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Research Article

Dynamics of an Impulsive Stochastic Nonautonomous Chemostat Model with Two Different Growth Rates in a Polluted Environment

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This paper proposes a novel impulsive stochastic nonautonomous chemostat model with the saturated and bilinear growth rates in a polluted environment. Using the theory of impulsive differential equations and Lyapunov functions method, we first investigate the dynamics of the stochastic system and establish the sufficient conditions for the extinction and the permanence of the microorganisms. Then we demonstrate that the stochastic periodic system has at least one nontrivial positive periodic solution. The results show that both impulsive toxicant input and stochastic noise have great effects on the survival and extinction of the microorganisms. Furthermore, a series of numerical simulations are presented to illustrate the performance of the theoretical results.

1. Introduction

The chemostat is a continuous culture device that keeps the flow of the nutrient solution constant and makes the microorganism reproduce under the condition of its maximum growth rate. In a chemostat, the density of microorganisms is controlled by the concentration of the growth-limiting nutrient, and the growth rate is controlled by the washout rate which can be adjusted arbitrarily. The chemostat model is mainly used for laboratory theoretical research. So far, many scholars have obtained significant results for chemostat models [1–11]. While the populations in nature suffer from instantaneous discontinuous interference (for example, toxic input, seasonal harvest, and spraying pesticides), this interference phenomenon can be described as an impulse mathematically [12–16]. Furthermore, the theory and method of impulsive differential equations are widely used in many domains of biological science [17–20]. In reality, many populations are inevitably affected by stochastic disturbances [21–28], and many scholars have explored many stochastic dynamic systems and obtained some new results [29–33]. Recently, impulsive stochastic dynamics models attract the research interests of scholars [34–37]. Therefore, we also consider the influence of stochastic disturbances and impulsive toxic input on the chemostat.

In previous studies, there are almost works on research of chemostat models with one of some different growth rates. In fact, microorganisms take different forms to absorb nutrients at different times, so the growth rates of microorganisms are different. For example, some take saturated growth rate and some adopt bilinear growth rate. Consequently, a model can utilize the transfer function as $T(S,x) = \sum_{i=1}^{n} p_i T_i(S,x)$, where p_i $(i = 1,2,\ldots,n)$ represent the probability of occurrence of $T_i(S,x)$ $(i = 1,2,\ldots,n)$ and $\sum_{i=1}^{n} p_i = 1$ [38]. In this paper, we consider two types of growth rates, bilinear growth rate and saturated growth rate. Hence we assume $T(S,x) = p\beta_1 Sx + (1-p)(\beta_2 Sx/(a+S))$, where p $(0 < p < 1)$ denotes the probability of occurrence of bilinear growth rate and $1-p$ is correspondingly the probability of occurrence of saturated growth rate. In addition, we also consider the impact of time on the system. Then the system becomes nonautonomous:

$$\begin{aligned}\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= D(t)\left(S_0(t)-S(t)\right)-p(t)\beta_1(t)S(t)x(t)\\ &\quad-\left(1-p(t)\right)\frac{\beta_2(t)S(t)x(t)}{a(t)+S(t)},\\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= p(t)\beta_1(t)S(t)x(t)\\ &\quad+\left(1-p(t)\right)\frac{\beta_2(t)S(t)x(t)}{a(t)+S(t)}-D(t)x(t),\end{aligned} \tag{1}$$

where $S(t)$ represents the concentration of the unconsumed nutrient at time t and $x(t)$ represents the biomass of the population of microorganism at time t. $D(t)$ is the washout rate at time t, $S_0(t)$ is the concentration of the growth-limiting nutrient at time t, $T(S,x)$ represents the nutrient intake rate of each individual in microbial population, and it also describes the conversion rate from nutrient to microorganism. $a(t)$ is the semisaturated rate at time t.

As we know, environmental pollution is one of the most important social and ecological problems at present. It affects the quality of life of human beings, the persistence of species, and the ecological vicissitude of habitat. In order to reasonably apply and control toxic substances, we must evaluate the degree of toxicant's damage to the population. In recent years, many scholars have already studied the effects of toxicant on various ecosystems [39–44]. In this paper, we use impulsive differential equations to describe the effects of toxicant on microbial population. Besides toxicant, we also consider the effects of environmental noises. In this paper, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in two types of growth response rate, so that $\beta_i \longrightarrow \beta_i + \sigma_i \dot{B}_i$ $(i = 1,2)$. Then, a nonautonomous impulsive chemostat model with stochastic perturbation in a polluted environment takes the following form:

$$\begin{aligned}\mathrm{d}S(t) &= \Big(D(t)\left(S_0(t)-S(t)\right)-p(t)\beta_1(t)S(t)x(t)\\ &\quad-\left(1-p(t)\right)\frac{\beta_2(t)S(t)x(t)}{a(t)+S(t)}\Big)\mathrm{d}t-p(t)\sigma_1(t)\\ &\quad\cdot S(t)x(t)\,\mathrm{d}B_1(t)-\left(1-p(t)\right)\\ &\quad\cdot\frac{\sigma_2(t)S(t)x(t)}{a(t)+S(t)}\mathrm{d}B_2(t),\\ \mathrm{d}x(t) &= \Big(p(t)\beta_1(t)S(t)+\left(1-p(t)\right)\frac{\beta_2(t)S(t)}{a(t)+S(t)}\\ &\quad-r(t)C_0(t)-D(t)\Big)x(t)\,\mathrm{d}t+p(t)\sigma_1(t)S(t)\\ &\quad\cdot x(t)\,\mathrm{d}B_1(t)+\left(1-p(t)\right)\\ &\quad\cdot\frac{\sigma_2(t)S(t)x(t)}{a(t)+S(t)}\mathrm{d}B_2(t),\\ \mathrm{d}C_0(t) &= \left(kC_e(t)-gC_0(t)-mC_0(t)\right)\mathrm{d}t,\\ \mathrm{d}C_e(t) &= -hC_e(t)\,\mathrm{d}t,\\ &\qquad t\neq n\tau,\ n\in Z^+,\\ \Delta S(t) &= 0,\\ \Delta x(t) &= 0,\\ \Delta C_0(t) &= 0,\\ \Delta C_e(t) &= u,\\ &\qquad t=n\tau,\ n\in Z^+,\end{aligned} \tag{2}$$

where $C_0(t)$ and $C_e(t)$ denote the concentrations of the toxicant in the organism and in the environment at time t, respectively. $r(t) > 0$ is the rate of decrease of the intrinsic growth rate at time t because of toxicant, k represents environmental toxicant uptake rate per unit mass organism, g and m are organismal net ingestion and depuration rates of toxicant, respectively, h denotes the loss rate of toxicant from the environment itself by volatilization, and u is the amount of pulsed input concentration of the toxicant at each τ. Moreover, all the above parameters are positive and $S(t)$, $x(t)$, $C_0(t)$, and $C_e(t)$ are both positive functions according to its biomathematics meaning, and $D(t)$, $S_0(t)$, $p(t)$, $\beta_1(t)$, $\beta_2(t)$, $a(t)$, $r(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are both periodic functions with $T = n\tau$.

The paper is organized as follows. Preliminaries are provided in Section 2. Existence and uniqueness of the global positive solution are demonstrated in Section 3. In Section 4, we explore the sufficient conditions for the extinction and the permanence of system (2). In Section 5, we show that the stochastic system has at least one nontrivial positive periodic solution by constructing a suitable Lyapunov function and a rectangular set. Finally, some numerical simulations are developed to illustrate the performance of the theoretical results.

2. Preliminaries

In this section, we give some notations and lemmas which will be used for the following reasoning process. Throughout this paper, we assume that $S(t)$, $x(t)$, and $C_0(t)$ are continuous at $t = n\tau$ and $C_e(t)$ is left continuous at $t = n\tau$ and $C_e(n\tau^+) = \lim_{t\longrightarrow n\tau^+} C_e(t)$, and let $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0}, \mathscr{P})$ be a complete probability space with a filtration $\mathscr{F}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathscr{P}$-null sets). For an integrable function $f(t)$ on $[0,\infty)$, define $f^u = \sup_{t\in[0,\infty)} f(t)$, $f^l = \inf_{t\in[0,\infty)} f(t)$, and $\langle f(t)\rangle_t = (1/t)\int_0^t f(s)\mathrm{d}s$; here $f(t)$ is a bounded function on $[0,\infty)$. Due to the properties of continuous positive periodic functions, it is obvious that the coefficients of the system (2) satisfy $\max\{D^u, S_0^u, p^u, \beta_1^u, \beta_2^u, a^u, r^u, \sigma_1^u, \sigma_2^u\} < \infty$ and $\min\{D^l, S_0^l, p^l, \beta_1^l, \beta_2^l, a^l, r^l, \sigma_1^l, \sigma_2^l\} \geq 0$.

Lemma 1. *For any positive solution $(S(t), x(t), C_0(t), C_e(t))$ of system (2) with initial value $(S(0), x(0), C_0(0), C_e(0^+)) \in \mathbb{R}_+^4$, we have $\limsup_{t\longrightarrow+\infty} S(t) \leq S_0^u$, $\limsup_{t\longrightarrow+\infty} x(t) \leq S_0^u$.*

Proof. From the first two equations of system (2), we have

$$\begin{aligned}\frac{\mathrm{d}(S(t)+x(t))}{\mathrm{d}t} &= D(t)S_0(t) - r(t)C_0(t)x(t) \\ &\quad - D(t)(s(t)+x(t)) \\ &\le D(t)S_0(t) - D(t)(s(t)+x(t)) \\ &\le D(t)S_0^u - D(t)(s(t)+x(t)),\end{aligned} \tag{3}$$

which implies that $\limsup_{t\longrightarrow+\infty} S(t) \le S_0^u$, $\limsup_{t\longrightarrow+\infty} x(t) \le S_0^u$. This completes the proof of Lemma 1. □

Lemma 2 (see [3]). *(i) The microorganism $x(t)$ is said to be extinctive if $\lim_{t\longrightarrow+\infty} x(t) = 0$.*

(ii) The species $x(t)$ is said to be permanent in the mean if there exists a positive constant λ such that $\lim_{t\longrightarrow+\infty}\langle x(t)\rangle \ge \lambda$.

Lemma 3 (see [45]). *Assume that*

(i) system has a unique global solution;

(ii) there is a function $V(t,x) \in C^2$ which is τ-periodic in t and satisfies the following conditions:

$$\inf_{|x|>\mathscr{R}} V(t,x) \longrightarrow \infty \quad \text{as } \mathscr{R} \longrightarrow \infty, \tag{4}$$

and

$$\mathscr{L}V \le -1 \quad \text{outside some compact set}, \tag{5}$$

where the operator $\mathscr{L}$ is given by

$$\begin{aligned}\mathscr{L}V &= V_t(t,x) + V_x(t,x)f(t,x) \\ &\quad + \frac{1}{2}\operatorname{trace}\left(g^T(t,x)V_{xx}(t,x)g(t,x)\right).\end{aligned} \tag{6}$$

Then the system has a τ-periodic solution.

Now we consider the following subsystem of system (2):

$$\begin{aligned}&\mathrm{d}C_0(t) = \left(kC_e(t) - gC_0(t) - mC_0(t)\right)\mathrm{d}t, \\ &\mathrm{d}C_e(t) = -hC_e(t)\,\mathrm{d}t, \\ &\qquad\qquad t \ne n\tau,\ n \in Z^+, \\ &\Delta C_0(t) = 0, \\ &\Delta C_e(t) = u, \\ &\qquad\qquad t = n\tau,\ n \in Z^+.\end{aligned} \tag{7}$$

Lemma 4 (see [3, 20]). *System (7) has a unique positive τ-periodic solution $(C_0^*(t), C_e^*(t))^T$ and for each solution $(C_0(t), C_e(t))^T$ of (7), $C_0(t) \longrightarrow C_0^*(t)$, and $C_e(t) \longrightarrow C_e^*(t)$ as $t \longrightarrow +\infty$. Moreover, $C_0(t) > C_0^*(t)$, $C_e(t) > C_e^*(t)$ for all $t \ge 0$ if $C_0(0) > C_0^*(0)$, and $C_e(0) > C_e^*(0)$, where*

$$\begin{aligned}C_0^*(t) &= C_0^*(0)e^{-(g+m)(t-n\tau)} \\ &\quad + \frac{ku\left(e^{-(g+m)(t-n\tau)} - e^{-h(t-n\tau)}\right)}{(h-g-m)\left(1-e^{-h\tau}\right)}, \\ C_e^*(t) &= \frac{ue^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ C_0^*(0) &= \frac{ku\left(e^{-(g+m)\tau} - e^{-h\tau}\right)}{(h-g-m)\left(1-e^{-(g+m)\tau}\right)\left(1-e^{-h\tau}\right)}, \\ C_e^*(0) &= \frac{u}{1-e^{-h\tau}},\end{aligned} \tag{8}$$

for $t \in (n\tau, (n+1)\tau]$ and $n \in Z^+$.

From Lemma 4, we have

$$\lim_{t\longrightarrow+\infty}\langle C_0(t)\rangle_t = \frac{ku}{h(g+m)\tau} \triangleq \overline{C_0}. \tag{9}$$

3. Existence and Uniqueness of the Global Positive Solution

This section discusses the existence and uniqueness of the global positive solution as an important prerequisite for further research. We take into account the following subsystem of system (2):

$$\begin{aligned}\mathrm{d}S(t) &= \Bigg(D(t)\left(S_0(t) - S(t)\right) - p(t)\beta_1(t)S(t)x(t) \\ &\quad - (1-p(t))\frac{\beta_2(t)S(t)x(t)}{a(t)+S(t)}\Bigg)\mathrm{d}t - p(t)\sigma_1(t) \\ &\quad \cdot S(t)x(t)\,\mathrm{d}B_1(t) - (1-p(t)) \\ &\quad \cdot \frac{\sigma_2(t)S(t)x(t)}{a(t)+S(t)}\mathrm{d}B_2(t), \\ \mathrm{d}x(t) &= \Bigg(p(t)\beta_1(t)S(t)x(t) \\ &\quad + (1-p(t))\frac{\beta_2(t)S(t)x(t)}{a(t)+S(t)} - r(t)C_0(t)x(t) \\ &\quad - D(t)x(t)\Bigg)\mathrm{d}t + p(t)\sigma_1(t)S(t)x(t)\,\mathrm{d}B_1(t) \\ &\quad + (1-p(t))\frac{\sigma_2(t)S(t)x(t)}{a(t)+S(t)}\mathrm{d}B_2(t),\end{aligned} \tag{10}$$

where the bounded function $C_0(t)$ is defined by system (7).

Theorem 5. *For any initial value $(S(0), x(0)) \in \mathbb{R}_+^2$, there is a unique positive solution $(S(t), x(t))$ of system (2) on $t \ge 0$, and the solution will remain in $\mathbb{R}_+^2$ with probability one.*

Proof. Owing to the coefficients of the system (10) obey local Lipschitz conditions; there exists a unique solution on $[0, \tau_e)$, where τ_e denotes the explosion time. Now, let us show that $(S(t), x(t))$ is global, i.e., $\tau_e = \infty$.

Let $n_0 \geq 1$ be sufficiently large such that $S(t)$ and $x(t)$ all lie within the interval $[1/n_0, n_0]$, and define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{S(t), x(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), x(t)\} \geq n\right\}, \tag{11}$$

for each integer $n > n_0$. We set $\inf \emptyset = \infty$($\emptyset$ denotes the empty set), and τ_n is distinctly increasing as $n \longrightarrow \infty$. Let $\tau_\infty = \lim_{n\longrightarrow\infty}\tau_n$, and then $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ is true, then $\tau_e = \infty$ a.s. and $(S(t), x(t)) \in \mathbb{R}_+^2$ a.s. for $t \geq 0$. If this hypothesis is not true, there exist a constant $T > 0$ and an $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \tag{12}$$

Hence, there is an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon \quad \text{for all } n \geq n_1. \tag{13}$$

Define a C^2 function $V : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, by

$$V(t) = S - \gamma - \gamma \ln\frac{S}{\gamma} + x - \gamma - \gamma \ln\frac{x}{\gamma}, \tag{14}$$

where $\gamma = S_0^u$. The nonnegativity of this function can be seen from

$$e - 1 - \ln e \geq 0 \quad \text{for any } e > 0. \tag{15}$$

Applying the Itô's formula to V leads to

$$\begin{aligned}
\mathscr{L}V = {} & \left(1 - \frac{\gamma}{S(t)}\right)\Big[D(t)(S_0(t) - S(t)) \\
& - p(t)\beta_1(t)S(t)x(t) \\
& - (1 - p(t))\frac{\beta_2(t)S(t)x(t)}{a(t) + S(t)}\Big] + \left(1 - \frac{\gamma}{x(t)}\right) \\
& \cdot \Big[p(t)\beta_1(t)S(t) + (1 - p(t))\frac{\beta_2(t)S(t)}{a(t) + S(t)} \\
& - r(t)C_0(t) - D(t)\Big]x(t) - \frac{\gamma}{2}p^2(t)\sigma_1^2(t)x^2(t) \\
& - \frac{\gamma}{2}(1 - p(t))^2\frac{\sigma_2^2(t)x^2(t)}{(a(t) + S(t))^2} + \frac{\gamma}{2}p^2(t)\sigma_1^2(t) \\
& \cdot S^2(t) + \frac{\gamma}{2}(1 - p(t))^2\frac{\sigma_2^2(t)S^2(t)}{(a(t) + S(t))^2} \leq D(t) \\
& \cdot S_0(t) + D(t)\gamma + p(t)\beta_1(t)x(t)\gamma + (1 - p(t)) \\
& \cdot \frac{\beta_2(t)x(t)\gamma}{a(t) + S(t)} + (r(t)C_0(t) + D(t))\gamma + \frac{\gamma}{2}p^2(t) \\
& \cdot \sigma_1^2(t)S^2(t) + \frac{\gamma}{2}(1 - p(t))^2\frac{\sigma_2^2(t)S^2(t)}{(a(t) + S(t))^2} \\
& \leq 3D^u\gamma + p^u\beta_1^u\gamma^2 + (1 - p^l)\frac{\beta_2^u\gamma^2}{a^l} + r^uC_0^*(0)\gamma \\
& + \frac{\gamma^3}{2}p^{2u}\sigma_1^{2u} + \frac{\gamma^3}{2}(1 - p^l)^2\frac{\sigma_2^{2u}}{a^{2l}} =: K.
\end{aligned} \tag{16}$$

Here K is a positive constant which is independent of $S(t)$ and $x(t)$. Therefore,

$$\begin{aligned}
\mathrm{d}V \leq {} & K\mathrm{d}t - (S(t) - \gamma)p(t)\sigma_1(t)x(t)\mathrm{d}B_1(t) \\
& - (S(t) - \gamma)\frac{(1 - p(t))\sigma_2(t)x(t)}{a(t) + S(t)}\mathrm{d}B_2(t) \\
& + (x(t) - \gamma)p(t)\sigma_1(t)S(t)\mathrm{d}B_1(t) \\
& + (x(t) - \gamma)\frac{(1 - p(t))\sigma_2(t)S(t)}{a(t) + S(t)}\mathrm{d}B_2(t).
\end{aligned} \tag{17}$$

Then, in the same way with [34], we get

$$\infty > V(S(0), x(0)) + KT = \infty, \quad n \longrightarrow \infty, \tag{18}$$

which is a contradiction, and then we have $\tau_\infty = \infty$. This completes the proof. □

4. Extinction and Persistence in Mean

4.1. Extinction. In this section, we establish sufficient conditions for the extinction of the microorganism, which implies that microculture failed.

Theorem 6. *Let $(S(t), x(t), C_0(t), C_e(t))$ be the solution of system (2) with any initial value $(S(0), x(0), C_0(0), C_e(0^+)) \in \mathbb{R}_+^4$. If one of the conditions holds*

(i) $\sigma_1(t) > \sqrt{\beta_1^l/p^uS_0^u}$, $\sigma_2(t) > \sqrt{a^l\beta_2^l/(1 - p^l)S_0^u}$, *and* $R_{11} = (\beta_1^{2u}\sigma_2^{2l} + \beta_2^{2u}\sigma_1^{2l})/2\sigma_1^{2l}\sigma_2^{2l}(D^l + r^l\overline{C_0}) < 1$,

(ii) $\sigma_1(t) > \sqrt{\beta_1^l/p^uS_0^u}$, $\sigma_2(t) < \sqrt{a^l\beta_2^l/(1 - p^u)S_0^u}$, *and* $R_{12} = (\beta_1^{2u}a^l + 2(1 - p^l)\beta_2^uS_0^u\sigma_1^{2l})/2\sigma_1^{2l}a^l(D^l + r^l\overline{C_0}) < 1$,

(iii) $\sigma_1(t) < \sqrt{\beta_1^u/p^lS_0^u}$, $\sigma_2(t) > \sqrt{a^l\beta_2^l/(1 - p^l)S_0^u}$, *and* $R_{21} = (2p^u\beta_1^uS_0^u\sigma_2^{2l} + \beta_2^{2u})/2\sigma_2^{2l}(D^l + r^l\overline{C_0}) < 1$,

(iv) $\sigma_1(t) < \sqrt{\beta_1^u/p^lS_0^u}$, $\sigma_2(t) < \sqrt{a^l\beta_2^u/(1 - p^u)S_0^u}$, *and* $R_{22} = [p^u\beta_1^ua^l + (1 - p^l)\beta_2^u]S_0^u/a^l(D^l + r^l\overline{C_0}) < 1$,

then the microorganism will die out almost surely, i.e., $\lim_{t\longrightarrow\infty}x(t) = 0$, *moreover,* $\lim_{t\longrightarrow\infty}S(t) = S_0^u$, $\lim_{t\longrightarrow\infty}C_0(t) = C_0^*(t)$, *and* $\lim_{t\longrightarrow\infty}C_e(t) = C_e^*(t)$.

Proof. Let $V(t) = \ln x$. Applying Itô's formula to system (2), we have

$$\begin{aligned}
\mathrm{d}V(t) &= p(t)\beta_1(t)S(t) + (1-p(t))\frac{\beta_2(t)S(t)}{a(t)+S(t)} \\
&\quad - r(t)C_0(t) - D(t) - \frac{1}{2}p^2(t)\sigma_1^2(t)S^2(t) - \frac{1}{2}(1 \\
&\quad - p(t))^2\frac{\sigma_2^2(t)S^2(t)}{(a(t)+S(t))^2} + p(t)\sigma_1(t)S(t)\mathrm{d}B_1(t) \\
&\quad + (1-p(t))\frac{\sigma_2(t)S(t)}{a(t)+S(t)}\mathrm{d}B_2(t) = \frac{\beta_1^2(t)}{2\sigma_1^2(t)} \\
&\quad - \frac{p^2(t)\sigma_1^2(t)}{2}\left(S(t) - \frac{\beta_1(t)}{p(t)\sigma_1^2(t)}\right)^2 + \frac{\beta_2^2(t)}{2\sigma_2^2(t)} \\
&\quad - \frac{(1-p(t))^2\sigma_2^2(t)}{2}\left(\frac{S(t)}{a(t)+S(t)}\right. \\
&\quad \left. - \frac{\beta_2(t)}{(1-p(t))\sigma_2^2(t)}\right)^2 - r(t)C_0(t) - D(t) + p(t) \\
&\quad \cdot \sigma_1(t)S(t)\mathrm{d}B_1(t) + (1-p(t))\frac{\sigma_2(t)S(t)}{a(t)+S(t)}\mathrm{d}B_2(t).
\end{aligned} \tag{19}$$

Integrating, respectively, from 0 to t and dividing both sides of (19) by t, one has

$$\begin{aligned}
\frac{\ln x(t)}{t} &= \frac{1}{t}\int_0^t\left[\frac{\beta_1^2(t)}{2\sigma_1^2(t)} - \frac{p^2(t)\sigma_1^2(t)}{2}\left(S(t)\right.\right. \\
&\quad \left. - \frac{\beta_1(t)}{p(t)\sigma_1^2(t)}\right)^2 + \frac{\beta_2^2(t)}{2\sigma_2^2(t)} \\
&\quad - \frac{(1-p(t))^2\sigma_2^2(t)}{2}\left(\frac{S(t)}{a(t)+S(t)}\right. \\
&\quad \left.\left. - \frac{\beta_2(t)}{(1-p(t))\sigma_2^2(t)}\right)^2 - r(t)C_0(t) - D(t)\right]\mathrm{d}t \\
&\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t},
\end{aligned} \tag{20}$$

where $M_1(t) = \int_0^t p(\theta)\sigma_1(\theta)S(\theta)\mathrm{d}B_1(\theta)$, $M_2(t) = \int_0^t(1 - p(\theta))(\sigma_2(\theta)S(\theta)/(a(\theta) + S(\theta)))\mathrm{d}B_2(\theta)$, and by the strong law of large numbers [46] we have $\lim_{t\longrightarrow\infty}(M_1(t)/t) = 0$ and $\lim_{t\longrightarrow\infty}(M_2(t)/t) = 0$.

Case 1. When $\sigma_1(t) > \sqrt{\beta_1^l/p^uS_0^u}$, $\sigma_2(t) > \sqrt{a^l\beta_2^l/(1-p^l)S_0^u}$ and $R_{11} < 1$, applying (20) we have

$$\begin{aligned}
\frac{\ln x(t)}{t} &\le \frac{1}{t}\int_0^t\left[\frac{\beta_1^2(t)}{2\sigma_1^2(t)} + \frac{\beta_2^2(t)}{2\sigma_2^2(t)} - D(t)\right]\mathrm{d}t - \frac{r(t)}{t} \\
&\quad \cdot \int_0^t C_0(s)\,\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t} \le \frac{\beta_1^{2u}}{2\sigma_1^{2l}} \\
&\quad + \frac{\beta_2^{2u}}{2\sigma_2^{2l}} - D^l - r^l\langle C_0(t)\rangle_t + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} \\
&\quad + \frac{\ln x(0)}{t} = \left(D^l + r^l\langle C_0(t)\rangle_t\right) \\
&\quad \cdot \left[\frac{\beta_1^{2u}\sigma_2^{2l} + \beta_2^{2u}\sigma_1^{2l}}{2\sigma_1^{2l}\sigma_2^{2l}\left(D^l + r^l\langle C_0(t)\rangle_t\right)} - 1\right] + \frac{M_1(t)}{t} \\
&\quad + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}.
\end{aligned} \tag{21}$$

Taking the limit superior on both sides of (21), we have

$$\limsup_{t\longrightarrow+\infty}\frac{\ln x(t)}{t} \le \left(D^l + r^l\overline{C_0}\right)(R_{11} - 1) < 0. \tag{22}$$

Thus $\lim_{t\longrightarrow\infty}x(t) = 0$.

Case 2. When $\sigma_1(t) > \sqrt{\beta_1^l/p^uS_0^u}$, $\sigma_2(t) < \sqrt{a^l\beta_2^u/(1-p^u)S_0^u}$ and $R_{12} < 1$, applying (20) we have

$$\begin{aligned}
\frac{\ln x(t)}{t} &\le \frac{1}{t} \\
&\quad \cdot \int_0^t\left[\frac{\beta_1^2(t)}{2\sigma_1^2(t)} + \frac{(1-p(t))\beta_2(t)S(t)}{a(t)+S(t)} - D(t)\right]\mathrm{d}t \\
&\quad - \frac{r(t)}{t}\int_0^t C_0(s)\,\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t} \\
&\le \frac{\beta_1^{2u}}{2\sigma_1^{2l}} + \frac{\left(1-p^l\right)\beta_2^uS_0^u}{a^l} - D^l - r^l\langle C_0(t)\rangle_t \\
&\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t} \\
&= \left(D^l + r^l\langle C_0(t)\rangle_t\right) \\
&\quad \cdot \left[\frac{\beta_1^{2u}a^l + 2\left(1-p^l\right)\beta_2^uS_0^u\sigma_1^{2l}}{2\sigma_1^{2l}a^l\left(D^l + r^l\langle C_0(t)\rangle_t\right)} - 1\right] + \frac{M_1(t)}{t} \\
&\quad + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}.
\end{aligned} \tag{23}$$

Taking the limit superior on both sides of (23), we have

$$\limsup_{t\longrightarrow+\infty}\frac{\ln x(t)}{t} \le \left(D^l + r^l\overline{C_0}\right)(R_{12} - 1) < 0. \tag{24}$$

Thus $\lim_{t\longrightarrow\infty}x(t) = 0$.

Case 3. When $\sigma_1(t) < \sqrt{\beta_1^u/p^l S_0^u}$, $\sigma_2(t) > \sqrt{a^l\beta_2^l/(1-p^l)S_0^u}$, with $R_{21} < 1$, applying (20) we have

$$\begin{aligned}
&\frac{\ln x(t)}{t} \\
&\le \frac{1}{t}\int_0^t \left[p(t)\beta_1(t)S(t) + \frac{\beta_2^2(t)}{2\sigma_2^2(t)} - D(t)\right]\mathrm{d}t \\
&\quad - \frac{r(t)}{t}\int_0^t C_0(s)\,\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} \\
&\quad + \frac{\ln x(0)}{t} \\
&\le p^u\beta_1^u S_0^u + \frac{\beta_2^{2u}}{2\sigma_2^{2l}} - D^l - r^l\langle C_0(t)\rangle_t + \frac{M_1(t)}{t} \\
&\quad + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t} \\
&= \left(D^l + r^l\langle C_0(t)\rangle_t\right)\left[\frac{2p^u\beta_1^u S_0^u\sigma_2^{2l} + \beta_2^{2u}}{2\sigma_2^{2l}\left(D^l + r^l\langle C_0(t)\rangle_t\right)} - 1\right] \\
&\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}.
\end{aligned} \tag{25}$$

Taking the limit superior on both sides of (25), we have

$$\limsup_{t\longrightarrow+\infty}\frac{\ln x(t)}{t} \le \left(D^l + r^l\overline{C_0}\right)(R_{21}-1) < 0. \tag{26}$$

Thus $\lim_{t\longrightarrow\infty}x(t) = 0$.

Case 4. When $\sigma_1(t) < \sqrt{\beta_1^u/p^l S_0^u}$, $\sigma_2(t) < \sqrt{a^l\beta_2^u/(1-p^u)S_0^u}$, with $R_{22} < 1$, applying (20) we have

$$\begin{aligned}
\frac{\ln x(t)}{t} &\le \frac{1}{t}\int_0^t \left[p(t)\beta_1(t)S(t)\right. \\
&\quad \left. + \frac{(1-p(t))\beta_2(t)S(t)}{a(t)+S(t)} - D(t)\right]\mathrm{d}t - \frac{r(t)}{t} \\
&\quad \cdot \int_0^t C_0(s)\,\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t} \\
&\le p^u\beta_1^u S_0^u + \frac{\left(1-p^l\right)\beta_2^u S_0^u}{a^l} - D^l - r^l\langle C_0(t)\rangle_t \\
&\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t} = \left(D^l\right. \\
&\quad \left. + r^l\langle C_0(t)\rangle_t\right)\left[\frac{\left[p^u\beta_1^u a^l + \left(1-p^l\right)\beta_2^u\right]S_0^u}{a^l\left(D^l + r^l\langle C_0(t)\rangle_t\right)} - 1\right] \\
&\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}.
\end{aligned} \tag{27}$$

Taking the limit superior on both sides of (27), we have

$$\limsup_{t\longrightarrow+\infty}\frac{\ln x(t)}{t} \le \left(D^l + r^l\overline{C_0}\right)(R_{22}-1) < 0. \tag{28}$$

This implies $\lim_{t\longrightarrow+\infty}x(t) = 0$, a.s.

Since the limit system of (2) is

$$\begin{aligned}
&\mathrm{d}S(t) = \left[D(t)\left(S_0(t) - S(t)\right)\right]\mathrm{d}t, \\
&\mathrm{d}C_0(t) = \left[kC_e(t) - gC_0(t) - mC_0(t)\right]\mathrm{d}t, \\
&\mathrm{d}C_e(t) = -hC_e(t)\,\mathrm{d}t, \\
&\qquad\qquad t \ne n\tau,\ n \in Z^+, \\
&\Delta S(t) = 0, \\
&\Delta C_0(t) = 0, \\
&\Delta C_e(t) = u, \\
&\qquad\qquad t = n\tau,\ n \in Z^+,
\end{aligned} \tag{29}$$

by Lemma 4, it is clear that $\lim_{t\longrightarrow\infty}S(t) = S_0^u$, $\lim_{t\longrightarrow\infty}C_0(t) = C_0^*(t)$, $\lim_{t\longrightarrow\infty}C_e(t) = C_e^*(t)$. This completes the proof of Theorem 6. □

4.2. Persistence in Mean

Theorem 7. *For any initial value* $(S(0), x(0), C_0(0), C_e(0^+)) \in \mathbb{R}_+^4$, *system (2) is permanent in the mean if*

$$R_1^* = \frac{D^l S_0^l\left(p^l\beta_1^l + (1-p^u)\beta_2^l/\left(a^u + S_0^u\right)\right)}{D^u\left(r^u\overline{C_0} + D^u + (1/2)p^{2u}\sigma_1^{2u}S_0^{2u} + (1/2)\left(\left(1-p^l\right)^2\sigma_2^{2u}S_0^{2u}/a^{2l}\right)\right)} > 1, \tag{30}$$

and the solution $(S(t), x(t), C_0(t), C_e(t))$ *of system (2) satisfies*

$$\begin{aligned}
&\liminf_{t\longrightarrow+\infty}\langle x(t)\rangle_t \\
&\ge \frac{r^u\overline{C_0} + D^u + (1/2)p^{2u}\sigma_1^{2u}S_0^{2u} + (1/2)\left(\left(1-p^l\right)^2\sigma_2^{2u}S_0^{2u}/a^{2l}\right)}{\left(r^u C_0^*(0) + D^u\right)\left(1 + \left(p^l\beta_1^l\left(a^u + S_0^u\right) + (1-p^u)\beta_2^l\right)/D^u\left(a^u + S_0^u\right)\right)}(R_1^* \\
&\quad - 1).
\end{aligned} \tag{31}$$

Proof. From system (2),

$$\begin{aligned}
\delta(t) &\triangleq \frac{S(t) - S(0)}{t} + \frac{x(t) - x(0)}{t} = \frac{1}{t} \\
&\quad \cdot \int_0^t \left[D(s)\left(S_0(s) - S(s)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&- \left(rC_0(s) + D(s)\right) x(s)\Big] \mathrm{d}s \geq \frac{1}{t}\int_0^t \Big[D^l S_0^l \\
&- D^u S(s) - \left(r^u C_0^*(0) + D^u\right) x(s)\Big] \mathrm{d}s \geq D^l S_0^l \\
&- D^u \langle S(t)\rangle_t - \left(r^u C_0^*(0) + D^u\right) \langle x(t)\rangle_t
\end{aligned} \tag{32}
$$

Obviously, $\lim_{t\longrightarrow+\infty}\delta(t) = 0$, and we obtain

$$
\langle S(t)\rangle_t \geq \frac{D^l S_0^l}{D^u} - \frac{r^u C_0^*(0) + D^u}{D^u}\langle x(t)\rangle_t - \frac{\delta(t)}{D^u}. \tag{33}
$$

Applying Itô's formula gives

$$
\begin{aligned}
&\mathrm{d}(\ln x(t) + x(t)) = \Bigg[(1 + x(t))\Bigg(p(t)\beta_1(t)S(t) \\
&+ (1 - p(t))\frac{\beta_2(t)S(t)}{a(t) + S(t)} - r(t)C_0(t) - D(t)\Bigg) \\
&- \frac{1}{2}p^2(t)\sigma_1^2(t)S^2(t) - \frac{1}{2} \\
&\cdot \frac{(1 - p(t))^2\sigma_2^2(t)S^2(t)}{(a(t) + S(t))^2}\Bigg]\mathrm{d}t + (1 + x(t))p(t) \\
&\cdot \sigma_1(t)S(t)\mathrm{d}B_1(t) + (1 + x(t)) \\
&\cdot \frac{(1 - p(t))\sigma_2(t)S(t)}{a(t) + S(t)}\mathrm{d}B_2(t) \geq \Bigg[(1 + x(t)) \\
&\cdot \left(p^l\beta_1^l S(t) + (1 - p^u)\frac{\beta_2^l S(t)}{a^u + S_0^u} - r^u C_0(t) - D^u\right) \\
&- \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2}\frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}}\Bigg]\mathrm{d}t + (1 \\
&+ x(t))p(t)\sigma_1(t)S(t)\mathrm{d}B_1(t) + (1 + x(t)) \\
&\cdot \frac{(1 - p(t))\sigma_2(t)S(t)}{a(t) + S(t)}\mathrm{d}B_2(t) \geq \Bigg[-r^u C_0(t) - D^u \\
&- \left(r^u C_0^*(0) + D^u\right)x(t) + p^l\beta_1^l S(t) + (1 - p^u) \\
&\cdot \frac{\beta_2^l S(t)}{a^u + S_0^u} - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2}\frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}}\Bigg]\mathrm{d}t \\
&+ (1 + x(t))p(t)\sigma_1(t)S(t)\mathrm{d}B_1(t) + (1 + x(t)) \\
&\cdot \frac{(1 - p(t))\sigma_2(t)S(t)}{a(t) + S(t)}\mathrm{d}B_2(t).
\end{aligned} \tag{34}
$$

Then

$$
\begin{aligned}
&\frac{\ln x(t) - \ln x(0)}{t} + \frac{x(t) - x(0)}{t} \geq -r^u\langle C_0(t)\rangle_t \\
&- D^u - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2}\frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}} \\
&- \left(r^u C_0^*(0) + D^u\right)\langle x(t)\rangle_t \\
&+ \left(p^l\beta_1^l + \frac{(1 - p^u)\beta_2^l}{a^u + S_0^u}\right)\langle S(t)\rangle_t + \frac{M_3(t)}{t} \\
&+ \frac{M_4(t)}{t} \geq -r^u\langle C_0(t)\rangle_t - D^u - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2} \\
&\cdot \frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}} + \frac{D^l S_0^l p^l\beta_1^l}{D^u} + \frac{(1 - p^u)D^l S_0^l\beta_2^l}{D^u\left(a^u + S_0^u\right)} \\
&- \left(r^u C_0^*(0) + D^u\right) \\
&\cdot \left(1 + \frac{p^l\beta_1^l\left(a^u + S_0^u\right) + (1 - p^u)\beta_2^l}{D^u\left(a^u + S_0^u\right)}\right)\langle x(t)\rangle_t \\
&- \frac{\delta(t)}{D^u}\left(p^l\beta_1^l + \frac{(1 - p^u)\beta_2^l}{a^u + S_0^u}\right) + \frac{M_3(t)}{t} + \frac{M_4(t)}{t},
\end{aligned} \tag{35}
$$

where $\lim_{t\longrightarrow+\infty}(M_3(t)/t) = 0$ and $\lim_{t\longrightarrow+\infty}(M_4(t)/t) = 0$. The inequality (35) can be rewritten as

$$
\begin{aligned}
\langle x(t)\rangle_t &\geq \frac{1}{\Delta}\left[-r^u\langle C_0(t)\rangle_t - D^u - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2}\frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}} + \frac{D^l S_0^l p^l\beta_1^l}{D^u} + \frac{(1 - p^u)D^l S_0^l\beta_2^l}{D^u\left(a^u + S_0^u\right)} - \frac{\delta(t)}{D^u}\left(p^l\beta_1^l + \frac{(1 - p^u)\beta_2^l}{a^u + S_0^u}\right) + \frac{M_3(t)}{t} + \frac{M_4(t)}{t} - \frac{\ln x(t) - \ln x(0)}{t} - \frac{x(t) - x(0)}{t}\right] \\
&\geq \begin{cases}
\frac{1}{\Delta}\left[-r^u\langle C_0(t)\rangle_t - D^u - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2}\frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}} + \frac{D^l S_0^l p^l\beta_1^l}{D^u} + \frac{(1 - p^u)D^l S_0^l\beta_2^l}{D^u\left(a^u + S_0^u\right)} - \frac{\delta(t)}{D^u}\left(p^l\beta_1^l + \frac{(1 - p^u)\beta_2^l}{a^u + S_0^u}\right) + \frac{M_3(t)}{t} + \frac{M_4(t)}{t} + \frac{\ln x(0)}{t} - \frac{x(t) - x(0)}{t}\right] & 0 < x(t) < 1; \\
\frac{1}{\Delta}\left[-r^u\langle C_0(t)\rangle_t - D^u - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2}\frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}} + \frac{D^l S_0^l p^l\beta_1^l}{D^u} + \frac{(1 - p^u)D^l S_0^l\beta_2^l}{D^u\left(a^u + S_0^u\right)} - \frac{\delta(t)}{D^u}\left(p^l\beta_1^l + \frac{(1 - p^u)\beta_2^l}{a^u + S_0^u}\right) + \frac{M_3(t)}{t} + \frac{M_4(t)}{t} - \frac{\ln x(t) - \ln x(0)}{t} - \frac{x(t) - x(0)}{t}\right] & 1 \leq x(t) < S_0^u,
\end{cases}
\end{aligned} \tag{36}
$$

where $\Delta = (r^u C_0^*(0) + D^u)(1 + (p^l\beta_1^l(a^u + S_0^u) + (1 - p^u)\beta_2^l)/D^u(a^u + S_0^u))$. According to Lemma 1, we see that $x(t) \leq S_0^u$; thus one has $\lim_{t\longrightarrow+\infty}(x(t)/t) = 0$, $\lim_{t\longrightarrow+\infty}(\ln x(t)/t) = 0$ as $x(t) \geq 1$, and $\lim_{t\longrightarrow+\infty}\delta(t) = 0$.

Taking the inferior limit of both sides of (36), one can yield that

$$
\begin{aligned}
&\liminf_{t\longrightarrow+\infty}\langle x\rangle_t \geq \frac{1}{\Delta}\Bigg[-r^u\overline{C_0} - D^u - \frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u} - \frac{1}{2} \\
&\cdot \frac{\left(1 - p^l\right)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}} + \frac{D^l S_0^l}{D^u}\Bigg(p^l\beta_1^l
\end{aligned}
$$

$$+\frac{(1-p^u)\beta_2^l}{a^u+S_0^u}\Bigg)\Bigg] = \frac{r^u\overline{C_0}+D^u+(1/2)p^{2u}\sigma_1^{2u}S_0^{2u}+(1/2)\left(\left(1-p^l\right)^2\sigma_2^{2u}S_0^{2u}/a^{2l}\right)}{\Delta}(R_1^*-1)>0. \tag{37}$$

By (ii) of Lemma 2, this completes the proof of Theorem 7. □

5. The Existence of Nontrivial Positive Periodic Solution

In this section, based on the theory of Has'minskii [45], we testify the existence of the nontrivial positive periodic solution.

Define

$$R_0^s := \frac{\langle p(t)\beta_1(t)S_0(t)\rangle_T}{\left\langle r(t)C_0(t)+D(t)+(1/2)p^{2u}\sigma_1^{2u}S_0^{2u}+(1/2)\left(\left(1-p^l\right)^2\sigma_2^{2u}S_0^{2u}/a^{2l}\right)\right\rangle_T}. \tag{38}$$

Theorem 8. *Assuming that $R_0^s>1$, then for any initial value $(S(0),x(0))\in\mathbb{R}_+^2$, system (10) has at least one positive τ-periodic solution.*

Proof. Define a nonnegative C^2–function $\widetilde{V}\colon\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$ as follows:

$$\begin{aligned}\widetilde{V} &= M(-\ln S(t)-\alpha\ln x(t)+\omega(t)) \\ &\quad+\frac{1}{\theta+1}(S(t)+x(t))^{\theta+1}-\ln S(t) \\ &=: M(V_1+\omega(t))+V_2+V_3,\end{aligned} \tag{39}$$

where $V_1=-\ln S(t)-a\ln x(t)$, $V_2=(1/(\theta+1))(S(t)+x(t))^{\theta+1}$, $V_3=-\ln S(t)$, α is positive constant which is defined in the following discussion, $\theta>1$ is a positive constant satisfying

$$D^l-\frac{\theta S_0^{2u}}{4}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]>0, \tag{40}$$

namely,

$$1<\theta<\frac{4D^l}{S_0^{2u}\left[p^{2u}\sigma_1^{2u}+\left(1-p^l\right)^2\sigma_2^{2u}/a^{2l}\right]}, \tag{41}$$

and $M>0$ obeys

$$-M\widetilde{\lambda}+E\le-2, \tag{42}$$

where

$$\widetilde{\lambda}=\langle D\rangle_T\left(R_0^s-1\right)>0 \tag{43}$$

and

$$\begin{aligned}B &= \sup_{(S,x)\in\mathbb{R}_+^2}\Bigg\{\left(D^uS_0^u-r^lS_0^lC_0^l\right)(S+x)^\theta-\frac{1}{2}\Bigg[D^l \\ &\quad-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\Bigg](S+x)^{\theta+1}\Bigg\} \\ &<\infty,\end{aligned}$$

$$\begin{aligned}C &= D^u+S_0^u\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]+\frac{S_0^{2u}}{2}\Bigg[p^{2u}\sigma_1^{2u} \\ &\quad+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\Bigg], \\ E &= \sup_{(S,x)\in\mathbb{R}_+^2}\Bigg\{-\frac{1}{2}\Bigg[D^l \\ &\quad-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\Bigg]\left(S^{\theta+1}\right. \\ &\quad\left.+x^{\theta+1}\right)+B+C\Bigg\}.\end{aligned} \tag{44}$$

In addition, $\widetilde{V}(S,x)$ must have a minimum point $(S_0,x_0)\in\mathbb{R}_+^2$ as $\widetilde{V}(S,x)$ is a continuous function; then we can define a nonnegative C^2–function $V\colon\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$ as follows:

$$V(S,x)=\widetilde{V}(S,x)-\widetilde{V}(S_0,x_0). \tag{45}$$

Applying Itô's formula, we can get

$$\begin{aligned}\mathscr{L}V_1 &= -\frac{D(t)S_0(t)}{S(t)}+D(t)+p(t)\beta_1(t)x(t) \\ &\quad+\frac{(1-p(t))\beta_2(t)x(t)}{a(t)+S(t)}+\frac{1}{2}p^2(t)\sigma_1^2(t)x^2(t) \\ &\quad+\frac{1}{2}\frac{(1-p(t))^2\sigma_2^2(t)x^2(t)}{(a(t)+S(t))^2}-\alpha p(t)\beta_1(t)S(t) \\ &\quad-\alpha\frac{(1-p(t))\beta_2(t)S(t)}{a(t)+S(t)}+\alpha r(t)C_0(t)+\alpha D(t)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{2}p^2(t)\sigma_1^2(t)S^2(t)+\frac{\alpha}{2}\frac{(1-p(t))^2\sigma_2^2(t)S^2(t)}{(a(t)+S(t))^2}\\
&\le -2\sqrt{\alpha D(t)p(t)\beta_1(t)S_0(t)}+\alpha\left[r(t)C_0(t)\right.\\
&\left.+D(t)+\frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u}+\frac{1}{2}\frac{(1-p^l)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}}\right]\\
&+D(t)+\left[p^u\beta_1^u+\frac{(1-p^l)\beta_2^u}{a^l}\right]x+\frac{1}{2}\left[p^{2u}\sigma_1^{2u}\right.\\
&\left.+\frac{(1-p^l)^2\sigma_2^{2u}}{a^{2l}}\right]x^2=:R_0(t)+\left[p^u\beta_1^u\right.\\
&\left.+\frac{(1-p^l)\beta_2^u}{a^l}\right]x+\frac{1}{2}\left[p^{2u}\sigma_1^{2u}+\frac{(1-p^l)^2\sigma_2^{2u}}{a^{2l}}\right]\\
&\cdot x^2.
\end{aligned}
\tag{46}
$$

Define a positive τ-periodic function $\omega(t)$ satisfying

$$
\omega'(t)=\langle R_0\rangle_T-R_0(t),
\tag{47}
$$

and then we obtain

$$
\begin{aligned}
\mathscr{L}(V_1+\omega(t))\le\langle R_0\rangle_T&+\left[p^u\beta_1^u+\frac{(1-p^l)\beta_2^u}{a^l}\right]x\\
&+\frac{1}{2}\left[p^{2u}\sigma_1^{2u}+\frac{(1-p^l)^2\sigma_2^{2u}}{a^{2l}}\right]x^2.
\end{aligned}
\tag{48}
$$

We choose

$$
\begin{aligned}
&\sqrt{\alpha}\\
&=\frac{\left\langle\sqrt{Dp\beta_1S_0}\right\rangle_T}{\left\langle rC_0+D+(1/2)p^{2u}\sigma_1^{2u}S_0^{2u}+(1/2)\left((1-p^l)^2\sigma_2^{2u}S_0^{2u}/a^{2l}\right)\right\rangle_T},
\end{aligned}
\tag{49}
$$

then

$$
\begin{aligned}
&\langle R_0\rangle_T\\
&=\frac{\langle Dp\beta_1S_0\rangle_T}{\left\langle rC_0+D+(1/2)p^{2u}\sigma_1^{2u}S_0^{2u}+(1/2)\left((1-p^l)^2\sigma_2^{2u}S_0^{2u}/a^{2l}\right)\right\rangle_T}\\
&\quad+\langle D\rangle_T=:-\langle D\rangle_T(R_0^s-1).
\end{aligned}
\tag{50}
$$

Similarly,

$$
\begin{aligned}
\mathscr{L}V_2&=(S(t)+x(t))^\theta\left(D(t)S_0(t)-D(t)S(t)\right.\\
&\left.-r(t)C_0(t)x(t)-D(t)x(t)\right)+\theta(S(t)\\
&+x(t))^{\theta-1}\left[p^2(t)\sigma_1^2(t)S(t)x(t)\right.\\
&\left.+\frac{(1-p(t))^2\sigma_2^2(t)S(t)x(t)}{(a(t)+S(t))^2}\right]S(t)x(t)\le(S\\
&+x)^\theta\left[D^uS_0^u-r^lC_0^lS_0^l-D^l(S+x)\right]+\frac{\theta}{4}(S+x)^{\theta+1}\\
&\cdot\left[p^{2u}\sigma_1^{2u}S_0^{2u}+\frac{(1-p^l)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}}\right]=\left(D^uS_0^u\right.\\
&\left.-r^lC_0^lS_0^l\right)(S+x)^\theta-\left[D^l\right.\\
&\left.-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{(1-p^l)^2\sigma_2^{2u}}{a^{2l}}\right)\right](S+x)^{\theta+1}\\
&\le B-\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{(1-p^l)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\
&\cdot(S+x)^{\theta+1}\le B-\frac{1}{2}\left[D^l\right.\\
&\left.-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{(1-p^l)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\left(S^{\theta+1}+x^{\theta+1}\right),
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
\mathscr{L}V_3&=-\frac{D(t)S_0(t)}{S(t)}+D(t)+p(t)\beta_1(t)x(t)\\
&\quad+\frac{(1-p(t))\beta_2(t)x(t)}{a(t)+S(t)}\\
&\quad+\frac{1}{2}p^2(t)\sigma_1^2(t)x^2(t)\\
&\quad+\frac{1}{2}\frac{(1-p(t))^2\sigma_2^2(t)x^2(t)}{(a(t)+S(t))^2}\\
&\le-\frac{D^lS_0^l}{S}+D^u+p^u\beta_1^uS_0^u+\frac{(1-p^l)\beta_2^uS_0^u}{a^l}\\
&\quad+\frac{1}{2}p^{2u}\sigma_1^{2u}S_0^{2u}+\frac{1}{2}\frac{(1-p^l)^2\sigma_2^{2u}S_0^{2u}}{a^{2l}}\\
&=:-\frac{D^lS_0^l}{S}+C.
\end{aligned}
\tag{52}
$$

Therefore, we get

$$\begin{aligned}\mathscr{L}V \le & -M\langle D\rangle_T\left(R_0^s-1\right)\\ & +M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]x^2-\frac{D^lS_0^l}{S}\\ & -\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot\left(S^{\theta+1}+x^{\theta+1}\right)+B+C.\end{aligned} \tag{53}$$

Define a bounded closed set

$$U_\epsilon=\left\{(S,x)\in\mathbb{R}_+^2:\epsilon\le S\le\frac{1}{\epsilon},\ \epsilon\le x\le\frac{1}{\epsilon}\right\}, \tag{54}$$

and here we can choose $\epsilon\in(0,1)$ sufficiently small which satisfies

$$-\frac{D^lS_0^l}{\epsilon}+A\le-1, \tag{55}$$

$$\begin{aligned}& -M\tilde{\lambda}+M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]\epsilon\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]\epsilon^2\le-1,\end{aligned} \tag{56}$$

$$\begin{aligned}& -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\frac{1}{\epsilon^{\theta+1}}\\ & +F\le-1,\end{aligned} \tag{57}$$

$$\begin{aligned}& -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\frac{1}{\epsilon^{\theta+1}}\\ & +G\le-1,\end{aligned} \tag{58}$$

where A, F, and G are both positive constants, which are defined in the following reasoning process. And for a sufficiently small ϵ, it is easy to see that if (42) holds, then (56) holds.

Divide $\mathbb{R}_+^2\setminus U_\epsilon$ into four domains:

$$\begin{aligned}U_1&=\left\{(S,x)\in\mathbb{R}_+^2:0<S<\epsilon\right\},\\ U_2&=\left\{(S,x)\in\mathbb{R}_+^2:0<x<\epsilon\right\},\\ U_3&=\left\{(S,x)\in\mathbb{R}_+^2:S>\frac{1}{\epsilon}\right\},\\ U_4&=\left\{(S,x)\in\mathbb{R}_+^2:x>\frac{1}{\epsilon}\right\},\end{aligned} \tag{59}$$

noting that $U_\epsilon^c=U_1\cup U_2\cup U_3\cup U_4$. Next we prove that $\mathscr{L}V\le-1$ on U_ϵ^c.

Case 1. If $(S,x)\in U_1$, then

$$\begin{aligned}\mathscr{L}V\le & -\frac{D^lS_0^l}{S}+M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]x^2\\ & -\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot\left(S^{\theta+1}+x^{\theta+1}\right)+B+C\le-\frac{D^lS_0^l}{S}+A\le-\frac{D^lS_0^l}{\epsilon}\\ & +A\le-1,\end{aligned} \tag{60}$$

where

$$\begin{aligned}A=\sup_{(S,x)\in\mathbb{R}_+^2} & M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]x^2\\ & -\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot\left(S^{\theta+1}+x^{\theta+1}\right)+B+C.\end{aligned} \tag{61}$$

According to (55), we get

$$\begin{aligned}&\mathscr{L}V\le-1,\\ &(S,x)\in U_1.\end{aligned} \tag{62}$$

Case 2. If $(S,x)\in U_2$, then

$$\begin{aligned}\mathscr{L}V\le & -M\langle D\rangle_T\left(R_0^s-1\right)\\ & +M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]x^2\\ & -\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot\left(S^{\theta+1}+x^{\theta+1}\right)+B+C\le-M\tilde{\lambda}\\ & +M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]\epsilon\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]\epsilon^2+E\le-1.\end{aligned} \tag{63}$$

By means of (56), we can conclude that

$$\begin{aligned}&\mathscr{L}V\le-1,\\ &(S,x)\in U_2.\end{aligned} \tag{64}$$

Case 3. If $(S,x)\in U_3$, then

$$\begin{aligned}\mathscr{L}V \le & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot S^{\theta+1}\\ & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]S^{\theta+1}\\ & +M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]x^2\\ & -\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]x^{\theta+1}\\ & +B+C\\ \le & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot S^{\theta+1}+F\\ \le & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot\frac{1}{\epsilon^{\theta+1}}+F\le -1,\end{aligned}\tag{65}$$

where

$$\begin{aligned}F=\sup_{(S,x)\in\mathbb{R}_+^2}\Bigg\{&-\frac{1}{4}\Bigg[D^l\\ & -\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\Bigg]S^{\theta+1}\\ & +M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x+\frac{M}{2}\Bigg[p^{2u}\sigma_1^{2u}\\ & +\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\Bigg]x^2-\frac{1}{2}\Bigg[D^l\\ & -\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\Bigg]x^{\theta+1}+B\\ & +C\Bigg\}.\end{aligned}\tag{66}$$

According to (57), one can get that

$$\begin{aligned}&\mathscr{L}V\le -1,\\ &(S,x)\in U_3.\end{aligned}\tag{67}$$

Case 4. If $(S,x)\in U_4$, then

$$\begin{aligned}\mathscr{L}V \le & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot x^{\theta+1}\\ & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot x^{\theta+1}+M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x\\ & +\frac{M}{2}\left[p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right]x^2\\ & -\frac{1}{2}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]S^{\theta+1}\\ & +B+C\\ \le & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot x^{\theta+1}+G\\ \le & -\frac{1}{4}\left[D^l-\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\right]\\ & \cdot\frac{1}{\epsilon^{\theta+1}}+G\le -1,\end{aligned}\tag{68}$$

where

$$\begin{aligned}G=\sup_{(S,x)\in\mathbb{R}_+^2}\Bigg\{&-\frac{1}{4}\Bigg[D^l\\ & -\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\Bigg]x^{\theta+1}\\ & +M\left[p^u\beta_1^u+\frac{\left(1-p^l\right)\beta_2^u}{a^l}\right]x+\frac{M}{2}\Bigg[p^{2u}\sigma_1^{2u}\\ & +\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\Bigg]x^2-\frac{1}{2}\Bigg[D^l\\ & -\frac{\theta S_0^{2u}}{4}\left(p^{2u}\sigma_1^{2u}+\frac{\left(1-p^l\right)^2\sigma_2^{2u}}{a^{2l}}\right)\Bigg]S^{\theta+1}+B\\ & +C\Bigg\}.\end{aligned}\tag{69}$$

In view of (58), we have

$$\begin{aligned}&\mathscr{L}V\le -1,\\ &(S,x)\in U_4.\end{aligned}\tag{70}$$

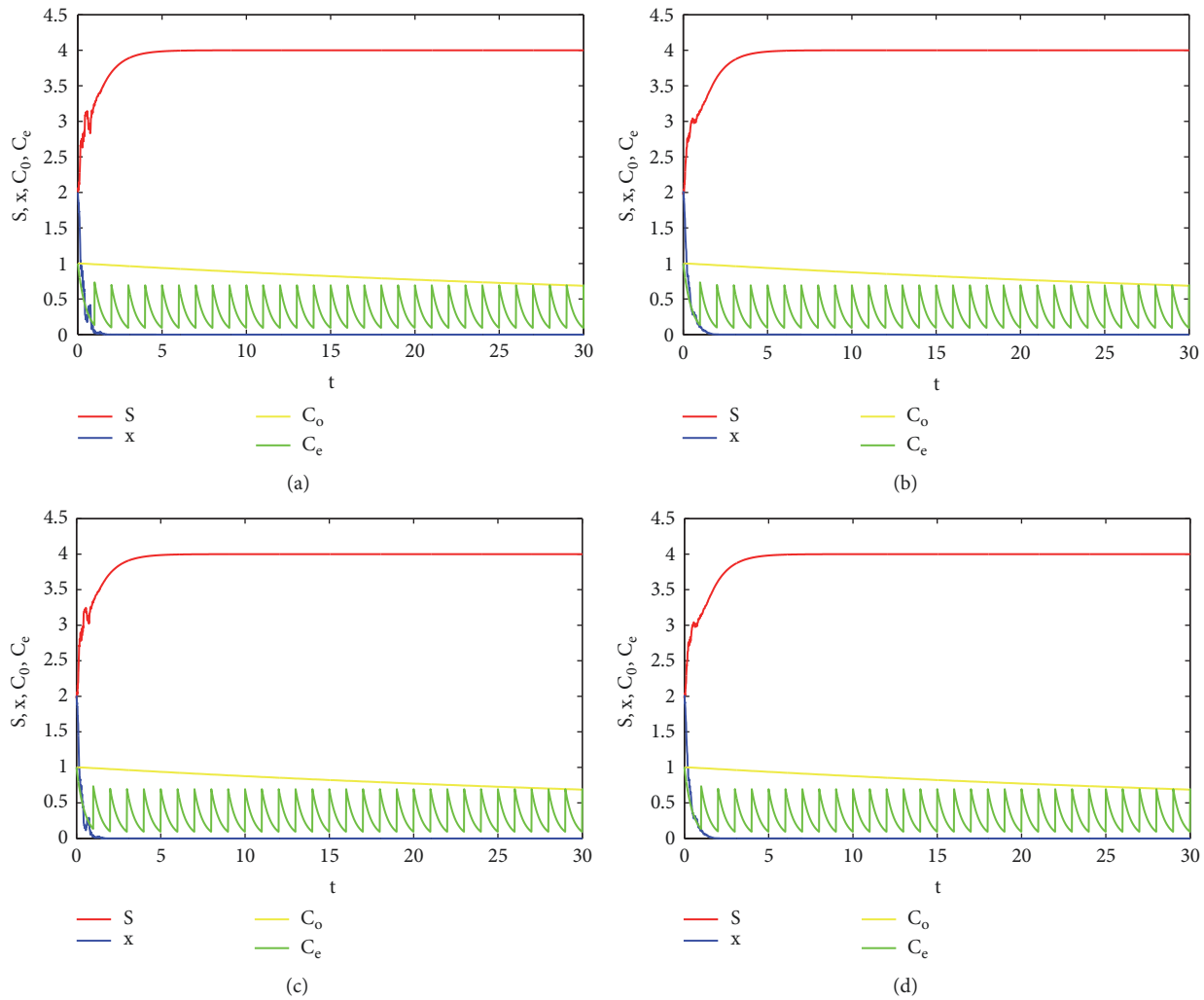

Figure 1: The solution of system (2) for extinction of the microorganism.

Thus, we have $\mathscr{L}V \leq -1$ on U_ϵ^c from the above four cases. By Lemma 3, we can get that system (2) admits a positive τ-periodic solution. This completes the proof of Theorem 8. □

6. Conclusions and Simulations

This paper investigates the dynamics of an impulsive stochastic chemostat model with two different growth rates in a polluted environment. The advantage of this paper is to take two different variable growth rates of microorganisms. We propose the novel model with the transfer function as $T(S,x) = \sum_{i=1}^{n} p_i T_i(S,x)$, where p_i $(i = 1,2,\ldots,n)$ represent the probability of occurrence of $T_i(S,x)$ $(i = 1,2,\ldots,n)$ and $\sum_{i=1}^{n} p_i = 1$. Hence we assume $T(S,x) = p\beta_1 Sx + (1-p)(\beta_2 Sx/(a+S))$, and it shows that the microorganism has two types of growth rates which are bilinear and saturated growth rate. Furthermore, if $p = 0$, this means the growth rate of the microorganism has only one growth rate, which is saturated growth rate. On the contrary, if the growth rate of the microorganism is only bilinear growth rate, we can assume $p = 1$. Thus our hypothesis is more extensive. And in this paper, by using the theory of impulsive differential equations and Lyapunov functions method, we establish the conditions for the extinction and the permanence of the microorganisms. Then we demonstrate that the stochastic system has at least one positive τ-periodic solution. The main results suggest that

(1) If one of the four conditions of Theorem 6 is satisfied, then $\lim_{t\longrightarrow+\infty} x(t) = 0$ *a.s.* This implies that the microorganism goes to extinction.

(2) If $R_1^* > 1$, system (2) is permanent in the mean.

(3) If $R_0^s > 1$, system (10) exists a nontrivial positive periodic solution.

Some numerical simulations are introduced to illustrate our main results [47].

In Figure 1, we set the parameter values in system (2) as $u = 0.6, k = 0.02, g = 0.01, m = 0.01, h = 0.02, \tau = 1, (S(0), x(0), C_0(0), C_e(0)) = (2,2,1,1)$, and (a) $\sigma_1^l > 0.202, \sigma_2^l > 0.128$ and $R_{11} = 0.908 < 1$; (b) $\sigma_1^l > 0.285, \sigma_2^u < 0.187$ and $R_{12} = 0.982 < 1$; (c) $\sigma_1^u < 1.104, \sigma_2^l > 0.114$ and $R_{21} = 0.999 < 1$; (d) $\sigma_1^u < 1.104, \sigma_2^u < 0.187$ and $R_{22} = 0.994 < 1$. It meets the conditions of Theorem 6

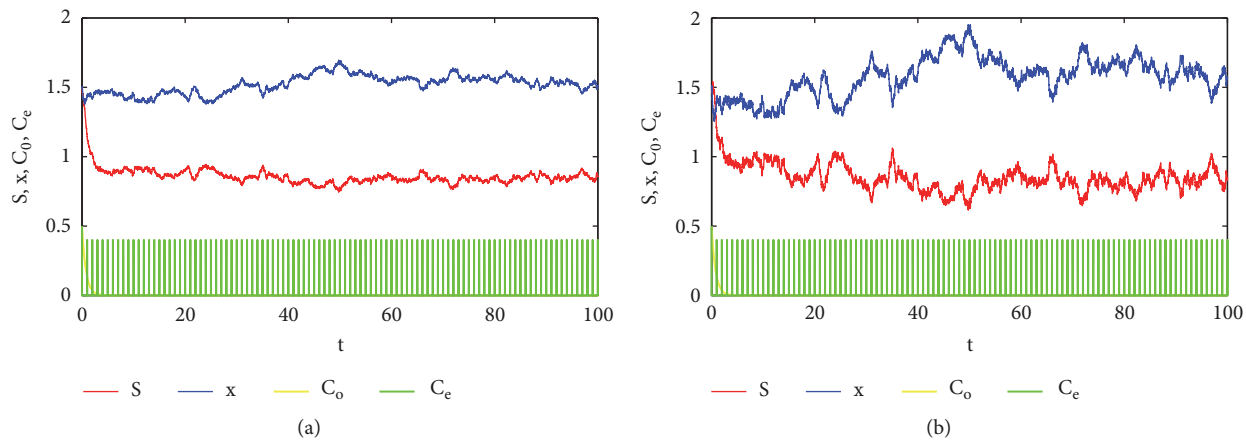

Figure 2: Numerical simulations of persistence in mean of system (2).

and the microorganisms in figure are all going extinct, so this is consistent with the conclusion in Theorem 6. In order to verify the condition of Theorems 7 and 8, we choose the parameter values as follows: $u = 0.4, k = 0.1, g = 0.8, m = 0.5, h = 1, \tau = 1, (S(0), x(0), C_0(0), C_e(0)) = (1.5, 1.5, 0.5, 0.5), 1 < \theta < 1.617$ in Figure 2. It obtains $R_1^* = 1.060 > 1$ and $R_0^s = 1.545 > 1$, and the microorganism in Figure 2 is persistence in mean, which is in line with the conclusion in Theorems 7 and 8, respectively.

The results show that both impulsive toxicant input and stochastic noise have great effects on the survival and extinction of the microorganisms. High concentration of toxicant or large intensity of noises can lead to extinction of microorganisms. Appropriate toxin input and noise intensity will make the microorganisms show periodic changes.

In this paper, we considered the effects of white noises. However, there are some other random perturbations in the environment, for example, regime-switching and Markovian switching. In fact, stochastic regime-switching or Markovian switching population models were studied by several authors recently [27, 43, 48, 49]. Therefore, it is interesting to consider regime-switching and Markovian switching models, and we will leave these cases as our future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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