Research Article

Localization of Δ-Mixing Property via Furstenberg Families

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1. Introduction

Throughout this paper, a topological dynamical system (or dynamical system, system for short) is a pair (X, f), where X is a compact metric space with metric ρ and f is a continuous map of X to itself. In some literature on nonautonomous discrete systems, the topological dynamical system is usually called the autonomous discrete system; for example, see [1].

Let (X, f) be a dynamical system. For two subsets U, V of X, we define the hitting time set of U and V by

$$N(U, V) = \{ n \in \mathbb{N} : U \cap f^{-n}(V) \neq \emptyset \}.$$  (1)

We say that (X, f) is transitive if, for every two nonempty open subsets U and V of X, the hitting time set N(U, V) is nonempty. Weakly mixing if the product system (X × X, f × f) is transitive, and strongly mixing if, for every two nonempty open subsets U and V of X, there exists N ∈ N such that {N, N + 1, ...} ⊂ N(U, V). In his seminal paper [2], Furstenberg showed that weak mixing implies n-fold transitivity for every positive integer n as follows.

(i) Let (X, f) be a topological dynamical system. Then (X × X, f × f) is transitive if and only if, for every n ∈ N, the product system (X × X × ... × X, f × f × ... × f) (n-times) is also transitive. For convenience, we denote (X × X × ... × X, f × f × ... × f) (n-times) by (X^n, f^n).

A closed subset C of X is called Δ-transitive if, for every d ≥ 2, there exists a residual subset C’ of C such that, for every x ∈ C’, the orbit closure of the diagonal d-tuple x(d), that is, x(d) = (x, x, ..., x), under the action f × f × ... × f

d-times

contains C’. A closed subset C of X with at least two points is Δ-weakly mixing if, for every n ≥ 1, C^n is Δ-transitive in the n-th product system (X^n, f^n). In [3], Huang et al. show that Δ-weakly mixing sets exhibit a nice characterization as follows.

Theorem 1 (see [3]). Let (X, f) be a dynamical system and E a closed subset of X but not a singleton. Then E is Δ-weakly mixing if and only if there exists a strictly increasing sequence of Cantor sets C_1 ⊂ C_2 ⊂ ... ⊂ E such that C := \bigcup_{k=1}^{\infty} C_k is dense in E and

(i) for any d ∈ N, any subset A of C, and any continuous functions g_j : A → E for j = 1, 2, ..., d, there exists a strictly increasing sequence {q_k}_{k=1}^{\infty} of positive integers such that

$$\lim_{k \to \infty} f^{q_k} x = g_j(x)$$  (2)

for every x ∈ A and j = 1, 2, ..., d;
For recent results, see ties goes back at least to Gottschalk and Hedlund [4] and was developed further by Furstenberg [5].

In this paper, we will extend the notion of \( \Delta \)-weakly mixing sets via Furstenberg family and show that \( \Delta \)-weakly mixing sets with respect to a sequence also share the same characterization under some considerable conditions.

2. Preliminary

In this section, we provide some definitions which will be used later.

Let \( \mathcal{F} \) denote the collection of all subsets of \( \mathbb{N} \). A subset \( \mathcal{F} \) of \( \mathcal{P} \) is called a Furstenberg family (or family for short), if it admits the property of hereditary upward; that is to say,

\[
F_1 \subset F_2 \text{ and } F_1 \in \mathcal{F} \implies F_2 \in \mathcal{F}.
\]

A family \( \mathcal{F} \) is called proper if it is neither empty nor all of \( \mathcal{P} \).

For a family \( \mathcal{F} \), the dual family of \( \mathcal{F} \) is defined by

\[
\kappa \mathcal{F} = \left\{ F \in \mathcal{P} : F \cap F' \neq \emptyset, \text{ for every } F' \in \mathcal{F} \right\}.
\]

Let \( \mathcal{F}_{inf} \) be the family of all infinite subsets of \( \mathbb{N} \). It is easy to see that its dual family \( \kappa \mathcal{F}_{inf} \), denoted by \( \mathcal{F}_{df} \), is the family of all cofinite subsets.

Any nonempty collection \( \mathcal{A} \) of subsets of \( \mathbb{N} \) naturally generates a family

\[
[\mathcal{A}] = \left\{ F \subset \mathbb{N} : A \subset F \text{ for some } A \in \mathcal{A} \right\}.
\]

We say that \( \mathcal{F} \) is countable generated if \( \mathcal{F} = [\mathcal{A}] \) for a collection \( \mathcal{A} \) consisting of countable subsets of \( \mathbb{N} \).

The idea of using families to describe dynamical properties goes back at least to Gottschalk and Hedlund [4] and was developed further by Furstenberg [5]. For recent results, see [6–9].

Let \( (X, f) \) be a dynamical system, \( \mathcal{F} \) a family, and \( a := \{a_i\}_{i=0}^{\infty} \) a strictly increasing sequence of nonnegative integers with \( a_0 = 0 \).

For every \( d \geq 2 \) and subsets \( U_1, U_2, \ldots, U_d \) of \( X \), we define the hitting time set of \( U_1, U_2, \ldots, U_d \) by

\[
N_a(U_1, U_2, \ldots, U_d) := \left\{ n \in \mathbb{N} : \exists j \leq d \text{ such that } f^{a_j(n)} U_j \neq \emptyset \right\}.
\]

We say that \( A \) is \( \Delta \mathcal{F} \)-weakly mixing with respect to \( a \) if, for every \( n, d \geq 2 \), and nonempty open subsets \( U_{i,j} \) of \( X \) intersecting \( A \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, d \), we have

\[
\bigcap_{i=1}^{n} N_a(U_{i,1} \cap A, U_{i,2}, \ldots, U_{i,d}) \in \mathcal{F}.
\]

Let \( (X, \rho) \) be a compact metric space. Denote \( 2^X \) by the collection of all nonempty closed subsets of \( X \) and endow \( 2^X \) with the Hausdorff metric

\[
\rho_H(A, B) = \max \left\{ \max_{x \in A} \min_{y \in B} \rho(x, y), \max_{y \in B} \min_{x \in A} \rho(x, y) \right\}
\]

for any \( A, B \in 2^X \). Then the metric space \( (2^X, \rho_H) \) is compact whenever \( X \) is compact. For any nonempty subsets \( S_1, \ldots, S_n \subset X \), denote

\[
\langle S_1, \ldots, S_n \rangle = \left\{ A \in 2^X : A \subset \bigcup_{i=1}^{n} S_i \text{ and } A \cap S_i \neq \emptyset \text{ for each } i = 1, \ldots, n \right\},
\]

then the following family

\[
\{ \langle U_1, \ldots, U_n \rangle : U_1, \ldots, U_n \text{ are non-empty open subsets of } X, n \in \mathbb{N} \}
\]
forms a basis for a topology of \( 2^X \), which is called the Vietoris topology. It is well known that the Hausdorff topology induced by the Hausdorff metric \( \rho_H \) coincides with the Vietoris topology for \( 2^X \).

3. Key Lemmas

In this section, we provide some lemmas which will be used later.

Let \( (X, f) \) be a dynamical system, \( \mathcal{F} \) a family, and \( a := \{a_i\}_{i=0}^{\infty} \) a strictly increasing sequence of nonnegative integers with \( a_0 = 0 \), and \( E \) a closed subset of \( X \). For \( \varepsilon > 0 \), \( d \geq 1 \), and any \( S \in k \mathcal{F} \), we say that a subset \( A \) of \( X \) is \( (S, \varepsilon, \rho) \)-spread with respect to \( a \), if there exist \( \delta \in (0, \varepsilon) \), \( n \in \mathbb{N} \), and distinct points \( x_1, x_2, \ldots, x_n \in X \) such that \( A \subset \bigcup_{i=1}^{n} B(x_i, \delta) \) and for any maps

\[
g_j : [x_1, x_2, \ldots, x_n] \longrightarrow E
\]

where \( j = 1, 2, \ldots, d \), there exists \( k \in S \) such that \( 1/k < \varepsilon \) and

\[
\left( f^{a_{d-1}(n)} B(x_1, \delta) \times \cdots \times B(x_n, \delta) \right) \subset B \left( g_j(x_1), \varepsilon \right) \times \cdots \times B \left( g_j(x_n), \varepsilon \right)
\]

for \( j = 1, 2, \ldots, d \). For any \( S \in k \mathcal{F} \), denote by \( \delta_{\mathcal{F}a}(\varepsilon, \rho, E) \) the collection of all closed sets that are \( (S, \varepsilon, \rho) \)-spread in \( E \) with respect to \( a \).

Remark 2. It is not hard to check that the \( \delta_{\mathcal{F}a}(\varepsilon, \rho, E) \) is hereditary; that is, if \( A \) is \( (S, \varepsilon, \rho) \)-spread in \( E \) with respect to \( a \) and \( B \) is a nonempty closed subset of \( A \), then \( B \) is also \( (S, \varepsilon, \rho) \)-spread in \( E \) with respect to \( a \).
Let

\[
\delta_{S,a}(E) = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \delta_{S,a}\left(\frac{1}{k}, \rho, E\right),
\]
\[
\delta_{\mathcal{F},a}(E) = \bigcap_{S \in \mathcal{F}} \delta_{S,a}(E).
\]

**Lemma 3.** If \(E\) is a \(\Delta-\mathcal{F}\)-weakly mixing subset of \(X\) but not a singleton, then \(\delta_{S,a}(E) \cap \mathbb{Z}^2\) is a dense open subset of \(\mathbb{Z}^2\) for every \(S \in k\mathcal{F}\).

**Proof.** Fix \(\varepsilon > 0\), \(d \in \mathbb{N}\), and \(S \in k\mathcal{F}\). We will divide our discussion into three claims to show that \(\delta_{S,a}(E) \cap \mathbb{Z}^2\) is a dense open subset of \(\mathbb{Z}^2\).

**Claim 4.** \(\delta_{S,a}(E, \rho, E)\) is open in \(\mathbb{Z}^2\).

**Proof of Claim 4.** Let \(A \in \delta_{S,a}(E, \rho, E)\). For any \(S \in k\mathcal{F}\), let \(\delta > 0\) and \(x_1, x_2, \ldots, x_n \in X\) as in the definition of set \((S, \varepsilon, \rho)\)-spread in \(E\) with respect to \(a\). For \(i = 1, 2, \ldots, n\), put \(U_i = B(x_i, \delta)\). Then, by Remark 2, every closed subset \(B \subset U_i\) is \((S, \varepsilon, \rho)\)-spread in \(E\) with respect to \(a\). It follows that

\[
A \in (U_1, \ldots, U_n) \subset \delta_{S,a}(E, \rho, E).
\]

Hence \(\delta_{S,a}(E, \rho, E)\) is open in \(\mathbb{Z}^2\).

**Claim 5.** \(E\) is a perfect set.

**Proof of Claim 5.** If \(E\) is not perfect, then there exists an isolated point \(x\) of \(E\). Note that the set \(\{x\}\) is open in \(E\). It follows that there are two nonempty open subsets \(U_1, U_2 \subset X\) such that

\[
U_1 \cap E = \{x\},
\]
\[
U_2 \cap E \neq \emptyset
\]

and \(U_1 \cap U_2 = \emptyset\) since the set \(E\) is not a singleton. So, for any \(m \in S\), such that \(f^{a,m}\{x\} \cap U_1 \neq \emptyset\), one has \(f^{a,m}\{x\} \cap U_2 = \emptyset\). This is a contradiction, and thus \(E\) is perfect.

**Claim 6.** \(\delta_{S,a}(E, \rho, E) \cap \mathbb{Z}^2\) is dense in \(\mathbb{Z}^2\).

**Proof of Claim 6.** Fix \(n\) nonempty open subsets \(U_1, \ldots, U_n\) of \(X\) intersecting \(E\). We want to show that

\[
\langle U_1, \ldots, U_n \rangle \cap \left(\delta_{S,a}(E, \rho, E) \cap \mathbb{Z}^2\right) \neq \emptyset.
\]

Since \(E\) is compact, there exists a finite subset \(\{x_1, x_2, \ldots, x_m\}\) of \(E\) such that \(U_m = B(x_i, \varepsilon/2)\) for \(i = 1, 2, \ldots, m\). For convenience, denote \(B_i = B(x_i, \varepsilon/2)\) for \(i = 1, 2, \ldots, m\). Since \(E\) is \(\Delta-\mathcal{F}\)-weakly mixing with respect to \(a\), the set \(E\) is perfect, so we may assume that \(m \geq 1/\varepsilon\). The collection of \(n\)-tuples on the set \(\{1, 2, \ldots, m\}^n\) can be arranged as the following finite sequence:

\[
\{(y_{1,1}, \ldots, y_{1,n}), (y_{2,1}, \ldots, y_{2,n}), \ldots, (y_{n,1}, \ldots, y_{n,n})\}.
\]

For \((y_{1,1}, \ldots, y_{1,n})\), as \(E\) is \(\Delta-\mathcal{F}\)-weakly mixing with respect to \(a\), there exists \(k_1 \in S\) with \(k_1 > m\), such that

\[
(U_1 \cap E) \cap f^{-a_k^{r_k}B_{y_{r,1}, \varepsilon/2}} \cap \cdots \cap f^{-a_k^{r_k}B_{y_{r,d}, \varepsilon/2}} \neq \emptyset
\]

for \(i = 1, 2, \ldots, n\). By continuity of \(f\), we can choose a nonempty subset \(W_i^{1}\) of \(U_i\) intersecting \(E\) such that \(f^{a_k^{r_k}}W_i^{1} \subset B_{y_{r,j}, \varepsilon/2}\) for \(j = 1, 2, \ldots, d\). Similarly, for \((y_{2,1}, \ldots, y_{2,n})\), since \(E\) is a strictly increasing sequence of positive integers, there exist \(k_2 \in S\) with \(k_2 > k_1\) and nonempty subsets \(W_i^{2}\) of \(W_i^{1}\) intersecting \(E\) such that

\[
W_i^{1} \subset \cdots \subset W_i^{d} \subset U_i
\]

for \(i = 1, 2, \ldots, n\), and

\[
(f^{a_k^{r_k}})^{(m)}(W_i^{1} \times W_i^{2} \times \cdots \times W_i^{d})
\]

for \(r = 1, 2, \ldots, l\), and \(j = 1, 2, \ldots, d\).

For each \(i = 1, 2, \ldots, n\), pick \(x_i \in W_i^{d}\) \(\cap E\). Since \(E\) is perfect, it is reasonable to assume that those \(x_i\)'s are distinct. It is clear that

\[
\{x_1, \ldots, x_n\} \subset \langle U_1, \ldots, U_n \rangle \cap \mathbb{Z}^2.
\]

Choose \(0 < \delta < \varepsilon\) such that \(B(x_i, \delta) \subset W_i^{d}\) for \(i = 1, 2, \ldots, n\). For any map

\[
g_j : \{x_1, x_2, \ldots, x_n\} \longrightarrow E
\]

for \(j = 1, 2, \ldots, d\), there exists an \(n\)-tuple \((y_{1,1}, \ldots, y_{1,n})\) such that \(V_{y_{r,j}} \subset B(g_j(x_i), \varepsilon)\) for \(i = 1, 2, \ldots, n\), and \(j = 1, 2, \ldots, d\). Thus there is \(k \in \{k_1, \ldots, k_n\} \subset S\) such that

\[
f^{a_k^{r_k}}(B(x_i, \delta)) \subset f^{a_k^{r_k}}(W_i^{1}) \subset V_{y_{r,j}} \subset B(g_j(x_i), \varepsilon)
\]

for \(i = 1, 2, \ldots, n\), and \(j = 1, 2, \ldots, d\). This implies \(\{x_1, \ldots, x_n\}\) is \((S, \varepsilon, \rho)\)-spread in \(E\) with respect to \(a\) and hence \(\delta_{S,a}(E, \rho, E) \cap \mathbb{Z}^2\) is dense in \(\mathbb{Z}^2\).

**Lemma 7.** Let \((X, f)\) be a dynamical system and \(\mathcal{F}\) a Furstenberg family with \(k\mathcal{F}\) being countable generated. If \(E\) is a \(\Delta-\mathcal{F}\)-weakly mixing subset with respect to \(a\) but not a singleton, then \(\delta_{\mathcal{F}}(E) \cap \mathbb{Z}^2\) is a residual subset of \(\mathbb{Z}^2\).

**Proof.** Since \(k\mathcal{F}\) is countable generated, there exists a sequence \(\{S_i\}_{i=1}^{\infty}\) of \(\mathcal{F}\) such that

\[
k\mathcal{F} = \{S \subset \mathbb{N} : S \subset S \text{ for some } i \in \mathbb{N}\}.
\]
Proof. Let $A \subset C$, $d \in \mathbb{N}$, and $g_j : A \rightarrow E$, $j = 1, 2, \ldots, d$, be continuous functions. For $k \geq 1$, take $A_k = A \cap C_k$. Since $\delta_{S,a}(E)$ is hereditary, the closure $A_k$ of $A_k$ is also in $\delta_{S,a}(E)$ for all $k \geq 1$.

For any $k \in \mathbb{N}$, the set $A_k$ is $(1/k, \rho)$-spread in $E$. Let $\gamma_{1,k}, \ldots, \gamma_{d,k}$, and $0 < \delta_k < 1/k$ be as in the definition of $A_k$ which is $(1/k, \rho)$-spread in $E$. Then there exists $q_k \in S$ with $q_k > k$ such that

$$\rho \left( (f^{\gamma_{1,k}})_j(B(\gamma_{1,k}, \delta_k) \times \cdots \times B(\gamma_{d,k}, \delta_k)) \right) < \frac{1}{k}$$

for $j = 1, 2, \ldots, d$. Without lose of generality, it can be assumed that $|q_k|_{k=1}$ is increasing. We are going to show that the sequence $|q_k|_{k=1}$ is as required.

For any $x \in A$, there exists $y_{k,n}$ such that $\rho(x, y_{k,n}) < \delta_k < 1/k$. Then

$$\rho \left( f^{\gamma_{1,k}}(x), y_{k,n} \right) < \frac{1}{k}$$

for $j = 1, 2, \ldots, d$. Thus

$$\lim_{k \rightarrow \infty} f^{\gamma_{1,k}}(x) = g_j(x)$$

for $j = 1, 2, \ldots, d$. This completes the proof of Lemma 9.

4. The Characterization of $\mathcal{F}$-$\Delta$-Mixing Sets

In this section, we will show the following theorem which generalizes the result in [3].

Theorem 10. Let $(X, f)$ be a dynamical system, $\mathcal{F}$ a Furstenberg family with $\mathcal{F}$ being countable generated, $E$ a closed subset of $X$ but not a singleton, and $a = \{a_i\}_{i=0}^\infty$ a strictly increasing sequence of nonnegative integers with $a_0 = 0$. Then $E$ is $\Delta$-$\mathcal{F}$-weakly mixing with respect to $a$ if and only if, for any $S \in \mathcal{F}$, there exists an increasing sequence of Cantor sets $C_1 \subset C_2 \subset \cdots \subset \mathcal{F}$ such that $C := \cup_{k=1}^\infty C_k$ is dense in $E$ and

(i) for any $d \in \mathbb{N}$, any subset $A$ of $C$, and any continuous functions $g_j : A \rightarrow E$ for $j = 1, 2, \ldots, d$, there exists a strictly increasing sequence $\{q_k\}_{k=1}^\infty \subset S$ such that

$$\lim_{k \rightarrow \infty} f^{\gamma_{1,k}}(x) = g_j(x)$$

for every $x \in A$ and $j = 1, 2, \ldots, d$;
(ii) for any \( d \in \mathbb{N}, k \in \mathbb{N} \), any closed set \( B \) of \( C_k \), and continuous functions \( h_j : B \to E \) for \( j = 1, 2, \ldots, d \), there exists a strictly increasing sequence \( \{ q_k \}_{k=1}^{\infty} \) of \( S \) such that
\[
\lim_{k \to \infty} f^{q_k} h_j x = h_j (x)
\]
uniformly on \( x \in B \) and \( j = 1, 2, \ldots, d \).

A subset \( Q \) of \( 2^X \) is called hereditary if \( 2^A \subset Q \) for every set \( A \in Q \). The following lemma, which is a consequence of the Kuratowski-Mycielski Theorem, is cited from [3].

**Lemma 11.** Let \( X \) be a perfect compact space. If a hereditary subset \( Q \) of \( 2^X \) is residual then there exists an increasing sequence of Cantor sets \( C_1 \subset C_2 \subset \cdots \) of \( X \) such that

(i) \( C_i \in Q \) for every \( i \geq 1 \);

(ii) \( C = \bigcup_{i=1}^{\infty} C_i \) is dense in \( X \).

**Proof of Theorem 10.**

Necessity. Since \( E \in \mathcal{F} \)-\( \Delta \)-weakly mixing with respect to \( a \), by Lemma 7, \( \mathcal{F}(E) \cap 2^E \) is a residual subset of \( 2^E \). Now, by Lemma 11, there exists a strictly increasing sequence of Cantor sets \( C_1 \subset C_2 \subset \cdots \) of \( E \) such that \( C_i \in \mathcal{F}(E) \) for every \( i \geq 1 \) and \( C = \bigcup_{i=1}^{\infty} C_i \) is dense in \( E \). The conclusion follows from Lemmas 8 and 9.

Sufficiency. Fix any \( S \in k\mathcal{F} \). Let \( C \) be the set satisfying the requirement. Let \( n, d \geq 2 \) and \( U_{i,j} \) be nonempty open subsets of \( X \) intersecting \( E \) for \( i = 1, 2, \ldots, n \), and \( j = 1, 2, \ldots, d \). It is not hard to see that \( E \) is perfect. It follows that there exist pairwise distinct points \( x_{i,j} \in U_{i,j} \cap E \) for \( i = 1, 2, \ldots, n \), and \( j = 1, 2, \ldots, d \). For \( j = 1, 2, \ldots, d-1 \), define \( g_j : A = \{ x_{i,j} : i = 1, 2, \ldots, n \} \to E \) by \( g_j (x_{i,j}) = x_{i,j+1} \) for \( i = 1, 2, \ldots, n \). Choose \( \varepsilon > 0 \) such that \( B(x_{i,j}, \varepsilon) \subset U_{i,j} \) for all \( i, j \). It is clear that \( g_j \) is continuous; thus we can find \( k \in S \) such that \( \rho(f^{n \cdot k} x_{i,j}, g_j (x_{i,j})) < \varepsilon \) for all \( i, j \). Then \( f^{n \cdot k} (x_{i,j}) \in U_{i,j+1} \) for all \( i, j \), which implies

\[
S \cap N_a \left( U_{1,1} \cap A \times \cdots \times U_{n,1} \cap A, U_{1,2} \times \cdots \times U_{n,2} \times \cdots \right)
\]

\[
\cdots \times U_{1,d} \times \cdots \times U_{n,d}
\]

\[
\neq \emptyset.
\]

Therefore, the set \( E \) is \( \mathcal{F} \)-\( \Delta \)-weakly mixing with respect to \( a \).

\[\square\]

**References**


