 and constant C such that

$$\mathrm{w}(x,y) \leq C(1+x)^{\alpha_1}(1+y)^{\alpha_2}. \tag{12}$$

Theorem 2. *Condition 2 guarantees the existence of the Fourier transform of Φ, and under Conditions 2 and 3, we have $\Phi \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$.*

Proof. It is easily seen that f, $T_\rho(f) \in L^2(\mathbb{R}_+)$. Under Condition 2 we have ω, $T_\rho\omega \in L^1(\mathbb{R}_+)$ and

$$\sup_{u\geq 0}\left\{T_\rho\omega(u)\right\} \leq \int_0^\infty \int_0^\infty \mathrm{w}(x,y) f(x+y)\,dydx < \infty, \tag{13}$$

thus

$$\int_0^\infty \left(T_\rho\omega(u)\right)^2 du \leq \sup_{u\geq 0}\left\{T_\rho\omega(u)\right\} \times \int_0^\infty T_\rho\omega(u)\,du < \infty. \tag{14}$$

Furthermore, under Condition 3 we have

$$\begin{aligned} \int_0^\infty (\omega(u))^2 du &= \sup_{u\geq 0}\{\omega(u)\} \times \int_0^\infty \omega(u)\,du \\ &\leq C\mathbb{E}\left[X^{\alpha_1+\alpha_2}\right] \times \int_0^\infty \omega(u)\,du < \infty, \end{aligned} \tag{15}$$

so $\omega \in L^2(\mathbb{R}_+)$, and then $H, f_\delta \in L^2(\mathbb{R}_+)$. By Theorem 1.4.5 in Stenger [59], we have $\int_0^u \Phi(u-x) f_\delta(x)dx \in L^2(\mathbb{R}_+)$. Using the inequality $(x+y)^2 \leq 2x^2 + 2y^2$, we finally derive $\Phi \in L^2(\mathbb{R}_+)$ due to (6). □

Now, we compute the Fourier transform of the Gerber-Shiu function. Applying the Fourier transform on both sides of (6) gives

$$\begin{aligned}
\mathscr{F}\Phi(s) &= \int_0^\infty e^{isu}\Phi(u)\,du \\
&= \int_0^\infty e^{isu}\int_0^u \Phi(u-x) f_\delta(x)\,dx\,du \\
&\quad + \int_0^\infty e^{isu}H(u)\,du \\
&= \int_0^\infty \int_0^\infty e^{is(z+x)}\Phi(z)\,dz\,dx + \mathscr{F}H(s) \\
&= \mathscr{F}\Phi(s)\,\mathscr{F}f_\delta(s) + \mathscr{F}H(s),
\end{aligned} \tag{16}$$

leading to

$$\mathscr{F}\Phi(s) = \frac{\mathscr{F}H(s)}{1-\mathscr{F}f_\delta(s)}. \tag{17}$$

For the Fourier transform $\mathscr{F}H(s)$, we have

$$\begin{aligned}
\mathscr{F}H(s) &= \int_0^\infty e^{isu}H(u)\,du \\
&= \frac{\lambda}{\lambda+\mu+\delta}\left[\int_0^\infty e^{isu}\omega(u)\,du \right.\\
&\quad \left. + (\beta-\rho)\int_0^\infty e^{isu}T_\rho\omega(u)\,du\right] \\
&= \frac{\lambda}{\lambda+\mu+\delta}\left[\mathscr{F}\omega(s) \right.\\
&\quad \left. + (\beta-\rho)\int_0^\infty e^{isu}T_\rho\omega(u)\,du\right],
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\int_0^\infty e^{isu}T_\rho\omega(u)\,du &= \int_0^\infty e^{isu}\int_u^\infty e^{-\rho(x-u)}\omega(x)\,dx\,du \\
&= \int_0^\infty \int_u^\infty e^{(is+\rho)u}\cdot e^{-\rho x}\omega(x)\,dx\,du \\
&= \int_0^\infty e^{-\rho y}\omega(y)\int_0^y e^{(is+\rho)u}\,du\,dy \\
&= \frac{1}{\rho+is}\left(\mathscr{F}\omega(s) - \mathscr{L}\omega(\rho)\right).
\end{aligned} \tag{19}$$

Then we obtain

$$\begin{aligned}
\mathscr{F}H(s) &= \frac{\lambda}{\lambda+\mu+\delta}\left[\mathscr{F}\omega(s) \right.\\
&\quad \left. + \frac{(\beta-\rho)}{\rho+is}\left(\mathscr{F}\omega(s) - \mathscr{L}\omega(\rho)\right)\right].
\end{aligned} \tag{20}$$

Similarly, for $\mathscr{F}f_\delta(s)$ we obtain

$$\begin{aligned}
\mathscr{F}f_\delta(s) &= \frac{\lambda}{\lambda+\mu+\delta}\left[\mathscr{F}f(s) \right.\\
&\quad \left. + \frac{(\beta-\rho)}{\rho+is}\left(\mathscr{F}f(s) - \mathscr{L}f(\rho)\right)\right].
\end{aligned} \tag{21}$$

2.2. Fourier-Cosine Series Expansion. In this subsection, we present some known results on the Fourier-cosine series expansion method and give the estimating formula of the Gerber-Shiu function derived by Fourier-cosine series expansion.

By [48], any real function has a cosine expansion when it is finitely supported. Therefore, for a integrable function h defined on $[a_1, a_2]$, it has the following cosine series expansion:

$$\begin{aligned}
h(x) &= {\sum_{k=0}^{\infty}}' \left\{\frac{2}{a_2-a_1}\int_{a_1}^{a_2} h(x)\cos\left(k\pi\frac{x-a_1}{a_2-a_1}\right)dx\right\} \\
&\quad \cdot \cos\left(k\pi\frac{x-a_1}{a_2-a_1}\right),
\end{aligned} \tag{22}$$

where $\sum'$ means the first term of the summation has half weight. For a function $h\in L^1(\mathbb{R}_+)$, we introduce an auxiliary function

$$h_a(x) = h(x)\cdot \mathbf{1}_{(0\le x\le a)}, \quad a>0, \tag{23}$$

then h_a has a finite domain $[0,a]$ and $h(x) = h_a(x)$ when $x\in[0,a]$. By (22) we have

$$\begin{aligned}
h(x) &= h_a(x) \\
&= {\sum_{k=0}^{\infty}}' \left\{\frac{2}{a}\int_0^a h(x)\cos\left(k\pi\frac{x}{a}\right)dx\right\}\cos\left(k\pi\frac{x}{a}\right) \\
&= {\sum_{k=0}^{\infty}}' \frac{2}{a}\operatorname{Re}\left\{\int_0^a h(x)e^{i(k\pi/a)x}dx\right\}\cos\left(k\pi\frac{x}{a}\right), \\
&\qquad 0\le x\le a.
\end{aligned} \tag{24}$$

Due to the existence of the Fourier transform of h, the integrand of $\mathscr{F}h(\cdot)$ has to decay to zero at $+\infty$ and we can truncate the integration range with a large a; then we have

$$\begin{aligned}
\mathscr{F}h\left(\frac{k\pi}{a}\right) &= \int_0^{+\infty} h(x)e^{i(k\pi/a)x}dx \\
&\approx \int_0^a h(x)e^{i(k\pi/a)x}dx.
\end{aligned} \tag{25}$$

Thus

$$\begin{aligned}
h(x) &\approx {\sum_{k=0}^{\infty}}' \frac{2}{a}\operatorname{Re}\left\{\mathscr{F}h\left(\frac{k\pi}{a}\right)\right\}\cos\left(k\pi\frac{x}{a}\right), \\
&\qquad 0\le x\le a.
\end{aligned} \tag{26}$$

Furthermore, for a large integer K, the above summation can be truncated as follows:

$$h(x) \approx \sum_{k=0}^{K-1}{}' \frac{2}{a} \operatorname{Re}\left\{\mathcal{F}h\left(\frac{k\pi}{a}\right)\right\} \cos\left(k\pi\frac{x}{a}\right), \quad 0 \leq x \leq a. \tag{27}$$

We now consider the Gerber-Shiu function Φ. It follows from (27) that, for $0 \leq u \leq a$,

$$\Phi(u) \approx \Phi_{K,a}(u) := \sum_{k=0}^{K-1}{}' \frac{2}{a} \operatorname{Re}\left\{\mathcal{F}\Phi\left(\frac{k\pi}{a}\right)\right\} \cos\left(k\pi\frac{x}{a}\right). \tag{28}$$

We can easily see that the key of approximating the Gerber-Shiu function by Fourier-cosine series expansion method lies in the calculation of the Fourier transform $\mathcal{F}\Phi(s)$ for $s \in \{k\pi/a : k = 0, 1, \ldots, K-1\}$. Therefore, we derive a specific expression of Fourier transform $\mathcal{F}\Phi(\cdot)$ in the next section.

3. Estimation Procedure

In this section, we study how to estimate the Gerber-Shiu function by Fourier-cosine series expansion based on the discretely observed information of the surplus process, the aggregate claims and premiums processes. According to (28), we know that the key is to construct an estimation of $\mathcal{F}\Phi(s)$ based on this discrete information. In reality, for an insurance company, even though the relevant probability characteristics are unknown, it is easy to obtain the data sets of surplus levels, the sizes of claims and premiums, and the number of claims and premiums through observation.

Assume that we can observe the surplus process over a long time interval $[0, T]$. Let $\Delta > 0$ be a fixed interobservation interval. Furthermore, w.l.o.g. we assume T/Δ is an integer denoted as n.

Suppose that the insurer can get the following data sets:

(1) Data set of surplus levels:

$$\left\{U_{j\Delta} : j = 0, 1, 2, \ldots, n\right\}. \tag{29}$$

where $U_{j\Delta}$ is the observed surplus level at time $t = j\Delta$.

(2) Data set of claim numbers and claim sizes:

$$\left\{N_{j\Delta}, X_1, X_2, \ldots, X_{N_{j\Delta}}\right\}, \quad j = 1, \ldots, n, \tag{30}$$

where $N_{j\Delta}$ is the total claim number up to time $t = j\Delta$.

(3) Data set of premium numbers and claim sizes:

$$\left\{M_{j\Delta}, Y_1, Y_2, \ldots, Y_{N_{j\Delta}}\right\}, \quad j = 1, \ldots, n, \tag{31}$$

where $M_{j\Delta}$ is the total premium number up to time $t = j\Delta$.

Next, we study how to estimate the Fourier transform $\mathcal{F}\Phi(s)$ based on the above data sets. To estimate $\mathcal{F}\Phi(s)$, by (17), (20), and (21) we should first estimate the following characteristics:

$$\lambda,\ \mu,\ \rho,\ \beta,\ \mathcal{F}f,\ \mathcal{F}\omega,\ \mathcal{L}f(\rho),\ \mathcal{L}\omega(\rho). \tag{32}$$

First, we can estimate $\mathcal{F}f(s)$ by the empirical characteristic function

$$\widehat{\mathcal{F}f}(s) = \frac{1}{N_T}\sum_{j=1}^{N_T} e^{isX_j}. \tag{33}$$

Similarly, $\mathcal{L}f(s)$ can be estimated by

$$\widehat{\mathcal{L}f}(s) = \frac{1}{N_T}\sum_{j=1}^{N_T} e^{-sX_j}. \tag{34}$$

Next, for the function $\omega(u)$, we have

$$\begin{aligned}\mathcal{F}\omega(s) &= \int_0^\infty e^{isu} \int_u^\infty \mathrm{w}(u, x-u)\,dx\,du \\ &= \int_0^\infty \int_0^x e^{isu}\mathrm{w}(u, x-u)\,du f(x)\,dx \\ &= \mathbb{E}\left(\int_0^X e^{isu}\mathrm{w}(u, X-u)\,du\right).\end{aligned} \tag{35}$$

Similarly,

$$\mathcal{L}\omega(s) = \mathbb{E}\left(\int_0^X e^{-su}\mathrm{w}(u, X-u)\,du\right). \tag{36}$$

Then, $\mathcal{F}\omega(s)$ and $\mathcal{L}\omega(s)$ can be, respectively, estimated by

$$\begin{aligned}\widehat{\mathcal{F}\omega}(s) &= \frac{1}{N_T}\sum_{j=1}^{N_T}\int_0^{X_j} e^{isu}\mathrm{w}\left(u, X_j - u\right)du, \\ \widehat{\mathcal{L}\omega}(s) &= \frac{1}{N_T}\sum_{j=1}^{N_T}\int_0^{X_j} e^{-su}\mathrm{w}\left(u, X_j - u\right)du.\end{aligned} \tag{37}$$

According to the property of Poisson distribution, λ and μ can be estimated by

$$\begin{aligned}\widehat{\lambda} &= \frac{1}{T}N_T, \\ \widehat{\mu} &= \frac{1}{T}M_T.\end{aligned} \tag{38}$$

It is easily seen that

$$\begin{aligned}\widehat{\lambda} - \lambda &= O_p\left(T^{-1/2}\right), \\ \widehat{\mu} - \mu &= O_p\left(T^{-1/2}\right).\end{aligned} \tag{39}$$

Since the premium size Y follows exponential distribution with parameter β, we have $\mathbb{E}[Y] = 1/\beta$; then we can estimate β by

$$\widehat{\beta} = \frac{1}{(1/M_T)\sum_{i=1}^{M_T} Y_i}. \tag{40}$$

It is also easily seen that $\widehat{\beta}-\beta = O_p(T^{-1/2})$. The estimator of ρ, denoted as $\widehat{\rho}$, is defined to be the positive root of the following equation (in s):

$$\left[\widehat{\lambda}+\widehat{\mu}+\delta-\widehat{\lambda}\widehat{\mathscr{L}f}(s)\right]\left(\widehat{\beta}-s\right)-\widehat{\mu}\widehat{\beta}=0. \tag{41}$$

Since $\rho = 0$ as $\delta = 0$, we set $\widehat{\rho} = 0$ as $\delta = 0$. Furthermore, we can estimate $\mathscr{L}f(\rho)$, $\mathscr{L}\omega(\rho)$, respectively, by $\widehat{\mathscr{L}f}(\widehat{\rho})$, $\widehat{\mathscr{L}\omega}(\widehat{\rho})$.

Remark 3. Let

$$\widehat{\chi}(s)=\left[\widehat{\lambda}+\widehat{\mu}+\delta-\widehat{\lambda}\widehat{\mathscr{L}f}(s)\right]\left(s-\widehat{\beta}\right)+\widehat{\mu}\widehat{\beta}. \tag{42}$$

It is clear that $\widehat{\chi}(0) = -\delta\widehat{\beta} \le 0$, $\widehat{\chi}(\widehat{\beta}) = \widehat{\mu}\widehat{\beta} > 0$, and if $\delta = 0$, $\widehat{\chi}(0) < 0$. In addition, since $\mu\mathbb{E}[Y] > \lambda\mathbb{E}[X]$ and $\mathbb{E}[Y] = 1/\beta$, it follows from the strong law of large numbers that, for any $s > 0$,

$$\begin{aligned}\widehat{\chi}'(s) &= \widehat{\lambda}+\widehat{\mu}+\delta-\widehat{\lambda}\widehat{\mathscr{L}f}(s)-s\widehat{\lambda}\widehat{\mathscr{L}f}'(s)\\&\quad+\widehat{\lambda}\widehat{\beta}\widehat{\mathscr{L}f}'(s)\\&=\widehat{\lambda}+\widehat{\mu}+\delta-\widehat{\lambda}\frac{1}{N_T}\sum_{j=1}^{N_T}e^{-sX_j}\\&\quad+\widehat{\lambda}\frac{1}{N_T}\sum_{j=1}^{N_T}sX_je^{-sX_j}\\&\quad-\widehat{\lambda}\widehat{\beta}\frac{1}{N_T}\sum_{j=1}^{N_T}X_je^{-sX_j}\\&\ge\widehat{\mu}-\widehat{\lambda}\widehat{\beta}\frac{1}{N_T}\sum_{j=1}^{N_T}X_j\xrightarrow{a.s.}\\&\mu-\lambda\beta\mathbb{E}[X]>0,\end{aligned} \tag{43}$$

which shows that the probability that $\widehat{\chi}(s) = 0$ has a unique positive root $\widehat{\rho}$ tends to one as $T \longrightarrow \infty$. Thus we conclude that $\widehat{\rho}$ is located in the interval $[0,\widehat{\beta})$ with probability tending to one as $T \longrightarrow \infty$.

Proposition 4. *Suppose that Condition 1 holds true, then we have* $\widehat{\rho}\xrightarrow{p}\rho$.

Proof. By Remark 1, $\chi(s)$ is an increasing function and $\chi(s) = 0$ has unique nonnegative root ρ. Then for any $\varepsilon > 0$, we have $\chi(\rho-\varepsilon) < 0 < \chi(\rho+\varepsilon)$. In addition, by Remark 3, $\widehat{\chi}(s)$ is nondecreasing on event $\{\widehat{\mu} > \widehat{\lambda}\widehat{\beta}(1/N_T)\sum_{j=1}^{N_T}X_j\}$ and $\mathbb{P}(\widehat{\mu} > \widehat{\lambda}\widehat{\beta}(1/N_T)\sum_{j=1}^{N_T}X_j) \longrightarrow 1$. Also, we find that, for any $s > 0$, $\widehat{\chi}(s)\xrightarrow{p}\chi(s)$. Thus it follows from Lemma 5.10 in Van der Vaart [60] that $\widehat{\rho}\xrightarrow{p}\rho$. □

Once we have obtained the estimation of the above characteristics, by (17), (20), and (21), the estimation of Fourier transform $\mathscr{F}\Phi(s)$, denoted as $\widehat{\mathscr{F}\Phi}(s)$, can be defined by

$$\widehat{\mathscr{F}\Phi}(s)=\frac{\widehat{\mathscr{F}H}(s)}{1-\widehat{\mathscr{F}f_\delta}(s)}, \tag{44}$$

where

$$\begin{aligned}\widehat{\mathscr{F}H}(s)&=\frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\left[\widehat{\mathscr{F}\omega}(s)\right.\\&\quad\left.+\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\left(\widehat{\mathscr{F}\omega}(s)-\widehat{\mathscr{L}\omega}(\widehat{\rho})\right)\right],\\\widehat{\mathscr{F}f_\delta}(s)&=\frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\left[\widehat{\mathscr{F}f}(s)\right.\\&\quad\left.+\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\left(\widehat{\mathscr{F}f}(s)-\widehat{\mathscr{L}f}(\widehat{\rho})\right)\right].\end{aligned} \tag{45}$$

Finally, replacing $\mathscr{F}\Phi(\cdot)$ in (28) by its estimation $\widehat{\mathscr{F}\Phi}(\cdot)$, the Gerber-Shiu function can be estimated by

$$\widehat{\Phi}_{K,a}(u) := {\sum_{k=0}^{K-1}}'\frac{2}{a}\operatorname{Re}\left\{\widehat{\mathscr{F}\Phi}\left(\frac{k\pi}{a}\right)\right\}\cos\left(k\pi\frac{x}{a}\right),\quad 0\le u\le a. \tag{46}$$

4. Asymptotic Properties

In this section, we study the asymptotic properties of the estimation $\widehat{\Phi}_{K,a}$. For any function $h \in L^2(\mathbb{R}_+)$, its L^2-norm is defined by $\|h\| = (\int_0^\infty h^2(x)dx)^{1/2}$. Throughout this section, C represents a positive generic constant that may take different values at different steps. For two nonnegative functions h_1 and h_2 with common domain $\Theta \subset \mathbb{R}_+$, $h_1 \lesssim h_2$ means that $h_1(x) \le Ch_2(x)$ uniformly for every $x \in \Theta$. In addition, we define

$$H_j(x)=\int_0^x u^j\mathrm{w}(u,x-u)\,du,\quad j=0,1,2. \tag{47}$$

It is easy to see that

$$\int_0^\infty u^j\omega(u)\,du=\mathbb{E}\left[H_j(X)\right]. \tag{48}$$

For reader's convenience, we introduce some definitions in empirical process theory, which are used to study the asymptotic properties. For any measurable function h, its $L^r(P)$-norm is defined by $\|h\|_{P,r} = (\int|h(\omega)|^r dP(\omega))^{1/r}$. Given two functions l and u, the bracket $[l,u]$ is the set of all functions h with $l \le h \le u$. An ε-bracket in $L^r(P)$ is a bracket $[l,u]$ with $\|u-l\|_{P,r} < \varepsilon$. For a class $\mathscr{G} \subset L^r(P)$, the bracketing number $N_\diamond(\varepsilon,\mathscr{G},L^r(P))$ is the minimum number of ε-brackets needed to cover $\mathscr{G}$. For $\beta > 0$, the bracketing integral is defined by $J_\diamond(\beta,\mathscr{G},L^r(P)) = \int_0^\beta\sqrt{N_\diamond(\varepsilon,\mathscr{G},L^r(P))}d\varepsilon$.

We shall use the L^2-norm to study the asymptotic properties of the estimator.

Put $\Phi_{K,a} = \widehat{\Phi}_{K,a} = 0$ when $u > a$. By triangle inequality, we have

$$\left\|\Phi - \widehat{\Phi}_{K,a}\right\| \leq \left\|\Phi - \Phi_{K,a}\right\| + \left\|\Phi_{K,a} - \widehat{\Phi}_{K,a}\right\|, \tag{49}$$

where the first term $\|\Phi - \Phi_{K,a}\|$ is the bias caused by Fourier-cosine series approximation, and the second term $\|\Phi_{K,a} - \widehat{\Phi}_{K,a}\|$ is the bias caused by statistical estimation.

For the bias $\|\Phi-\Phi_{K,a}\|$, by the similar arguments in Zhang [54] we obtain the following result.

Theorem 5. *Suppose that $\int_0^\infty |\Phi'(u)|du < \infty$ and for some integer m, $\Phi(u) \leq Cu^{-(m+1)}$; then under Conditions 1, 2, and 3 we have*

$$\left\|\Phi - \Phi_{K,a}\right\| \leq C\left\{\frac{K+1}{a^{2m+1}} + \frac{a}{K-1}\right\}. \tag{50}$$

Next, we study the error $\|\Phi_{K,a} - \widehat{\Phi}_{K,a}\|$. For the estimator $\widehat{\rho}$, we derive the following result.

Theorem 6. *Suppose that Condition 1 holds and $\mathbb{E}[X^2] < \infty$, then we have*

$$\widehat{\rho} - \rho = O_p\left(T^{-1/2}\right). \tag{51}$$

Proof. By the mean value theorem, we have

$$\widehat{\chi}(\rho) = \widehat{\chi}(\widehat{\rho}) + \widehat{\chi}'(\rho^*)(\rho - \widehat{\rho}) = \widehat{\chi}'(\rho^*)(\rho - \widehat{\rho}), \tag{52}$$

where ρ^* is a random number between ρ and $\widehat{\rho}$. Since $\chi(\rho) = 0$, we obtain

$$\widehat{\rho} - \rho = \frac{\chi(\rho) - \widehat{\chi}(\rho)}{\widehat{\chi}'(\rho^*)}, \tag{53}$$

where

$$\begin{aligned}
\chi(\rho) - \widehat{\chi}(\rho) &= \left[\lambda + \mu + \delta - \lambda\mathscr{L}f(\rho)\right](\rho - \beta) \\
&\quad - \left[\widehat{\lambda} + \widehat{\mu} + \delta - \widehat{\lambda}\widehat{\mathscr{L}f}(\rho)\right]\left(\rho - \widehat{\beta}\right) \\
&= \left(\lambda - \widehat{\lambda}\right)(\rho - \beta)\left(1 - \widehat{\mathscr{L}f}(\rho)\right) \\
&\quad + \left(\beta - \widehat{\beta}\right)\left[\delta - \widehat{\lambda}\left(\widehat{\mathscr{L}f}(\rho) - 1\right)\right] \\
&\quad + \lambda(\rho - \beta)\left(\widehat{\mathscr{L}f}(\rho) - \mathscr{L}f(\rho)\right) \\
&\quad + \rho(\mu - \widehat{\mu}).
\end{aligned} \tag{54}$$

Since $\widehat{\lambda} - \lambda = O_p(T^{-1/2})$, $\widehat{\mu} - \mu = O_p(T^{-1/2})$, and $\widehat{\beta} - \beta = O_p(T^{-1/2})$, it is easily seen that $\chi(\rho) - \widehat{\chi}(\rho) = O_p(T^{-1/2})$.

In addition, we introduce the following set:

$$A_T = \left\{\left|\widehat{\chi}'(\rho^*)\right| > \frac{1}{2}(\mu - \lambda\beta\mathbb{E}[X])\right\}. \tag{55}$$

Since $\widehat{\mu} - \widehat{\lambda}\widehat{\beta}(1/N_T)\sum_{j=1}^{N_T} X_j \xrightarrow{p} \mu - \lambda\beta$ and $|\widehat{\chi}'(\rho^*)| \geq \widehat{\mu} - \widehat{\lambda}\widehat{\beta}(1/N_T)\sum_{j=1}^{N_T} X_j$, we have

$$\begin{aligned}
\mathbb{P}(A_T) &\geq \mathbb{P}\left(\widehat{\mu} - \widehat{\lambda}\widehat{\beta}\frac{1}{N_T}\sum_{j=1}^{N_T} X_j \geq \frac{1}{2}(\mu - \lambda\beta\mathbb{E}[X])\right) \\
&= 1 - \mathbb{P}\left(\widehat{\mu} - \widehat{\lambda}\widehat{\beta}\frac{1}{N_T}\sum_{j=1}^{N_T} X_j < \frac{1}{2}(\mu - \lambda\beta\mathbb{E}[X])\right) \\
&= 1 - \mathbb{P}\left(\mu - \lambda\beta\mathbb{E}[X] - \widehat{\mu} + \widehat{\lambda}\widehat{\beta}\frac{1}{N_T}\sum_{j=1}^{N_T} X_j \right. \\
&\quad \left. \geq \frac{1}{2}(\mu - \lambda\beta\mathbb{E}[X])\right) \longrightarrow 1, \quad \text{as } T \longrightarrow \infty.
\end{aligned} \tag{56}$$

Furthermore,

$$\begin{aligned}
&\mathbb{P}\left(\left|\widehat{\rho} - \rho\right| > CT^{-1/2}\right) \\
&\quad = \mathbb{P}\left(\frac{\left|\chi(\rho) - \widehat{\chi}(\rho)\right|}{\left|\widehat{\chi}'(\rho^*)\right|} > CT^{-1/2}\right) \\
&\quad \leq \mathbb{P}\left(\left\{\frac{\left|\chi(\rho) - \widehat{\chi}(\rho)\right|}{\left|\widehat{\chi}'(\rho^*)\right|} > CT^{-1/2}\right\} \cap A_T\right) \\
&\qquad + \mathbb{P}\left(A_T^c\right) \\
&\quad = \mathbb{P}\left(\left|\chi(\rho) - \widehat{\chi}(\rho)\right| > CT^{-1/2}\right) + \mathbb{P}\left(A_T^c\right).
\end{aligned} \tag{57}$$

As a result, since $\chi(\rho) - \widehat{\chi}(\rho) = O_p(T^{-1/2})$ and $\mathbb{P}(A_T^c) \longrightarrow 0$, we derive that $\widehat{\rho} - \rho = O_p(T^{-1/2})$. □

The following two theorems give the uniform convergence rates of $\mathscr{F}H$ and $\mathscr{F}f_\delta$.

Theorem 7. *Suppose that Condition 1 holds, $\|H_j(X)\|_{P,1} < \infty$, $j = 0, 1$, and $\|H_j(X)\|_{P,2} < \infty$, $j = 1, 2$. Then for large a, K and T, we have*

$$\sup_{s\in[0,K\pi/a]}\left|\mathscr{F}H(s) - \widehat{\mathscr{F}H}(s)\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \tag{58}$$

Proof. By (20) and (44),

$$\begin{aligned}
&\widehat{\mathscr{F}H}(s) - \mathscr{F}H(s) \\
&\quad = \frac{\widehat{\lambda}}{\widehat{\lambda} + \widehat{\mu} + \delta}\widehat{\mathscr{F}\omega}(s) - \frac{\lambda}{\lambda + \mu + \delta}\mathscr{F}\omega(s) \\
&\qquad + \frac{\widehat{\lambda}}{\widehat{\lambda} + \widehat{\mu} + \delta}\frac{\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\rho} + is}\left[\widehat{\mathscr{F}\omega}(s) - \widehat{\mathscr{L}\omega}(\widehat{\rho})\right] \\
&\qquad - \frac{\lambda}{\lambda + \mu + \delta}\frac{(\beta - \rho)}{\rho + is}\left[\mathscr{F}\omega(s) - \mathscr{L}\omega(\rho)\right] \\
&\quad := I_1 + I_2,
\end{aligned} \tag{59}$$

where

$$I_1 = \frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta}\widehat{\mathscr{F}\omega}(s) - \frac{\lambda}{\lambda+\mu+\delta}\mathscr{F}\omega(s),$$

$$I_2 = \frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta}\frac{(\hat{\beta}-\hat{\rho})}{\hat{\rho}+is}\left[\widehat{\mathscr{F}\omega}(s) - \widehat{\mathscr{L}\omega}(\hat{\rho})\right] - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\left[\mathscr{F}\omega(s) - \mathscr{L}\omega(\rho)\right]. \tag{60}$$

We first study I_1.

$$\begin{aligned} I_1 &= \frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta}\widehat{\mathscr{F}\omega}(s) - \frac{\lambda}{\lambda+\mu+\delta}\mathscr{F}\omega(s) \\ &= \frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[\int_0^{X_j} e^{isu}\mathrm{w}(u, X_j-u)\,du - \mathbb{E}\left(\int_0^X e^{isu}\mathrm{w}(u,X-u)\,du\right)\right] + \left(\frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\mathbb{E}\left(\int_0^X e^{isu}\mathrm{w}(u,X-u)\,du\right) \\ &= \frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{1,s}(X_j) - \mathbb{E}(g_{1,s}(X))\right] + \left(\frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\mathbb{E}(g_{1,s}(X)), \end{aligned} \tag{61}$$

where

$$g_{1,s}(x) = \int_0^x e^{isu}\mathrm{w}(u,x-u)\,du, \quad x \ge 0. \tag{62}$$

For $g_{1,s}(x)$, we have

$$\begin{aligned} &\sup_{s\in[0,K\pi/a]}\left|\mathbb{E}(g_{1,s}(X))\right| \\ &\quad = \sup_{s\in[0,K\pi/a]}\left|\mathbb{E}\left(\int_0^X e^{isu}\mathrm{w}(u,X-u)\,du\right)\right| \\ &\quad \le \left|\mathbb{E}\left(\int_0^X \mathrm{w}(u,X-u)\,du\right)\right| = \|H_0(X)\|_{P,1} < \infty, \end{aligned} \tag{63}$$

then

$$\begin{aligned} &\sup_{s\in[0,K\pi/a]}\left|\left(\frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\mathbb{E}(g_{1,s}(X))\right| \\ &\quad \le \left|\left(\frac{\hat{\lambda}}{\hat{\lambda}+\hat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\right| \cdot \|H_0(X)\|_{p,1} \\ &\quad = O_p\left(T^{-1/2}\right) \end{aligned} \tag{64}$$

due to $\hat{\lambda} - \lambda = O_p(T^{-1/2})$ and $\hat{\mu} - \mu = O_p(T^{-1/2})$.

Now we introduce the following two classes of real-valued functions:

$$\begin{aligned} \mathscr{G}_{K,R} &= \left\{g : g = \operatorname{Re}(g_{1,s}),\ s \in \left[0, \frac{K\pi}{a}\right]\right\}, \\ \mathscr{G}_{K,I} &= \left\{g : g = \operatorname{Im}(g_{1,s}),\ s \in \left[0, \frac{K\pi}{a}\right]\right\}. \end{aligned} \tag{65}$$

Then we have

$$\begin{aligned} &\sup_{s\in[0,K\pi/a]}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{1,s}(X_j) - \mathbb{E}(g_{1,s}(X))\right]\right| \\ &\quad \le \sup_{g\in\mathscr{G}_{K,R}}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{1,s}(X_j) - \mathbb{E}(g_{1,s}(X))\right]\right| \\ &\quad + \sup_{g\in\mathscr{G}_{K,I}}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{1,s}(X_j) - \mathbb{E}(g_{1,s}(X))\right]\right|. \end{aligned} \tag{66}$$

We only study the convergence rate of the first term $\sup_{g\in\mathscr{G}_{K,R}}|(1/N_T)\sum_{j=1}^{N_T}[g_{1,s}(X_j) - \mathbb{E}(g_{1,s}(X))]|$, since the second term follows similarly.

For any real-valued function $g \in \mathscr{G}_{K,R}$, we have

$$\begin{aligned} |g(x)| &\le \sup_{s\in[0,K\pi/a]}\left|\int_0^x e^{isu}\mathrm{w}(u,x-u)\,du\right| \\ &\le \int_0^x \mathrm{w}(u,x-u)\,du = H_0(x), \end{aligned} \tag{67}$$

which implies that $\mathscr{G}_{K,R}$ is contained in the single bracket $[-H_0, H_0]$. For two functions g_{1,s_1}, g_{1,s_2}, where $s_j \in [0, K\pi/a]$, $j = 1, 2$, the mean value theorem gives

$$\begin{aligned} &\left|\operatorname{Re}(g_{1,s_1}) - \operatorname{Re}(g_{1,s_2})\right| \\ &\quad = \left|\int_0^x (\cos(s_1 u) - \cos(s_2 u))\,\mathrm{w}(u,x-u)\,du\right| \\ &\quad = \left|-\int_0^x \sin(s^* u)\cdot u\cdot(s_1 - s_2)\,\mathrm{w}(u,x-u)\,du\right| \\ &\quad \le \int_0^x u\mathrm{w}(u,x-u)\,du \cdot |s_1 - s_2| \\ &\quad = H_1(x)\,|s_1 - s_2|, \end{aligned} \tag{68}$$

where s^* is a number between s_1 and s_2. Under the condition $\|H_1(X)\|_{p,2} < \infty$, it follows from Example 19.7 in Van der Vaart [60] that, for any $0 < \varepsilon < K\pi/a$, there exists a constant C such that the bracket number for $\mathscr{G}_{K,R}$ satisfies

$$N_\diamond\left(\varepsilon, \mathscr{G}_{K,R}, L^2(P)\right) \le C\frac{K\pi}{\varepsilon a}\|H_1(x)\|_{P,2}. \tag{69}$$

As a result, for every $\delta > 0$, the bracketing integral

$$\begin{aligned} &J_\diamond\left(\delta, \mathscr{G}_{K,R}, L^2(P)\right) \\ &\quad \le \int_0^\delta \sqrt{\log\left(C\frac{K\pi}{\varepsilon a}\|H_1(x)\|_{p,2}\right)d\varepsilon} \lesssim \sqrt{\log\left(\frac{K}{a}\right)}. \end{aligned} \tag{70}$$

Furthermore, by Corollary 19.35 in Van der Vaart [60] we have

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\sqrt{N_T}}\sup_{g\in\mathcal{G}_{K,R}}\left|\sum_{j=1}^{N_T}\left[g\left(X_j\right)-\mathbb{E}\left(g\left(X\right)\right)\right]\right| \mid N_T\right) &\leq J_{\diamond}\left(\delta,\mathcal{G}_{K,R},L^2\left(P\right)\right) \\ &\leq \int_0^{\delta}\sqrt{\log\left(C\frac{K\pi}{\varepsilon a}\left\|H_1\left(x\right)\right\|_{P,2}\right)d\varepsilon} \lesssim \sqrt{\log\left(\frac{K}{a}\right)},\end{aligned} \tag{71}$$

then

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N_T}\sup_{g\in\mathcal{G}_{K,R}}\left|\sum_{j=1}^{N_T}\left[g\left(X_j\right)-\mathbb{E}\left(g\left(X\right)\right)\right]\right|\right) &= \mathbb{E}\left(\frac{1}{\sqrt{N_T}}\mathbb{E}\left(\frac{1}{\sqrt{N_T}}\cdot\sup_{g\in\mathcal{G}_{K,R}}\left|\sum_{j=1}^{N_T}\left[g\left(X_j\right)-\mathbb{E}\left(g\left(X\right)\right)\right]\right| \mid N_T\right)\right) \\ &\lesssim \frac{\sqrt{\log\left(K/a\right)}}{\mathbb{E}\left[\sqrt{N_T}\right]} \leq \frac{\sqrt{\log\left(K/a\right)}}{\sqrt{\mathbb{E}\left[N_T\right]}} = \sqrt{\frac{\log\left(K/a\right)}{\lambda T}}.\end{aligned} \tag{72}$$

Therefore,

$$\sup_{g\in\mathcal{G}_{K,R}}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g\left(X_j\right)-\mathbb{E}\left(g\left(X\right)\right)\right]\right| = O_p\left(\sqrt{\frac{\log\left(K/a\right)}{T}}\right). \tag{73}$$

Similarly,

$$\sup_{g\in\mathcal{G}_{K,I}}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g\left(X_j\right)-\mathbb{E}\left(g\left(X\right)\right)\right]\right| = O_p\left(\sqrt{\frac{\log\left(K/a\right)}{T}}\right). \tag{74}$$

Combining (66), (73), and (74), we obtain

$$\sup_{s\in[0,K\pi/a]}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{1,s}\left(X_j\right)-\mathbb{E}\left(g_{1,s}\left(X\right)\right)\right]\right| = O_p\left(\sqrt{\frac{\log\left(K/a\right)}{T}}\right). \tag{75}$$

As a result, (64) and (75) give

$$\sup_{s\in[0,K\pi/a]}\left|I_1\right| = O_p\left(\sqrt{\frac{\log\left(K/a\right)}{T}}\right). \tag{76}$$

Next, we consider I_2. We have

$$\begin{aligned}I_2 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\left[\widehat{\mathcal{F}\omega}\left(s\right)-\widehat{\mathcal{L}\omega}\left(\widehat{\rho}\right)\right] - \frac{\lambda}{\lambda+\mu+\delta}\frac{\left(\beta-\rho\right)}{\rho+is}\left[\mathcal{F}\omega\left(s\right)-\mathcal{L}\omega\left(\rho\right)\right] \\ &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\cdot\sum_{j=1}^{N_T}\int_0^{X_j}\left(e^{isu}-e^{-\widehat{\rho}u}\right)\mathrm{w}\left(u,X_j-u\right)du - \frac{\lambda}{\lambda+\mu+\delta}\frac{\left(\beta-\rho\right)}{\rho+is}\cdot\mathbb{E}\left[\int_0^{X}\left(e^{isu}-e^{-\rho u}\right)\mathrm{w}\left(u,X-u\right)du\right] \\ &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\cdot\sum_{j=1}^{N_T}\int_0^{X_j}\left(e^{-\rho u}-e^{-\widehat{\rho}u}\right)\mathrm{w}\left(u,X_j-u\right)du \\ &\quad + \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\cdot\sum_{j=1}^{N_T}\int_0^{X_j}\left(e^{isu}-e^{-\rho u}\right)\mathrm{w}\left(u,X_j-u\right)du - \frac{\lambda}{\lambda+\mu+\delta}\frac{\left(\beta-\rho\right)}{\rho+is}\cdot\mathbb{E}\left[\int_0^{X}\left(e^{isu}-e^{-\rho u}\right)\mathrm{w}\left(u,X-u\right)du\right] \coloneqq l_1+l_2,\end{aligned} \tag{77}$$

where

$$\begin{aligned}l_1 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\cdot\sum_{j=1}^{N_T}\int_0^{X_j}\left(e^{-\rho u}-e^{-\widehat{\rho}u}\right)\mathrm{w}\left(u,X_j-u\right)du, \\ l_2 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\cdot\sum_{j=1}^{N_T}\int_0^{X_j}\left(e^{isu}-e^{-\rho u}\right)\mathrm{w}\left(u,X_j-u\right)du - \frac{\lambda}{\lambda+\mu+\delta}\frac{\left(\beta-\rho\right)}{\rho+is}\cdot\mathbb{E}\left[\int_0^{X}\left(e^{isu}-e^{-\rho u}\right)\mathrm{w}\left(u,X-u\right)du\right].\end{aligned} \tag{78}$$

For l_1, by the mean value theorem we have

$$\left|e^{-\rho u} - e^{-\widehat{\rho} u}\right| \leq \left|\rho - \widehat{\rho}\right| u e^{-\rho^* u} \leq \left|\rho - \widehat{\rho}\right| u, \quad (79)$$

where ρ^* is a number between ρ and $\widehat{\rho}$. Then

$$\begin{aligned} \left|l_1\right| &\leq \frac{\widehat{\lambda}}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\rho} + is} \frac{1}{N_T} \\ &\cdot \sum_{j=1}^{N_T} \int_0^{X_j} u \left|\rho - \widehat{\rho}\right| \mathrm{w}\left(u, X_j - u\right) du = \frac{\widehat{\lambda}}{\widehat{\lambda} + \widehat{\mu} + \delta} \\ &\cdot \frac{\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\rho} + is} \left|\rho - \widehat{\rho}\right| \frac{1}{N_T} \sum_{j=1}^{N_T} H_1\left(X_j\right). \end{aligned} \quad (80)$$

Since $(1/N_T)\sum_{j=1}^{N_T} H_1(X_j) \xrightarrow{p} \|H_1(X)\|_{P,1} < \infty$, $\rho - \widehat{\rho} = O_p(T^{-1/2})$, we have

$$\left|l_1\right| = O_p\left(T^{-1/2}\right). \quad (81)$$

For l_2, we have

$$\begin{aligned} l_2 &= \frac{\widehat{\lambda}}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\rho} + is} \frac{1}{N_T} \\ &\cdot \sum_{j=1}^{N_T} \int_0^{X_j} \left(e^{isu} - e^{-\rho u}\right) \mathrm{w}\left(u, X_j - u\right) du \\ &- \frac{\lambda}{\lambda + \mu + \delta} \frac{(\beta - \rho)}{\rho + is} \\ &\cdot \mathbb{E}\left[\int_0^X \left(e^{isu} - e^{-\rho u}\right) \mathrm{w}(u, X - u) du\right] \\ &= \frac{\widehat{\lambda}\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{\rho + is}{\widehat{\rho} + is} \frac{1}{N_T} \\ &\cdot \sum_{j=1}^{N_T} \left[g_{2,s}\left(X_j\right) - \mathbb{E}\left(g_{2,s}(X)\right)\right] + (\rho + is) \\ &\cdot \left(\frac{\widehat{\lambda}\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{1}{\widehat{\rho} + is} - \frac{\lambda(\beta - \rho)}{\lambda + \mu + \delta} \frac{1}{\rho + is}\right) \\ &\cdot \mathbb{E}\left(g_{2,s}(X)\right), \end{aligned} \quad (82)$$

where

$$\begin{aligned} g_{2,s}(x) &= \int_0^x \frac{e^{isu} - e^{-\rho u}}{\rho + is} \mathrm{w}(u, x - u) du \\ &= \int_0^x \int_0^u e^{is(u-y)-\rho y} dy \mathrm{w}(u, x - u) du, \quad x \geq 0. \end{aligned} \quad (83)$$

Under condition $\|H_2(X)\|_{P,2} < \infty$, by the similar arguments on $g_{1,s}$ we conclude that

$$\begin{aligned} &\sup_{s \in [0, K\pi/a]} \left|\frac{1}{N_T} \sum_{j=1}^{N_T} \left[g_{2,s}\left(X_j\right) - \mathbb{E}\left(g_{2,s}(X)\right)\right]\right| \\ &= O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \end{aligned} \quad (84)$$

In addition, we have

$$\begin{aligned} &(\rho + is)\left(\frac{\widehat{\lambda}\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{1}{\widehat{\rho} + is} - \frac{\lambda(\beta - \rho)}{\lambda + \mu + \delta} \frac{1}{\rho + is}\right) \\ &= O_p\left(T^{-1/2}\right) \end{aligned} \quad (85)$$

due to $\widehat{\lambda} - \lambda = O_p(T^{-1/2})$, $\widehat{\beta} - \beta = O_p(T^{-1/2})$, $\widehat{\mu} - \mu = O_p(T^{-1/2})$, and $\widehat{\rho} - \rho = O_p(T^{-1/2})$. For $g_{2,s}(x)$, we have

$$\begin{aligned} &\sup_{s \in [0, K\pi/a]} \left|\mathbb{E}\left[g_{2,s}(X)\right]\right| \\ &\leq \int_0^\infty \int_0^x \int_0^u e^{-\rho y} dy \mathrm{w}(u, x - u) du f(x) dx \\ &\leq \int_0^\infty \int_0^x u \mathrm{w}(u, x - u) du f(x) dx = \left\|H_1(X)\right\|_{P,1} \\ &< \infty, \end{aligned} \quad (86)$$

thus

$$\begin{aligned} &\sup_{s \in [0, K\pi/a]} \left|(\rho + is)\right. \\ &\cdot \left(\frac{\widehat{\lambda}\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{1}{\widehat{\rho} + is} - \frac{\lambda(\beta - \rho)}{\lambda + \mu + \delta} \frac{1}{\rho + is}\right) \\ &\left.\cdot \mathbb{E}\left(g_{2,s}(X)\right)\right| \leq \left|(\rho + is)\right. \\ &\left.\cdot \left(\frac{\widehat{\lambda}\left(\widehat{\beta} - \widehat{\rho}\right)}{\widehat{\lambda} + \widehat{\mu} + \delta} \frac{1}{\widehat{\rho} + is} - \frac{\lambda(\beta - \rho)}{\lambda + \mu + \delta} \frac{1}{\rho + is}\right)\right| \\ &\cdot \left\|H_1(X)\right\|_{p,1} = O_p\left(T^{-1/2}\right). \end{aligned} \quad (87)$$

By (84) and (87), we derive that

$$\sup_{s \in [0, K\pi/a]} \left|l_2\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \quad (88)$$

Furthermore, (81) and (88) give

$$\begin{aligned} &\sup_{s \in [0, K\pi/a]} \left|I_2\right| \leq \sup_{s \in [0, K\pi/a]} \left|l_1\right| + \sup_{s \in [0, K\pi/a]} \left|l_2\right| \\ &= O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \end{aligned} \quad (89)$$

Combining (76) and (89), we eventually obtain the desired result. □

Theorem 8. *Suppose that Condition 1 holds,* $\mathbb{E}[X^k] < \infty$, $k = 1,2,3,4$. *Then for large* a, K, *and* T, *we have*

$$\sup_{s\in[0,K\pi/a]}\left|\mathscr{F} f_\delta(s) - \widehat{\mathscr{F} f_\delta}(s)\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \quad (90)$$

Proof. By (21) and (44),

$$\begin{aligned}
&\widehat{\mathscr{F} f_\delta}(s) - \mathscr{F} f_\delta(s)\\
&\quad = \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\widehat{\mathscr{F} f}(s) - \frac{\lambda}{\lambda+\mu+\delta}\mathscr{F} f(s)\\
&\qquad + \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{(\widehat{\beta}-\widehat{\rho})}{\widehat{\rho}+is}\left[\widehat{\mathscr{F} f}(s) - \widehat{\mathscr{L} f}(\widehat{\rho})\right]\\
&\qquad - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\left[\mathscr{F} f(s) - \mathscr{L} f(\rho)\right]\\
&\quad := II_1 + II_2,
\end{aligned} \quad (91)$$

where

$$\begin{aligned}
II_1 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\widehat{\mathscr{F} f}(s) - \frac{\lambda}{\lambda+\mu+\delta}\mathscr{F} f(s),\\
II_2 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{(\widehat{\beta}-\widehat{\rho})}{\widehat{\rho}+is}\left[\widehat{\mathscr{F} f}(s) - \widehat{\mathscr{L} f}(\widehat{\rho})\right]\\
&\quad - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\left[\mathscr{F} f(s) - \mathscr{L} f(\rho)\right].
\end{aligned} \quad (92)$$

First, we study II_1. Note that

$$\begin{aligned}
II_1 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\widehat{\mathscr{F} f}(s) - \frac{\lambda}{\lambda+\mu+\delta}\mathscr{F} f(s)\\
&= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{isX_j} - \mathbb{E}\left[e^{isX}\right]\right]\\
&\quad + \left(\frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\mathbb{E}\left[e^{isX}\right]\\
&= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{3,s}(X_j) - \mathbb{E}\left[g_{3,s}(X)\right]\right]\\
&\quad + \left(\frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\mathbb{E}\left[g_{3,s}(X)\right],
\end{aligned} \quad (93)$$

where

$$g_{3,s}(x) = e^{isx}, \quad x \ge 0. \quad (94)$$

For $g_{3,s}(x)$, we have

$$\begin{aligned}
\sup_{s\in[0,K\pi/a]}\left|\mathbb{E}\left[g_{3,s}(X)\right]\right| &= \sup_{s\in[0,K\pi/a]}\left|\int_0^\infty e^{isx} f(x)\,dx\right|\\
&\le 1 < \infty,
\end{aligned} \quad (95)$$

leading to

$$\begin{aligned}
&\sup_{s\in[0,K\pi/a]}\left|\left(\frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right)\mathbb{E}\left[g_{3,s}(X)\right]\right|\\
&\quad \le \left|\frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta} - \frac{\lambda}{\lambda+\mu+\delta}\right| = O_p\left(T^{-1/2}\right).
\end{aligned} \quad (96)$$

By the similar arguments on $g_{1,s}$, when $E[X] < \infty$ we conclude that

$$\begin{aligned}
&\sup_{s\in[0,K\pi/a]}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{3,s}(X_j) - \mathbb{E}\left(g_{3,s}(X)\right)\right]\right|\\
&\quad = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right).
\end{aligned} \quad (97)$$

Then (96) and (97) give

$$\sup_{s\in[0,K\pi/a]}\left|II_1\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \quad (98)$$

Next, we study II_2. Note that

$$\begin{aligned}
II_2 &= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{(\widehat{\beta}-\widehat{\rho})}{\widehat{\rho}+is}\left[\widehat{\mathscr{F} f}(s) - \widehat{\mathscr{L} f}(\widehat{\rho})\right]\\
&\quad - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\left[\mathscr{F} f(s) - \mathscr{L} f(\rho)\right]\\
&= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{(\widehat{\beta}-\widehat{\rho})}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{isX_j} - e^{-\widehat{\rho}X_j}\right]\\
&\quad - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\mathbb{E}\left[e^{isX} - e^{-\rho X}\right]\\
&= \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{(\widehat{\beta}-\widehat{\rho})}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{-\rho X_j} - e^{-\widehat{\rho}X_j}\right]\\
&\quad + \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{(\widehat{\beta}-\widehat{\rho})}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{isX_j} - e^{-\rho X_j}\right]\\
&\quad - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\mathbb{E}\left[e^{isX} - e^{-\rho X}\right] := ll_1 + ll_2,
\end{aligned} \quad (99)$$

where

$$ll_1 = \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{-\rho X_j}-e^{-\widehat{\rho}X_j}\right],$$

$$\begin{aligned} ll_2 = {} & \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{isX_j}-e^{-\rho X_j}\right] \\ & - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\mathbb{E}\left[e^{isX}-e^{-\rho X}\right]. \end{aligned} \tag{100}$$

By the similar arguments on l_1 in the proof of Theorem 6, when $\mathbb{E}[X]<\infty$ we can derive that

$$\left|ll_1\right| = O_p\left(T^{-1/2}\right). \tag{101}$$

For ll_2, we have

$$\begin{aligned} ll_2 = {} & \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[e^{isX_j}-e^{-\rho X_j}\right] \\ & - \frac{\lambda}{\lambda+\mu+\delta}\frac{(\beta-\rho)}{\rho+is}\mathbb{E}\left[e^{isX}-e^{-\rho X}\right] = \frac{\widehat{\lambda}}{\widehat{\lambda}+\widehat{\mu}+\delta} \\ & \cdot \frac{\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\rho}+is}\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{4,s}\left(X_j\right)-\mathbb{E}\left(g_{4,s}(X)\right)\right] \\ & + (\rho+is)\left(\frac{\widehat{\lambda}\left(\widehat{\beta}-\widehat{\rho}\right)}{\widehat{\lambda}+\widehat{\mu}+\delta}\frac{1}{\widehat{\rho}+is} - \frac{\lambda(\beta-\rho)}{\lambda+\mu+\delta}\frac{1}{\rho+is}\right) \\ & \cdot \mathbb{E}\left(g_{4,s}(X)\right), \end{aligned} \tag{102}$$

where

$$g_{4,s}(x) = \frac{e^{isx}-e^{-\rho x}}{\rho+is} = \int_0^{\infty} e^{is(x-y)-\rho y}dy, \quad x \ge 0. \tag{103}$$

Under condition $\mathbb{E}[X^k]<\infty$, $k=1,2,3,4$, by the similar arguments on $g_{1,s}$ we conclude that

$$\begin{aligned} & \sup_{s\in[0,K\pi/a]}\left|\frac{1}{N_T}\sum_{j=1}^{N_T}\left[g_{4,s}\left(X_j\right)-\mathbb{E}\left(g_{4,s}(X)\right)\right]\right| \\ & = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \end{aligned} \tag{104}$$

In addition, it follows from (104) and the similar analysis of l_2 in the proof of Theorem 6 that

$$\sup_{s\in[0,K\pi/a]}\left|ll_2\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \tag{105}$$

Then (101) and (105) give

$$\begin{aligned} \sup_{s\in[0,K\pi/a]}\left|II_2\right| & \le \sup_{s\in[0,K\pi/a]}\left|ll_2\right| + \sup_{s\in[0,K\pi/a]}\left|ll_2\right| \\ & = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \end{aligned} \tag{106}$$

Finally, we can derive the desired result by (98) and (106). □

Based on the above conclusions we have the following result.

Theorem 9. *Suppose that Condition 1 holds,* $\mathbb{E}[X^k]<\infty$, $k=1,2,3,4$, $\|H_j(X)\|_{P,1}<\infty$, $j=0,1$, *and* $\|H_j(X)\|_{P,2}<\infty$, $j=1,2$. *Then for large* a, K *and* T, *we have*

$$\left\|\Phi_{K,a}-\widehat{\Phi}_{K,a}\right\|^2 = O_p\left(\frac{(K/a)\log(K/a)}{T}\right). \tag{107}$$

Proof. First, we have

$$\begin{aligned} \left\|\Phi_{K,a}-\widehat{\Phi}_{K,a}\right\|^2 & = \int_0^a\left|\Phi_{K,a}(u)-\widehat{\Phi}_{K,a}(u)\right|^2du \\ & \le \sum_{k=0}^{K-1}\frac{4}{a^2}\left(\operatorname{Re}\left\{\mathscr{F}\Phi\left(\frac{k\pi}{a}\right)-\widehat{\mathscr{F}\Phi}\left(\frac{k\pi}{a}\right)\right\}\right)^2 \\ & \cdot \int_0^a\left(\cos\left(\frac{k\pi}{a}u\right)\right)^2du = \frac{2}{a} \\ & \cdot \sum_{k=0}^{K-1}\left|\mathscr{F}\Phi\left(\frac{k\pi}{a}\right)-\widehat{\mathscr{F}\Phi}\left(\frac{k\pi}{a}\right)\right|^2 \le \frac{2K}{a} \\ & \cdot \sup_{s\in[0,K\pi/a]}\left|\mathscr{F}\Phi(s)-\widehat{\mathscr{F}\Phi}(s)\right|^2. \end{aligned} \tag{108}$$

Then by (17) and (44) we obtain

$$\begin{aligned} \sup_{s\in[0,K\pi/a]}\left|\mathscr{F}\Phi(s)-\widehat{\mathscr{F}\Phi}(s)\right| & = \sup_{s\in[0,K\pi/a]}\left|\frac{\mathscr{F}H(s)}{1-\mathscr{F}f_\delta(s)}-\frac{\widehat{\mathscr{F}H}(s)}{1-\widehat{\mathscr{F}f_\delta}(s)}\right| \\ & = \sup_{s\in[0,K\pi/a]}\left|\frac{\left(1-\mathscr{F}f_\delta(s)\right)\left(\mathscr{F}H(s)-\widehat{\mathscr{F}H}(s)\right)+\mathscr{F}H(s)\left(\mathscr{F}f_\delta(s)-\widehat{\mathscr{F}f_\delta}(s)\right)}{\left(1-\mathscr{F}f_\delta(s)\right)\left(1-\widehat{\mathscr{F}f_\delta}(s)\right)}\right|. \end{aligned} \tag{109}$$

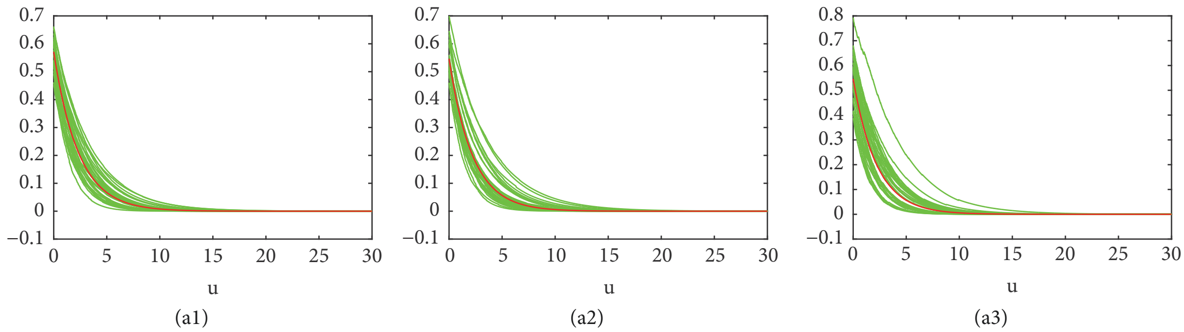

Figure 1: Estimation of the Gerber-Shiu function for exponential claim sizes: true value curves (red curves) and 30 estimated curves (green curves) when ($T = 52$). (a1) Ruin probability. (a2) Laplace transform of ruin time. (a3) Expected discounted deficit at ruin due to a claim.

Combining Theorems 6 and 7 gives

$$\sup_{s\in[0,K\pi/a]}\left|\mathscr{F}\Phi(s)-\widehat{\mathscr{F}\Phi}(s)\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \tag{110}$$

Finally, by (108) we derive that

$$\begin{aligned}\left\|\Phi_{K,a}-\widehat{\Phi}_{K,a}\right\|^2 &\le \frac{2K}{a}\sup_{s\in[0,K\pi/a]}\left|\mathscr{F}\Phi(s)-\widehat{\mathscr{F}\Phi}(s)\right|^2\\ &= O_p\left(\frac{(K/a)\log(K/a)}{T}\right).\end{aligned} \tag{111}$$

This completes the proof. □

Combining Theorems 5 and 9, we finally obtain the following convergence rate:

$$\begin{aligned}\left\|\Phi-\widehat{\Phi}_{K,a}\right\|^2 &= O\left(\frac{K+1}{a^{2m+1}}\right)+O\left(\frac{a}{K-1}\right)\\ &\quad + O_p\left(\frac{(K/a)\log(K/a)}{T}\right).\end{aligned} \tag{112}$$

We first find the optimal truncation parameter $a^* = O(K^{-(n+1)})$ to minimize the convergence rate $O((K+1)/a^{2m+1}) + O(a/(K-1))$. Replacing a by a^* in (112) gives

$$\begin{aligned}\left\|\Phi-\widehat{\Phi}_{K,a}\right\|^2 &= O\left(K^{-m/(m+1)}\right)\\ &\quad + O_p\left(\frac{K^{m/(m+1)}\log(K)}{T}\right).\end{aligned} \tag{113}$$

In addition, we find the optimal truncation $K^* = O(T^{(m+1)/2m})$. Thus we obtain the smallest convergence rate $\|\Phi-\widehat{\Phi}_{K,a}\|^2 = O_p(T^{-2m^2/(m+1)^2})$.

5. Simulations

In this section, we present some simulation examples to illustrate the performance of our estimator when sample is finite. Both the exponential claims with density function $f(x) = e^{-x}$, $x > 0$ and the Erlang(2) claims with density function $f(x) = 4xe^{-2x}$, $x > 0$ are considered in our simulation. We estimate the following three specific Gerber-Shiu functions: (1) ruin probability (RP: $w \equiv 1$, $\delta = 0$); (2) Laplace transform of ruin time (LT: $w \equiv 1$, $\delta = 0.1$); (3) expected discounted deficit at ruin (EDD) when ruin is due to a claim ($w(x,y) = y$, $\delta = 0.1$).

Suppose that, in a long time interval, the insurer will observe the data once a week; thus we set $\Delta = 1$, which can be explained as one week. In all cases, we set $\lambda = 2$, $\mu = 5$; that is to say, the expected claim number is 2 times per week, and the expected premium number is 5 times per week. Furthermore, since there are 52 business weeks every year, we assume that $T = 1 \times 52\times\Delta$, $T = 3 \times 52\times\Delta$, $T = 5 \times 52\times\Delta$, $T = 7 \times 52\times\Delta$. Then we use formula (46) to estimate RP, LT, EDD due to a claim. In all simulations, we set $a = 30$, $K = 2^{13}$, and we carry out the relevant analysis based on 300 times simulation results. In this section, the mean value and the mean relative error are, respectively, defined by

$$\begin{gathered}\frac{1}{300}\sum_{j=1}^{300}\widehat{\Phi}_{K,a,j}(u),\\ \frac{1}{300}\sum_{j=1}^{300}\left|\frac{\widehat{\Phi}_{K,a,j}(u)}{\Phi(u)}-1\right|,\end{gathered} \tag{114}$$

and the integrated mean square error (IMSE) is defined by

$$\begin{aligned}&\frac{1}{300}\sum_{j=1}^{300}\int_0^\infty\left(\widehat{\Phi}_{K,a,j}(u)-\Phi(u)\right)^2du\\ &\quad\approx\frac{1}{300}\sum_{j=1}^{300}\int_0^{30}\left(\widehat{\Phi}_{K,a,j}(u)-\Phi(u)\right)^2du,\end{aligned} \tag{115}$$

where $\widehat{\Phi}_{K,a,j}(u)$ is the estimate of Gerber-Shiu function in the j-th experiment. For IMSE, we compute the integral on the finite domain $[0,30]$ in that both the true value and the estimator will be very small when $u \ge 30$.

First, to show variability bands and illustrate the stability of our estimation procedures, when $T = 52$, we plot 30 consecutive estimate curves (green curves) of Gerber-Shiu functions on the same picture together with the associated truth curves (red curves) in Figures 1 and 2. In all examples,

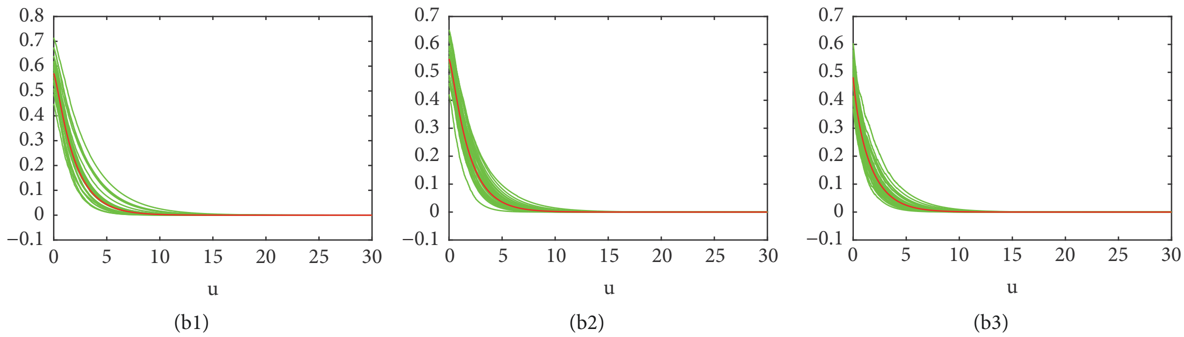

Figure 2: Estimation of the Gerber-Shiu function for Erlang(2) claim sizes: true value curves (red curves) and 30 estimated curves (green curves) when (T = 52). (b1) Ruin probability. (b2) Laplace transform of ruin time. (b3) Expected discounted deficit at ruin due to a claim.

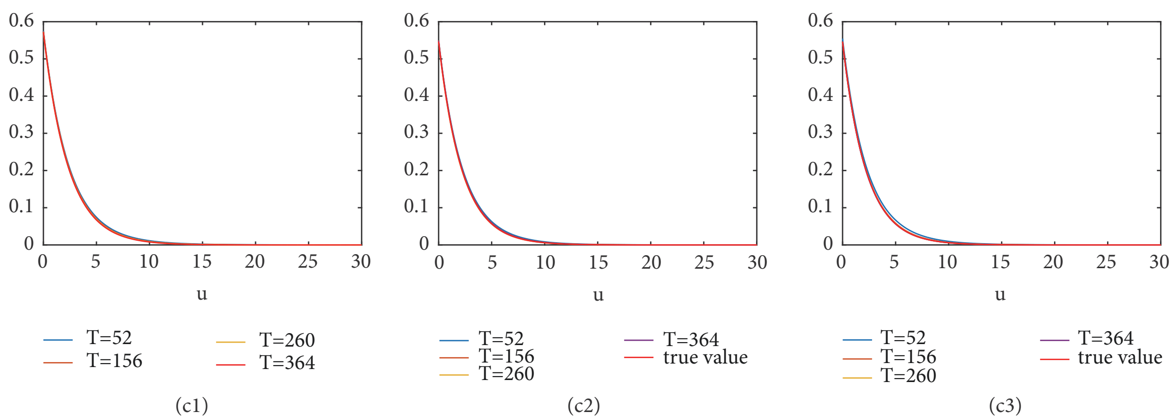

Figure 3: Estimation of the Gerber-Shiu function for exponential claim sizes: mean curves. (c1) Ruin probability. (c2) Laplace transform of ruin time. (c3) Expected discounted deficit at ruin due to a claim.

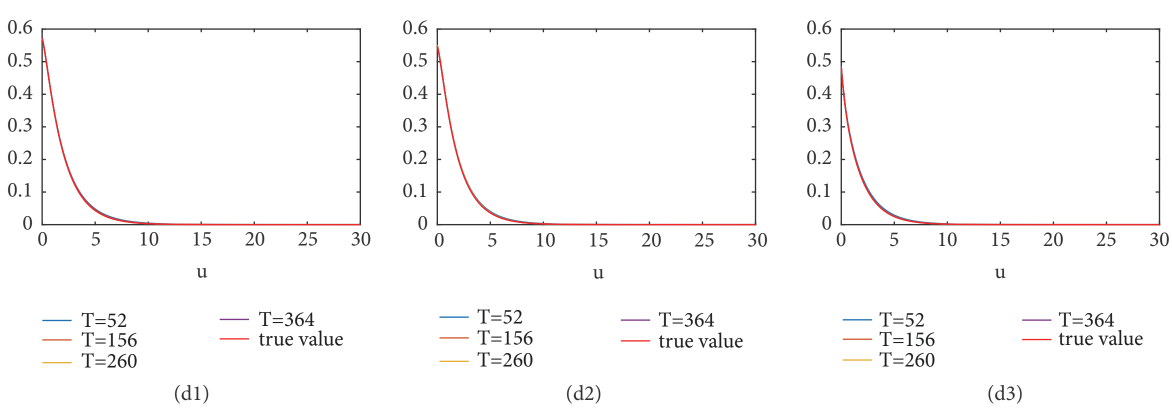

Figure 4: Estimation of the Gerber-Shiu function for Erlang(2) claim sizes: mean curves. (d1) Ruin probability. (d2) Laplace transform of ruin time. (d3) Expected discounted deficit at ruin due to a claim.

we observe that all estimate curves are very close and they have the same trend with their reference curves. This shows that our estimations can match the real functions perfectly and they have good stationarity. Next, in Figures 3 and 4, we show the mean value curves for different observe intervals (namely, different T) and compared them with the mean value curves. It is obvious that our estimations are accurate in the sense of mean and the mean value curves converge to the true value curves as T increases. In addition, Figures 5 and 6 display the mean relative error curves of our estimations for different observe intervals. We find that the mean relative errors increase with the increase of u. They are small when u is small; however, they are very large when u becomes large. We can also observe that the mean relative errors decrease as T increases. This trend shows that our estimations perform better with larger T.

Finally, we compare the Fourier-cosine series expansion method with the FFT method used in [45]. The parameter

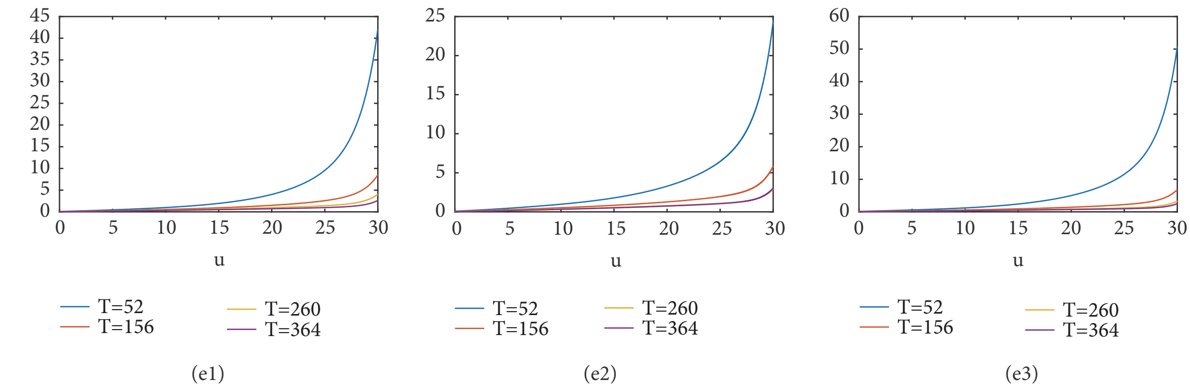

FIGURE 5: Estimation of the Gerber-Shiu function for exponential claim sizes: mean relative error curves. (e1) Ruin probability. (e2) Laplace transform of ruin time. (e3) Expected discounted deficit at ruin due to a claim.

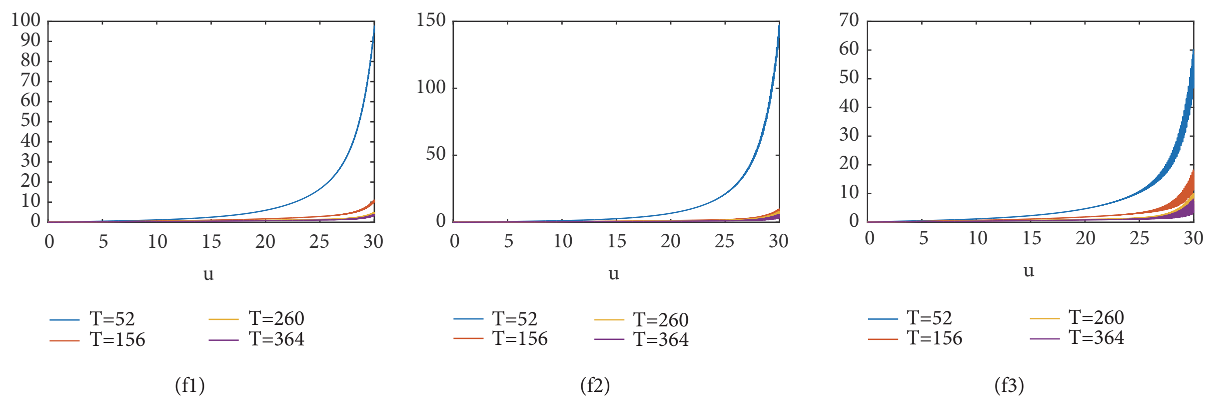

FIGURE 6: Estimation of the Gerber-Shiu function for Erlang(2) claim sizes: mean relative error curves. (f1) Ruin probability. (f2) Laplace transform of ruin time. (f3) Expected discounted deficit at ruin due to a claim.

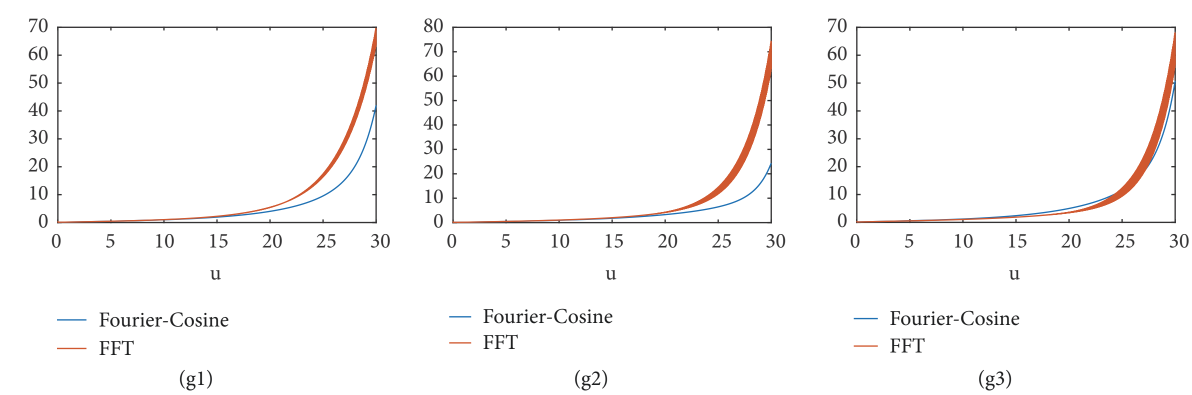

FIGURE 7: Comparing Fourier-cosine method with FFT method for exponential claim sizes when $T = 52$: mean relative error curves. (g1) Ruin probability. (g2) Laplace transform of ruin time. (g3) Expected discounted deficit at ruin due to a claim.

setting of FFT is the same as in [45]. First, we present the IMSE values for both methods in Table 1, and we find that the Fourier-cosine series expansion method can lead to smaller IMSEs compared with FFT method. Besides, we use the mean relative error to compare these two methods when $T = 52$. The results are illustrated in Figures 7 and 8. It is easily seen that Fourier-cosine series expansion method can yield smaller mean relative errors.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Table 1: IMSE for the estimation of Gerber-Shiu function.

		Exponential distribution			Erlang(2) distribution		
	T	RP	LT	EDD	RP	LT	EDD
Fouroer-Cosine	52	0.0242	0.0160	0.0378	0.0164	0.0130	0.0090
	156	0.0081	0.0056	0.0109	0.0056	0.0031	0.0038
	260	0.0047	0.0030	0.0064	0.0028	0.0023	0.0017
	364	0.0033	0.0021	0.0046	0.0021	0.0015	0.0014
FFT	52	0.0209	0.0171	0.0332	0.0169	0.0114	0.0090
	156	0.0086	0.0065	0.0117	0.0068	0.0052	0.0042
	260	0.0059	0.0051	0.0079	0.0049	0.0038	0.0031
	364	0.0048	0.0043	0.0054	0.0039	0.0032	0.0024

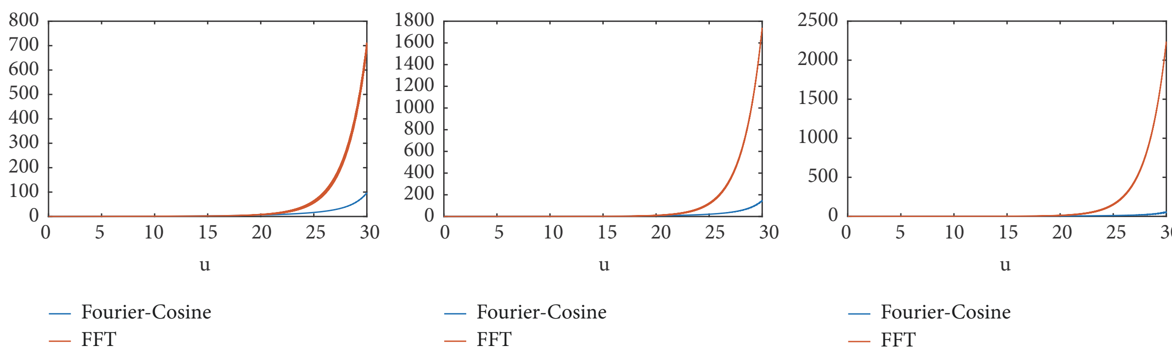

Figure 8: Comparing Fourier-cosine method with FFT method for Erlang(2) claim sizes when $T = 52$: mean relative error curves. (h1) Ruin probability. (h2) Laplace transform of ruin time. (h3) Expected discounted deficit at ruin due to a claim.

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