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Research Article

Localization of Δ-Mixing Property via Furstenberg Families

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This paper is devoted to studying the localization of Δ-mixing property via Furstenberg families. The notion of Δ-weakly mixing sets is extended to $\mathcal{F}$-Δ-weakly mixing sets with respect to a sequence, and the characterization of Δ-weakly mixing sets is also generalized.

1. Introduction

Throughout this paper, a *topological dynamical system* (or dynamical system, system for short) is a pair (X, f), where X is a compact metric space with metric ρ and f is a continuous map of X to itself.

In some literature on nonautonomous discrete systems, the topological dynamical system is usually called the autonomous discrete system; for example, see [1].

Let (X, f) be a dynamical system. For two subsets U, V of X, we define the *hitting time set of U and V* by

$$N(U,V) := \{n \in \mathbb{N} : U \cap f^{-n}(V) \neq \emptyset\}. \tag{1}$$

We say that (X, f) is *transitive* if, for every two nonempty open subsets U and V of X, the hitting time set $N(U,V)$ is nonempty, *weakly mixing* if the product system $(X \times X, f \times f)$ is transitive, and *strongly mixing* if, for every two nonempty open subsets U and V of X, there exists $N \in \mathbb{N}$ such that $\{N, N+1, \ldots\} \subset N(U,V)$. In his seminal paper [2], Furstenberg showed that weak mixing implies n-fold transitivity for every positive integer n as follows.

(i) Let (X, f) be a topological dynamical system. Then $(X \times X, f \times f)$ is transitive if and only if, for every $n \in \mathbb{N}$, the product system $(X \times X \times \cdots \times X, f \times f \times \cdots \times f)$ (n-times) is also transitive.

For convenience, we denote $(X \times X \times \cdots \times X, f \times f \times \cdots \times f)$ (n-times) by $(X^n, f^{(n)})$.

A closed subset C of X is called *Δ-transitive* if, for every $d \geq 2$, there exists a residual subset C' of C such that, for every $x \in C'$, the orbit closure of the diagonal d-tuple $x(d)$, that is, $x(d) = (\underbrace{x, x, \ldots, x}_{d-times})$, under the action $f \times f^2 \times \cdots \times f^d$ contains C^d. A closed subset C of X with at least two points is *Δ-weakly mixing* if, for every $n \geq 1$, C^n is Δ-transitive in the n-th product system $(X^n, f^{(n)})$. In [3], Huang et al. show that Δ-weakly mixing sets exhibit a nice characterization as follows.

Theorem 1 (see [3]). *Let (X, f) be a dynamical system and E a closed subset of X but not a singleton. Then E is Δ-weakly mixing if and only if there exists a strictly increasing sequence of Cantor sets $C_1 \subset C_2 \subset \cdots$ of E such that $C := \cup_{k=1}^{\infty} C_k$ is dense in E and*

(i) *for any $d \in \mathbb{N}$, any subset A of C, and any continuous functions $g_j : A \longrightarrow E$ for $j = 1, 2, \ldots, d$, there exists a strictly increasing sequence $\{q_k\}_{k=1}^{\infty}$ of positive integers such that*

$$\lim_{k \longrightarrow \infty} f^{j \cdot q_k} x = g_j(x) \tag{2}$$

for every $x \in A$ and $j = 1, 2, \ldots, d$;

(ii) *for any $d \in \mathbb{N}$, $k \in \mathbb{N}$, any closed subset B of C_k, and continuous functions $h_j : B \longrightarrow E$ for $j = 1, 2, \ldots, d$, there exists a strictly increasing sequence $\{q_k\}_{k=1}^{\infty}$ of positive integers such that*

$$\lim_{k \longrightarrow \infty} f^{j \cdot q_k} x = h_j(x) \tag{3}$$

uniformly on $x \in B$ and $j = 1, 2, \ldots, d$.

In this paper, we will extend the notion of Δ-weakly mixing sets via Furstenberg family and show that $\mathscr{F}$-Δ-weakly mixing sets with respect to a sequence also share the same characterization under some considerable conditions.

2. Preliminary

In this section, we provide some definitions which will be used later.

Let $\mathscr{P}$ denote the collection of all subsets of $\mathbb{N}$. A subset $\mathscr{F}$ of $\mathscr{P}$ is called a *Furstenberg family* (or *family* for short), if it admits the property of hereditary upward; that is to say,

$$F_1 \subset F_2 \text{ and } F_1 \in \mathscr{F} \text{ imply } F_2 \in \mathscr{F}. \tag{4}$$

A family $\mathscr{F}$ is called *proper* if it is neither empty nor all of $\mathscr{P}$.

For a family $\mathscr{F}$, the *dual family* of $\mathscr{F}$ is defined by

$$\kappa\mathscr{F} := \left\{F \in \mathscr{P} : F \cap F' \neq \emptyset, \text{ for every } F' \in \mathscr{F}\right\}. \tag{5}$$

Let $\mathscr{F}_{inf}$ be the family of all infinite subsets of $\mathbb{N}$. It is easy to see that its dual family $\kappa\mathscr{F}_{inf}$, denoted by $\mathscr{F}_{cf}$, is the family of all cofinite subsets.

Any nonempty collection $\mathscr{A}$ of subsets of $\mathbb{N}$ naturally generates a family

$$[\mathscr{A}] = \{F \subset \mathbb{N} : A \subset F \text{ for some } A \in \mathscr{A}\}. \tag{6}$$

We say that $\mathscr{F}$ is *countable generated* if $\mathscr{F} = [\mathscr{A}]$ for a collection $\mathscr{A}$ consisting of countable subsets of $\mathbb{N}$.

The idea of using families to describe dynamical properties goes back at least to Gottschalk and Hedlund [4] and was developed further by Furstenberg [5]. For recent results, see [6–9].

Let (X, f) be a dynamical system, $\mathscr{F}$ a family, and $\mathbf{a} := \{a_i\}_{i=0}^{\infty}$ a strictly increasing sequence of nonnegative integers with $a_0 = 0$.

For every $d \geq 2$ and subsets $U_1, U_2, \ldots, U_d$ of X, we define the *hitting time set* of $U_1, U_2, \ldots, U_d$ by

$$N_{\mathbf{a}}(U_1, U_2, \ldots, U_d) := \left\{n \in \mathbb{N} : \bigcap_{i=1}^{d} f^{-a_{i-1} n} U_i \neq \emptyset\right\} \tag{7}$$

We say that A is Δ-$\mathscr{F}$*-weakly mixing with respect to* $\mathbf{a}$ if, for every $n, d \geq 2$, and nonempty open subsets $U_{i,j}$ of X intersecting A for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, d$, we have

$$\bigcap_{i=1}^{n} N_{\mathbf{a}}(U_{i,1} \cap A, U_{i,2}, \ldots, U_{i,d}) \in \mathscr{F}. \tag{8}$$

Let (X, ρ) be a compact metric space. Denote 2^X by the collection of all nonempty closed subsets of X and endow 2^X with the Hausdorff metric

$$\begin{aligned} &\rho_H(A, B) \\ &\quad = \max\left\{\max_{x \in A} \min_{y \in B} \rho(x, y), \max_{y \in B} \min_{x \in A} \rho(x, y)\right\} \end{aligned} \tag{9}$$

for any $A, B \in 2^X$. Then the metric space $(2^X, \rho_H)$ is compact whenever X is compact. For any nonempty subsets $S_1, \ldots, S_n \subset X$, denote

$$\begin{aligned} \langle S_1, \ldots, S_n \rangle := \Bigg\{&A \in 2^X : A \subset \bigcup_{i=1}^{n} S_i \text{ and } A \cap S_i \\ &\neq \emptyset \text{ for each } i = 1, \ldots, n\Bigg\}, \end{aligned} \tag{10}$$

then the following family

$$\begin{aligned} &\{\langle U_1, \ldots, U_n \rangle : U_1, \ldots, \\ &\quad U_n \text{ are non-empty open subsets of } X,\ n \in \mathbb{N}\} \end{aligned} \tag{11}$$

forms a basis for a topology of 2^X, which is called the *Vietoris topology*. It is well known that the Hausdorff topology induced by the Hausdorff metric ρ_H coincides with the Vietoris topology for 2^X.

3. Key Lemmas

In this section, we provide some lemmas which will be used later.

Let (X, f) be a dynamical system, $\mathscr{F}$ a family, $\mathbf{a} := \{a_i\}_{i=0}^{\infty}$ a strictly increasing sequence of nonnegative integers with $a_0 = 0$, and E a closed subset of X. For $\varepsilon > 0$, $d \geq 1$, and any $S \in k\mathscr{F}$, we say that a subset A of X is (S, ε, ρ)*-spread with respect to* $\mathbf{a}$, if there exist $\delta \in (0, \varepsilon)$, $n \in \mathbb{N}$, and distinct points $x_1, x_2, \ldots, x_n \in X$ such that $A \subset \cup_{i=1}^{n} B(x_i, \delta)$ and for any maps

$$g_j : \{x_1, x_2, \ldots, x_n\} \longrightarrow E \tag{12}$$

where $j = 1, 2, \ldots, d$, there exists $k \in S$ such that $1/k < \varepsilon$ and

$$\begin{aligned} &\left(f^{a_j \cdot k}\right)^{(n)} (B(x_1, \delta) \times \cdots \times B(x_n, \delta)) \\ &\quad \subset B(g_j(x_1), \varepsilon) \times \cdots \times B(g_j(x_n), \varepsilon) \end{aligned} \tag{13}$$

for $j = 1, 2, \ldots, d$.

For any $S \in k\mathscr{F}$, denote by $\mathscr{S}_{S,\mathbf{a}}(\varepsilon, \rho, E)$ the collection of all closed sets that are (S, ε, ρ)-spread in E with respect to $\mathbf{a}$.

Remark 2. It is not hard to check that the $\mathscr{S}_{S,\mathbf{a}}(\varepsilon, \rho, E)$ is hereditary; that is, if A is (S, ε, ρ)-spread in E with respect to $\mathbf{a}$ and B is a nonempty closed subset of A, then B is also (S, ε, ρ)-spread in E with respect to $\mathbf{a}$.

Let

$$\mathcal{S}_{S,\mathbf{a}}(E) = \bigcap_{d=1}^{\infty}\bigcap_{k=1}^{\infty}\mathcal{S}_{S,\mathbf{a}}\left(\frac{1}{k},\rho,E\right),$$
$$\mathcal{S}_{\mathcal{F},\mathbf{a}}(E) = \bigcap_{S\in k\mathcal{F}}\mathcal{S}_{S,\mathbf{a}}(E). \tag{14}$$

Lemma 3. *If E is a Δ-$\mathcal{F}$-weakly mixing subset of X but not a singleton, then $\mathcal{S}_{S,\mathbf{a}}(E)\cap 2^E$ is a dense open subset of 2^E for every $S\in k\mathcal{F}$.*

Proof. Fix $\varepsilon > 0$, $d\in\mathbb{N}$, and $S\in k\mathcal{F}$. We will divide our discussion into three claims to show that $\mathcal{S}_{S,\mathbf{a}}(E)\cap 2^E$ is a dense open subset of 2^E.

Claim 4. $\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E)$ is open in 2^X.

Proof of Claim 4. Let $A\in\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E)$. For any $S\in k\mathcal{F}$, let $\delta>0$ and $x_1,x_2,\ldots,x_n\in X$ as in the definition of set (S,ε,ρ)-spread in E with respect to $\mathbf{a}$. For $i=1,2,\ldots,n$, put $U_i = B(x_i,\delta)$. Then, by Remark 2, every closed subset $B\subset\cup_{i=1}^n U_i$ is (S,ε,ρ)-spread in E with respect to $\mathbf{a}$. It follows that

$$A\in\langle U_1,\ldots,U_n\rangle\subset\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E). \tag{15}$$

Hence $\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E)$ is open in 2^X. □

Claim 5. E is a perfect set.

Proof of Claim 5. If E is not perfect, then there exists an isolated point x of E. Note that the set $\{x\}$ is open in E. It follows that there are two nonempty open subsets $U_1,U_2\subset X$ such that

$$U_1\cap E=\{x\},$$
$$U_2\cap E\neq\emptyset \tag{16}$$

and $U_1\cap U_2=\emptyset$ since the set E is not a singleton. So, for any $m\in S$, such that $f^{a_1 m}\{x\}\cap U_1\neq\emptyset$, one has $f^{a_1 m}\{x\}\cap U_2=\emptyset$. This is a contradiction, and thus E is perfect. □

Claim 6. $\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E)\cap 2^E$ is dense in 2^E.

Proof of Claim 6. Fix n nonempty open subsets $U_1,\ldots,U_n$ of X intersecting E. We want to show that

$$\langle U_1,\ldots,U_n\rangle\cap\left(\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E)\cap 2^E\right)\neq\emptyset. \tag{17}$$

Since E is compact, there exists a finite subset $\{x_1,x_2,\ldots,x_m\}$ of E such that $\cup_{i=1}^m B(x_i,\varepsilon/2)\supset E$. For convenience, denote $B_i = B(x_i,\varepsilon/2)$ for $i=1,2,\ldots,m$. Since E is Δ-$\mathcal{F}$-weakly mixing with respect to $\mathbf{a}$, the set E is perfect, so we may assume that $m\geq 1/\varepsilon$. The collection of n-tuples on the set $\{1,2,\ldots,m\}^n$ can be arranged as the following finite sequence:

$$\{(\gamma_{1,1},\ldots,\gamma_{1,n}),(\gamma_{2,1},\ldots,\gamma_{2,n}),\ldots,(\gamma_{l,1},\ldots,\gamma_{l,n})\}. \tag{18}$$

For $(\gamma_{1,1},\ldots,\gamma_{1,n})$, as E is Δ-$\mathcal{F}$-weakly mixing with respect to $\mathbf{a}$, there exists $k_1\in S$ with $k_1>m$, such that

$$(U_i\cap E)\cap f^{-a_1k_1}B_{\gamma_{1,1}}\cap f^{-a_2k_1}B_{\gamma_{1,2}}\cap\cdots\cap f^{-a_dk_1}B_{\gamma_{1,d}}\neq\emptyset \tag{19}$$

for $i=1,2,\ldots,n$. By continuity of f, we can choose a nonempty subset W_i^1 of U_i intersecting E such that $f^{a_j\cdot k_1}W_i^1\subset B_{\gamma_{1,j}}$ for $j=1,2,\ldots,d$. Similarly, for $(\gamma_{2,1},\ldots,\gamma_{2,n})$, since S is a strictly increasing sequence of positive integers, there exist $k_2\in S$ with $k_2>k_1$ and nonempty subsets W_i^2 of W_i^1 intersecting E such that $f^{a_j\cdot k_2}W_i^2\subset B_{\gamma_{2,j}}$ for $i=1,2,\ldots,n$ and $j=1,2,\ldots,d$. After repeating this process l times, we obtain l distinct positive integers $k_1,k_2,\ldots,k_l\in S$ with $k_1>k_2>\cdots>k_l$, and nonempty open subsets $W_i^1,W_i^2,\ldots,W_i^l$ intersecting E such that

$$W_i^l\subset W_i^{l-1}\subset\cdots\subset W_i^1\subset U_i, \tag{20}$$

$i=1,2,\ldots,n$, and

$$\left(f^{a_j\cdot k_r}\right)^{(n)}\left(W_1^r\times W_2^r\times\cdots\times W_n^r\right)\subset B_{\gamma_{r,j}}\times B_{\gamma_{r,j}}\times\cdots\times B_{\gamma_{r,j}} \tag{21}$$

for $r=1,2,\ldots,l$, and $j=1,2,\ldots,d$.

For each $i=1,2,\ldots,n$, pick $x_i\in W_i^l\cap E$. Since E is perfect, it is reasonable to assume that those x_i's are distinct. It is clear that

$$\{x_1,\ldots,x_n\}\in\langle U_1,\ldots,U_n\rangle\cap 2^E. \tag{22}$$

Choose $0<\delta<\varepsilon$ such that $B(x_i,\delta)\subset W_i^l$ for $i=1,2,\ldots,n$. For any map

$$g_j:\{x_1,x_2,\ldots,x_n\}\longrightarrow E \tag{23}$$

for $j=1,2,\ldots,d$, there exists an n-tuple $(\gamma_{0,1},\ldots,\gamma_{0,n})$ such that $V_{\gamma_{0,j}}\subset B(g_j(x_i),\varepsilon)$ for $i=1,2,\ldots,n$, and $j=1,2,\ldots,d$. Thus there is $k\in\{k_1,\ldots,k_r\}(\subset S)$ such that

$$f^{a_j\cdot k}(B(x_i,\delta))\subset f^{a_j\cdot k}\left(W_i^l\right)\subset V_{\gamma_{0,j}}\subset B\left(g_j(x_i),\varepsilon\right) \tag{24}$$

for $i=1,2,\ldots,n$, and $j=1,2,\ldots,d$. This implies $\{x_1,\ldots,x_n\}$ is (S,ε,ρ)-spread in E with respect to $\mathbf{a}$ and hence $\mathcal{S}_{S,\mathbf{a}}(\varepsilon,\rho,E)\cap 2^E$ is dense in 2^E. □

Lemma 7. *Let (X,f) be a dynamical system and $\mathcal{F}$ a Furstenberg family with $k\mathcal{F}$ being countable generated. If E is a Δ-$\mathcal{F}$-weakly mixing subset with respect to $\mathbf{a}$ but not a singleton, then $\mathcal{S}_{\mathcal{F}}(E)\cap 2^E$ is a residual subset of 2^E.*

Proof. Since $k\mathcal{F}$ is countable generated, there exists a sequence $\{S_i\}_{i=1}^{\infty}$ of $\mathcal{P}$ such that

$$k\mathcal{F}=\{S\subset\mathbb{N}:S_i\subset S\text{ for some }i\in\mathbb{N}\}. \tag{25}$$

By Lemma 3, it follows that

$$\begin{aligned}\mathcal{S}_{\mathcal{F},\mathbf{a}}(E)\cap 2^E &= \bigcap_{S\in k\mathcal{F}}\mathcal{S}_{S,\mathbf{a}}(E)\cap 2^E\\ &= \bigcap_{i\in\mathbb{N}}\mathcal{S}_{S_i,\mathbf{a}}(E)\cap 2^E.\end{aligned}\tag{26}$$

And this completes the proof of Lemma 7. □

Lemma 8. *If $C\in\mathcal{S}_{S,\mathbf{a}}(E)$, then, for any closed subset B of C, $d\in\mathbb{N}$, and continuous functions $h_j : B\longrightarrow E$ for $j=1,2,\ldots,d$, there exists a strictly increasing sequence $\{q_k\}_{k=1}^{\infty}\subset S$ such that*

$$\lim_{k\longrightarrow\infty} f^{a_j\cdot q_k}(x)=h_j(x)\tag{27}$$

uniformly on $x\in B$ and $j=1,2,\ldots,d$.

Proof. Let B be a closed subset of C, $d\in\mathbb{N}$. Let $h_j : B\longrightarrow E$, $j=1,2,\ldots,d$, be continuous functions. Then h_j is uniform continuous and this follows that, for any $k\in\mathbb{N}$, there exists $\eta>0$ such that

$$\rho(x,y)<\eta \text{ implies } \rho\left(h_j(x),h_j(y)\right)<\frac{1}{k},\quad j=1,2,\ldots,d.\tag{28}$$

Choose $l>k$ with $1/l<\eta$. Let $\{x_{k,1},\ldots,x_{k,n_k}\}$ and $0<\delta_k<1/l$ be as in the definition of B which is $(S,1/l,\rho)$-spread in E. By the definition of $(S,1/l,\rho)$-spread subset, there exists $q_k\in S$ with $q_k>k$ such that $f^{a_j\cdot q_k}(B(x_{k,i},\delta_k))\subset B(h_j(x_{k,i}),1/l)$ for $i=1,2,\ldots,n_k$ and $j=1,2,\ldots,d$. Without lose of generality, it can be assumed that $\{q_k\}_{k=1}^{\infty}$ is increasing.

We are going to show that the sequence $\{q_k\}_{k=1}^{\infty}$ is as required.

For any $x\in B$, there exists y_{k,n_k^x} such that $\rho(x,y_{k,n_k^x})<\delta_k<1/k<\eta$. Then

$$\begin{aligned}\rho\left(f^{a_j\cdot q_k}(x),h_j(x)\right)&\le\rho\left(f^{a_j\cdot q_k}(x),h_j\left(y_{k,n_k^x}\right)\right)\\&\quad+\rho\left(h_j\left(y_{k,n_k^x}\right),h_j(x)\right)\\&<\frac{1}{l}+\frac{1}{k}<\frac{2}{k}\end{aligned}\tag{29}$$

for $j=1,2,\ldots,d$. Thus

$$\lim_{k\longrightarrow\infty} f^{a_j\cdot q_k}(x)=h_j(x)\tag{30}$$

for $j=1,2,\ldots,d$. This completes the proof of Lemma 8. □

Lemma 9. *If $C_1\subset C_2\subset\cdots$ is a strictly increasing sequence of sets in $\mathcal{S}_{S,\mathbf{a}}(E)$, then, for any subset A of $C:=\cup_{i=1}^{\infty}C_i$, $d\in\mathbb{N}$ and continuous functions $g_j : A\longrightarrow E$ for $j=1,2,\ldots,d$, there exists a strictly increasing sequence $\{q_k\}_{k=1}^{\infty}\subset S$ such that*

$$\lim_{k\longrightarrow\infty} f^{a_j\cdot q_k}(x)=g_j(x)\tag{31}$$

for every $x\in A$ and $j=1,2,\ldots,d$.

Proof. Let $A\subset C$, $d\in\mathbb{N}$, and $g_j : A\longrightarrow E$, $j=1,2,\ldots,d$, be continuous functions. For $k\ge 1$, take $A_k=A\cap C_k$. Since $\mathcal{S}_{S,\mathbf{a}}(E)$ is hereditary, the closure $\overline{A_k}$ of A_k is also in $\mathcal{S}_{S,\mathbf{a}}(E)$ for all $k\ge 1$.

For any $k\in\mathbb{N}$, the set A_k is $(1/k,\rho)$-spread in E. Let $y_{k,1},\ldots,y_{k,n_k}$ and $0<\delta_k<1/k$ be as in the definition of A_k which is $(1/k,\rho)$-spread in E. Then there exists $q_k\in S$ with $q_k>k$ such that

$$\begin{aligned}&\left(f^{a_j\cdot q_k}\right)^{(n)}\left(B\left(y_{k,1},\delta_k\right)\times\cdots\times B\left(y_{k,n},\delta_k\right)\right)\\&\quad\subset B\left(g_j\left(y_{k,1}\right),\frac{1}{k}\right)\times\cdots\times B\left(g_j\left(y_{k,n}\right),\frac{1}{k}\right)\end{aligned}\tag{32}$$

for $j=1,2,\ldots,d$. Without lose of generality, it can be assumed that $\{q_k\}_{k=1}^{\infty}$ is increasing.

We are going to show that the sequence $\{q_k\}$ is as required.

Fix any $x\in A$. There exists $M_1\in\mathbb{N}$ such that $x\in A_k$ for all $k\ge M_1$. By the continuity of g_j, for any $\varepsilon>0$, there exists $M_2\in\mathbb{N}$ with $M_2>\varepsilon$ such that

$$\rho(x,y)<\frac{1}{M_2} \text{ implies } \rho\left(g_j(x),g_j(y)\right)<\varepsilon\quad\text{for } j=1,2,\ldots,d.\tag{33}$$

For every $k>\max\{M_1,M_2\}$, there exists y_{k,n_k^x} such that $\rho(x,y_{k,n_k^x})<\delta_k<1/k$. Then

$$\begin{aligned}\rho\left(f^{a_j\cdot q_k}(x),g_j(x)\right)&\le\rho\left(f^{a_j\cdot q_k}(x),g_j\left(y_{k,n_k^x}\right)\right)\\&\quad+\rho\left(g_j\left(y_{k,n_k^x}\right),g_j(x)\right)\\&<\frac{1}{k}+\varepsilon<2\varepsilon\end{aligned}\tag{34}$$

for $j=1,2,\ldots,d$. Thus

$$\lim_{k\longrightarrow\infty} f^{a_j\cdot q_k}(x)=g_j(x)\tag{35}$$

for $j=1,2,\ldots,d$. This completes the proof of Lemma 9. □

4. The Characterization of $\mathcal{F}$-Δ-Mixing Sets

In this section, we will show the following theorem which generalizes the result in [3].

Theorem 10. *Let (X,f) be a dynamical system, $\mathcal{F}$ a Furstenberg family with $k\mathcal{F}$ being countable generated, E a closed subset of X but not a singleton, and $\mathbf{a}:=\{a_i\}_{i=0}^{\infty}$ a strictly increasing sequence of nonnegative integers with $a_0=0$. Then E is Δ-$\mathcal{F}$-weakly mixing with respect to $\mathbf{a}$ if and only if, for any $S\in k\mathcal{F}$, there exists an increasing sequence of Cantor sets $C_1\subset C_2\subset\cdots$ of E such that $C:=\cup_{k=1}^{\infty}C_k$ is dense in E and*

(i) *for any $d\in\mathbb{N}$, any subset A of C, and any continuous functions $g_j : A\longrightarrow E$ for $j=1,2,\ldots,d$, there exists a strictly increasing sequence $\{q_k\}_{k=1}^{\infty}$ of S such that*

$$\lim_{k\longrightarrow\infty} f^{a_j\cdot q_k}x=g_j(x)\tag{36}$$

for every $x\in A$ and $j=1,2,\ldots,d$;

(ii) *for any* $d \in \mathbb{N}$, $k \in \mathbb{N}$, *any closed subset* B *of* C_k, *and continuous functions* $h_j : B \longrightarrow E$ *for* $j = 1, 2, \ldots, d$, *there exists a strictly increasing sequence* $\{q_k\}_{k=1}^{\infty}$ *of* S *such that*

$$\lim_{k \longrightarrow \infty} f^{a_j \cdot q_k} x = h_j(x) \tag{37}$$

uniformly on $x \in B$ *and* $j = 1, 2, \ldots, d$.

A subset Q of 2^X is called *hereditary* if $2^A \subset Q$ for every set $A \in Q$. The following lemma, which is a consequence of the Kuratowski-Mycielski Theorem, is cited from [3].

Lemma 11. *Let* X *be a perfect compact space. If a hereditary subset* Q *of* 2^X *is residual then there exists an increasing sequence of Cantor sets* $C_1 \subset C_2 \subset \cdots$ *of* X *such that*

(i) $C_i \in Q$ *for every* $i \geq 1$;

(ii) $C = \cup_{i=1}^{\infty} C_i$ *is dense in* X.

Proof of Theorem 10.

Necessity. Since E is $\mathscr{F}$-Δ-weakly mixing with respect to $\mathbf{a}$, by Lemma 7, $\mathcal{S}_{\mathscr{F},\mathbf{a}}(E) \cap 2^E$ is a residual subset of 2^E. Now, by Lemma 11, there exists a strictly increasing sequence of Cantor sets $C_1 \subset C_2 \subset \cdots$ of E such that $C_i \in \mathcal{S}_{\mathscr{F},\mathbf{a}}(E)$ for every $i \geq 1$ and $C = \cup_{i=1}^{\infty} C_i$ is dense in E. The conclusion follows from Lemmas 8 and 9.

Sufficiency. Fix any $S \in k\mathscr{F}$. Let C be the set satisfying the requirement. Let $n, d \geq 2$ and $U_{i,j}$ be nonempty open subsets of X intersecting E for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, d$. It is not hard to see that E is perfect. It follows that there exist pairwise distinct points $x_{i,j} \in U_{i,j} \cap E$ for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, d$.

For $j = 1, 2, \ldots, d-1$, define $g_j : A = \{x_{i,1} : i = 1, 2, \ldots, n\} \longrightarrow E$ by $g_j(x_{i,1}) = x_{i,j+1}$ for $i = 1, 2, \ldots, n$. Choose $\varepsilon > 0$ such that $B(x_{i,j}, \varepsilon) \subset U_{i,j}$ for all i, j. It is clear that g_j are continuous; thus we can find $k \in S$ such that $\rho(f^{a_j \cdot k} x_{i,1}, g_j(x_{i,1})) < \varepsilon$ for all i, j. Then $f^{a_j \cdot k}(x_{i,1}) \in U_{i,j+1}$ for all i, j, which implies

$$S \cap N_{\mathbf{a}}\left(\underbrace{U_{1,1} \cap A \times \cdots \times U_{n,1} \cap A}_{n-times}, \underbrace{U_{1,2} \times \cdots \times U_{n,2}}_{n-times}, \ldots, \underbrace{U_{1,d} \times \cdots \times U_{n,d}}_{n-times}\right) \neq \emptyset. \tag{38}$$

Therefore, the set E is $\mathscr{F}$-Δ-weakly mixing with respect to $\mathbf{a}$. □

It is not hard to see that a subset A of X is Δ-weakly mixing if and only if Δ-$\mathscr{F}_{inf}$-weakly mixing with repect to $\{n\}$, and since $\mathscr{F}_{cf} = \{S \subset \mathbb{N} : S_i \subset S \text{ for some } i \in \mathbb{N}\}$, where $S_i = \{i, i+1, i+2, \ldots\}$ for each $i \in \mathbb{N}$, it follows that $\mathscr{F}_{cf}$ is coutable generated. Thus our result extended [3, Theorem A].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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