

Research Article

Robust Optimal Excess-of-Loss Reinsurance and Investment Problem with Delay and Dependent Risks

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This paper investigates a robust optimal excess-of-loss reinsurance and investment problem with delay and dependent risks for an ambiguity-averse insurer (AAI). The AAI's wealth process is assumed to be two dependent classes of insurance business. He/she can purchase excess-of-loss reinsurance from the reinsurer and invest in a risk-free asset and a risky asset whose price follows Heston model. We obtain the explicit expressions of the optimal excess-of-loss reinsurance and investment strategy by maximizing the expected exponential utility of AAI's terminal wealth. Finally, we give the proof of the verification theorem. Our models and results posed here can be regarded as a generalization of the existing results in the literature.

1. Introduction

In the past decade, the topic about optimal investment and reinsurance problems has attracted a lot of attention. These optimal problems have been studied in terms of various objectives, for example, [1–4] considered the objective function of minimizing the ruin probability; [5–9] studied the optimal problems aiming to maximize the survival probability or the expected utility of terminal wealth; [10–13] investigated optimal reinsurance and investment problems under mean variance criterion.

Although so many notable scholars have considered the optimal reinsurance and investment problems, two important aspects are still being worthy of further exploration. One is lack of considering ambiguity, and the other one is the optimal control problems under delayed systems. On one hand, the model uncertainties do exist widely in finance, especially in insurance, the field of asset pricing, consumption, and portfolio selection. As a result, the ambiguity-averse insurer (AAI) has to look for a methodology to handle this uncertainty. One possible way is to use the robust approach, where some alternative models closed to the estimated model are introduced and the robust optimal strategy is obtained. Recently, some scholars paid more attention to

optimal investment-reinsurance problems with ambiguity. Reference [14] assumed that the insurer's wealth process follows a diffusion model, and they optimized a proportional reinsurance and investment problem with model uncertainty. Reference [15] obtained the robust optimal proportional reinsurance and investment strategies for an AAI; in their article, the surplus process is assumed to be a Cramér–Lundberg risk model and the risky asset's price follows a constant elasticity of variance (CEV) model. Reference [16] studied the robust optimal proportional reinsurance and investment strategies for both an insurer and a reinsurer. Reference [17] took default risk into account and derived the robust optimal control strategy under variance premium. Different from the above-mentioned literature, [18] analyzed a robust optimal problem of excess-of-loss reinsurance and investment in a model with jumps for an AAI.

On the other hand, the investors or the insurers make important future decisions only according to the present states of a system, but they do not consider the past states. However, the future states of a system usually may depend on its past states, which do exist in our real-world systems. For example, in the stock market, investors not only are concerned with the present stock price but also pay more attention to the trend of the stock price in the past periods.

Thus, it is more realistic to take some past information of the system into account. Due to the structure of infinite-dimensional state space, generally speaking, it is difficult to solve these stochastic control problems with delay analytically. As a result, there is an explicit solution for this problem. Only those problems, in which some special forms of delay information are considered in the state process, are found to be finite-dimensional and then can be solved (see, for example, [19–21]). Moreover, the delay is first introduced into the optimal proportional reinsurance and investment problems by [22] under mean-variance criterion. Reference [23] optimized the delayed problem of excess-of-loss reinsurance and investment under maximizing the expected exponential utility of the insurer's terminal wealth. Reference [24] took multiple dependent classes of insurance business into consideration, investigated the time-consistent reinsurance-investment problem with delay and derived the optimal strategy under the mean-variance criterion. As mentioned in [24], in fact, some insurance businesses are usually correlated by some way in practice. For example, a traffic accident (or fire accidents or car accidents or aviation accidents and so on) may cause property loss or medical claims or death claims; these insurance businesses will be correlated. Therefore, it is necessary to take dependent risks into account in the actuarial literature. References [25, 26] assumed that the insurer's surplus process consists of two or more dependent classes of insurance business and the claim number processes are correlated through a common shock component, and they discussed optimal proportional reinsurance problems under the criterion of maximizing the expected utility of terminal wealth. For other research about dependent risks, we refer the readers to [27–31] and the references therein.

This paper takes excess-of-loss reinsurance into account, which is better than proportional reinsurance in most situations; see [32]. Suppose that the insurer's wealth process consists of two dependent classes of insurance business. The insurer is allowed to purchase excess-of-loss reinsurance and invest in a financial market which consists of a risk-free asset and a risky asset. The risky asset's price is described by Heston model. Moreover, it is assumed that there exists capital inflow into or outflow from the insurer's current wealth. Given that the insurer's claim process and the risky asset price (true model) may deviate from a relative good estimated model (reference model) in real-world, the model uncertainty should be taken into consideration. On the basis of the above setup, we first formulate a robust optimal control problem with delay and dependent risks and then investigate the optimal strategy for an AAI by maximizing the expected exponential utility of terminal wealth. This paper has the following main contributions: (i) an optimal excess-of-loss reinsurance and investment problem with dependent risks is studied; (ii) both ambiguity and the capital inflow/outflow are introduced into this problem; (iii) some special cases are provided, such as the case of investment-only, ambiguity-neutral insurer, and no delay, which demonstrates that our model and results can be considered as a generalization of the existing results in some literature, e.g., [23, 25].

The rest of this paper is structured as follows. We present the formulation of our model in Section 2. Section 3 discusses the robust optimal strategy and derives the optimal results. Section 4 is devoted to proving the verification theorem. Some special cases of our model are provided in Section 5. Section 6 concludes the paper. In Appendix, technical proofs are presented.

2. Model Formulation

We consider a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$, where T represents the terminal time and is a positive finite constant and \mathcal{F}_t stands for the information of the market available up to time t . Assume that all processes introduced below are well-defined and adapted processes in this space. In addition, suppose that trading takes place continuously and involves no taxes or transaction costs and that all securities are infinitely divisible.

2.1. Surplus Process. This section presents a risk model consisting of two dependent classes of insurance business. The insurer's wealth process is modeled of

$$dU(t) = cdt - d\left(\sum_{i=1}^{N_1(t)+N(t)} X_i\right) - d\left(\sum_{i=1}^{N_2(t)+N(t)} Y_i\right), \quad (1)$$

where the positive constant c is the premium rate; X_i is the i th claim size from the first class; $\{X_i, i \geq 1\}$ are assumed to be i.i.d. positive random variables with common distribution function $F_X(\cdot)$, finite first moment $E(X_i) = \mu_X > 0$, and second moment $E(X_i^2) = \sigma_X^2$; Y_i is the i th claim size from the second class and $\{Y_i, i \geq 1\}$ are assumed to be i.i.d. positive random variables with common distribution function $F_Y(\cdot)$, finite first moment $E(Y_i) = \mu_Y > 0$, and second moment $E(Y_i^2) = \sigma_Y^2$; $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are three independent Poisson processes with positive intensity parameters λ_1 , λ_2 , and λ , respectively.

The compound Poisson processes $\sum_{i=1}^{N_1(t)+N(t)} X_i$ and $\sum_{i=1}^{N_2(t)+N(t)} Y_i$ represent the cumulative amount of claims for the first class and the second class in time interval $[0, t]$, respectively. $\{X_i, i \geq 1\}$, $\{Y_i, i \geq 1\}$, $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are mutually independent. Further, the insurer's premium rate is calculated according to the expected value principle; i.e.

$$c = (1 + \eta_1)(\lambda_1 + \lambda)\mu_X + (1 + \eta_2)(\lambda_2 + \lambda)\mu_Y, \quad (2)$$

where $\eta_i > 0$ is the insurer's safety loading from the i th claim.

In addition, we assume that $F_X(0) = 0$, $0 < F_X(x) < 1$ for $0 < x < D_1$ and $F_X(x) = 1$ for $x \geq D_1$, where $D_1 = \sup\{x : F_X(x) \leq 1\} < +\infty$ represents the maximum claim size from the first class; and $F_Y(0) = 0$, $0 < F_Y(y) < 1$ for $0 < y < D_2$, and $F_Y(y) = 1$ for $y \geq D_2$, where $D_2 = \sup\{y : F_Y(y) \leq 1\} < +\infty$ represents the maximum claim size from the second class.

2.2. Excess-of-Loss Reinsurance. Suppose that the insurer can purchase excess-of-loss reinsurance by reducing the

underlying claims risk. Denote by m_1 and m_2 the excess-of-loss retention levels, and let

$$\begin{aligned} X_i^m &= \min \{X_i, m_1\}, \\ Y_i^m &= \min \{Y_i, m_2\} \end{aligned} \quad (3)$$

be the parts of the first claims and the second claims held by the insurer, respectively. Then by (1), the wealth process becomes

$$\begin{aligned} dX^m(t) &= c^m dt - d \left(\sum_{i=1}^{N_1(t)+N(t)} X_i^m \right) \\ &\quad - d \left(\sum_{i=1}^{N_2(t)+N(t)} Y_i^m \right), \end{aligned} \quad (4)$$

with the premium rate

$$\begin{aligned} c^m &= c - (1 + \xi_1)(\lambda_1 + \lambda)(\mu_X - EX_i^m) \\ &\quad - (1 + \xi_2)(\lambda_2 + \lambda)(\mu_Y - EY_i^m) \\ &= (\lambda_1 + \lambda)[(\eta_1 - \xi_1)\mu_X + (1 + \xi_1)EX_i^m] \\ &\quad + (\lambda_2 + \lambda)[(\eta_2 - \xi_2)\mu_Y + (1 + \xi_2)EY_i^m], \end{aligned} \quad (5)$$

where ξ_i is the reinsurer's safety loading from the i th claim. Assume that $\xi_i > \eta_i$, which implies that the reinsurance is not cheap. According to [33], the wealth process (4) can be approximated by the following diffusion model:

$$\begin{aligned} dX^m(t) &= (c_1 + c_2) dt + \gamma_1 dW_X(t) + \gamma_2 dW_Y(t) \\ &\stackrel{d}{=} (c_1 + c_2) dt + \sqrt{\gamma_1^2 + \gamma_2^2 + 2\rho_{XY}\gamma_1\gamma_2} dW_0(t), \end{aligned} \quad (6)$$

where

$$c_1 = (\lambda_1 + \lambda)[(\eta_1 - \xi_1)\mu_X + \xi_1 EX_i^m], \quad (7)$$

$$\gamma_1 = \sqrt{(\lambda_1 + \lambda) E [X_i^m]^2},$$

$$c_2 = (\lambda_2 + \lambda)[(\eta_2 - \xi_2)\mu_Y + \xi_2 EY_i^m], \quad (8)$$

$$\gamma_2 = \sqrt{(\lambda_2 + \lambda) E [Y_i^m]^2},$$

$W_X(t)$ and $W_Y(t)$ are two standard Brownian motions whose correlation coefficient is

$$\rho_{XY} = \frac{\lambda}{\gamma_1\gamma_2} EX_i^m EY_i^m, \quad (9)$$

and $W_0(t)$ is another standard Brownian motion dependent of $W_X(t)$ and $W_Y(t)$.

For convenience, let

$$\begin{aligned} g_X(m_1) &= EX_i^m = \int_0^{m_1} \bar{F}_X(x) dx, \\ g_Y(m_2) &= EY_i^m = \int_0^{m_2} \bar{F}_Y(y) dy, \\ G_X(m_1) &= E[X_i^m]^2 = \int_0^{m_1} 2x\bar{F}_X(x) dx, \\ G_Y(m_2) &= E[Y_i^m]^2 = \int_0^{m_2} 2y\bar{F}_Y(y) dy, \end{aligned} \quad (10)$$

where $\bar{F}_X(x) = 1 - F_X(x)$, $\bar{F}_Y(y) = 1 - F_Y(y)$.

2.3. Financial Market. The insurer is assumed to invest in a risk-free asset whose price process $B(t)$ is governed by

$$dB(t) = rB(t) dt, \quad B(0) = b_0, \quad (11)$$

and a risky asset whose price process $S(t)$ follows Heston model,

$$\begin{aligned} dS(t) &= S(t) \left[(r + \alpha L(t)) dt + \sqrt{L(t)} dW_1(t) \right], \\ S(0) &= s_0, \end{aligned} \quad (12)$$

$$dL(t) = k(\delta - L(t)) dt + \sigma \sqrt{L(t)} d\bar{W}_2(t),$$

$$L(0) = l_0,$$

where positive constant r is the risk-free interest rate, α, k, δ , and σ are all positive constants, and $W_1(t)$ and $\bar{W}_2(t)$ are two standard Brownian motions with $E[W_1(t)\bar{W}_2(t)] = \rho t$, $\rho \in [-1, 1]$. By standard Gaussian linear regression, $\bar{W}_2(t)$ can be rewritten as

$$d\bar{W}_2(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t), \quad (13)$$

where $W_2(t)$ is another standard Brownian motion. We assume that $W_0(t)$, $W_1(t)$, and $W_2(t)$ are mutually independent. Moreover, we require $2k\delta \geq \sigma^2$ to ensure that $L(t)$ is almost surely nonnegative.

2.4. Wealth Process with Delay. Let $u := \{u(t) := m_1(t), m_2(t), \pi(t)\}_{t \in [0, T]}$ be the reinsurance-investment strategy, where $m_i(t)$ ($i = 1, 2$) is the excess-of-loss retention level for i th claim at time t , where $m_i(t) = D_i$ means "no reinsurance" and $m_i(t) = 0$ means "full reinsurance," and $\pi(t)$ is the money amount invested in the risky asset at time t , the amount of money invested in the risk-free asset at time t is $X^u(t) - \pi(t)$, and here $X^u(t)$ is the insurer's wealth after adopting strategy u . Thus, the evolution of $X^u(t)$ is governed by

$$\begin{aligned} dX^u(t) &= \pi(t) \frac{dS(t)}{S(t)} + (X^u(t) - \pi(t)) \frac{dB(t)}{B(t)} \\ &\quad + dX^m(t). \end{aligned} \quad (14)$$

It is noted that the wealth process is traditionally formulated as (14), which is a stochastic differential equation (SDE) without delay. In the sections below, we will formulate a wealth process with delay, which is caused by the instantaneous capital inflow into or outflow from the insurer's current wealth. The delayed wealth process is still denoted by $X^u(\cdot)$. Let $Y^u(t)$, $\bar{Y}^u(t)$, and $Z^u(t)$ be the delayed wealth and average and pointwise performance of the wealth in the past horizon $[t-h, t]$, respectively, i.e.

$$Y^u(t) = \int_{-h}^0 e^{\beta s} X^u(t+s) ds, \quad (15)$$

$$\bar{Y}^u(t) = \frac{Y^u(t)}{\int_{-h}^0 e^{\beta s} ds},$$

$$Z^u(t) = X^u(t-h), \quad (16)$$

for $\forall t \in [0, T]$, where $\beta \geq 0$ is an average parameter and $h > 0$ is the delay parameter. Denote by the function

$$\psi(t, X^u(t) - \bar{Y}^u(t), X^u(t) - Z^u(t)) \quad (17)$$

the capital inflow/outflow amount, where $X^u(t) - Z^u(t)$ represents the absolute performance of wealth between t and $t-h$, and $X^u(t) - \bar{Y}^u(t)$ stands for the average performance of the wealth in $[t-h, t]$. Such capital inflow or outflow, which is related to the past performance of the wealth, may come out in various situations. For example, a good past performance of the wealth may bring the insurer more gain. On the contrary, a poor past performance of the wealth may force the insurer to seek further capital injection to cover the loss so as to achieve the final performance objective. Following [22, 23], when we consider such a capital inflow/outflow function, the wealth process $X^u(t)$ can be given as follows:

$$\begin{aligned} dX^u(t) &= \pi(t) \frac{dS(t)}{S(t)} + (X^u(t) - \pi(t)) \frac{dB(t)}{B(t)} \\ &+ dX^m(t) \\ &- \psi(t, X^u(t) - \bar{Y}^u(t), X^u(t) - Z^u(t)) dt. \end{aligned} \quad (18)$$

Such capital inflow/outflow, which is related to the past performance of the wealth, may come out in various situations. For example, a good past performance of the wealth may bring the insurer more gain and further the insurer can pay a part of the gain as dividend to his/her stakeholders, in this case $\psi < 0$. On the contrary, a poor past performance of the wealth may force the insurer to seek further capital injection to cover the loss so as to achieve the final performance objective and in this case $\psi > 0$. To make this problem solvable, we assume that the amount of the capital inflow/outflow is proportional to the past performance of the insurer's wealth, i.e.

$$\begin{aligned} \psi(t, X^u(t) - \bar{Y}^u(t), X^u(t) - Z^u(t)) \\ = a_1 (X^u(t) - \bar{Y}^u(t)) + a_2 (X^u(t) - Z^u(t)) \end{aligned} \quad (19)$$

where a_1 and a_2 are two nonnegative constants. Inserting (6), (11), (12), and (19) into (18) leads to the following stochastic differential delay equation (SDDE):

$$\begin{aligned} dX^u(t) &= [AX^u(t) + c_1 + c_2 + \alpha\pi(t)L(t) + \bar{a}_1 Y^u(t) \\ &+ a_2 Z^u(t)] dt + \pi(t) \sqrt{L(t)} dW_1(t) \\ &+ \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} dW_0(t), \end{aligned} \quad (20)$$

where

$$\begin{aligned} A &= r - a_1 - a_2, \\ \bar{a}_1 &= \frac{a_1}{\int_{-h}^0 e^{\beta s} ds}. \end{aligned} \quad (21)$$

Furthermore, assume that $X^u(t) = x_0 > 0$, $\forall t \in [-h, 0]$, which means that the insurer is endowed with the initial wealth x_0 at time $-h$ and does not start the investment and (re)insurance business until time 0. Therefore, the initial value of the average performance wealth $Y^u(0)$ is

$$Y^u(0) = \frac{x_0(1 - e^{-\beta h})}{\beta}. \quad (22)$$

Given that the investment performance has an effect on the insurer's wealth, this paper assumes that the insurer is concerned with $X^u(T)$ and $\bar{Y}^u(T)$ in the time interval $[T-h, T]$. Moreover, suppose that the insurer has the following exponential utility function defined by

$$U(X^u(T), \bar{Y}^u(T)) = -\frac{1}{\nu} e^{-\nu(X^u(T) + \bar{\varepsilon}\bar{Y}^u(T))}, \quad (23)$$

where $\nu > 0$ and $0 < \bar{\varepsilon} < 1$ are constants. Here ν represents a constant absolute risk aversion coefficient which plays a vital role in insurance practice and actuarial mathematics. Note that $\bar{\varepsilon}$ is the weight of $\bar{Y}^u(T)$, so $\bar{\varepsilon}\bar{Y}^u(T)$ will impact the average performance of the terminal wealth. We consider the integrated delayed wealth $Y^u(t)$ rather than the average one $\bar{Y}^u(t)$ directly. Define $\varepsilon = \bar{\varepsilon} / \int_{-h}^0 e^{\beta s} ds$ as the transformed weight, which can be considered as the weight between $X^u(T)$ and $Y^u(T)$. Thus (23) is equivalent to considering

$$U(X^u(T), Y^u(T)) = -\frac{1}{\nu} e^{-\nu(X^u(T) + \varepsilon Y^u(T))}. \quad (24)$$

For simplicity, we call the combination $X^u(T) + \varepsilon Y^u(T)$ the terminal wealth. In fact, our modeling framework for the term $X^u(T) + \varepsilon Y^u(T)$ is consistent with the classical literature (e.g., [34, 35]) on stochastic control problem with delay. Reference [23] considered a utility function, which is similar to (23) in our paper and studied an optimal problem with delay for an insurer; in their article, the insurer's surplus process is assumed to follow the classical Cramér-Lundberg model. Compared with [23], we not only incorporate model uncertainty into our study which will be introduced later but also assume that the wealth process consists of two classes

of insurance business, in which the two claim processes are dependent.

In traditional, it is assumed that the insurer is ambiguity-neutral with the following objective function:

$$\begin{aligned} & \sup_{u \in \bar{\Pi}} E[U(X^u(T), Y^u(T)) \mid X^u(t) = x, Y^u(t) \\ & = y, L(t) = l]. \end{aligned} \quad (25)$$

where $E(\cdot)$ is the expectation under the probability measure P and $\bar{\Pi}$ is the set of admissible strategies u which will be defined in Definition 1. However, many insurers are ambiguity-averse and always try to guard themselves against worse-case scenarios. Thus, it is reasonable to suppose an insurer is ambiguity-averse in the field of insurance. In what follows, we present a robust portfolio choice with uncertainty for an AAI. Suppose that the AAI has a relative good estimated model (also called reference model) to describe the risky assets prices and his/her claim process, but he/she is always skeptical about this reference model and hopes to take alternative models into account. According to [36], the alternative models are defined by a set of probability measures Q which are equivalent to the P as follows:

$$Q := \{Q \mid Q \sim P\}. \quad (26)$$

Definition 1. For any fixed $t \in [0, T]$, the strategy u is said to be admissible if it is F_t -progressively measurable and satisfies

- (i) $m_i(t) \in [0, D_i]$, $i = 1, 2$;
- (ii) $E(\int_0^T \|u(s)\|^2 ds) < \infty$, where $\|u(t)\|^2 = m_1^2(t) + m_2^2(t) + \pi^2(t)$;
- (iii) $\forall(t, x, y, l) \in [0, T] \times R \times R \times R^+$, the SDDE (20) has a pathwise unique solution $\{X^u(t), t \in [0, T]\}$ with $E_{t,x,y,l}^{Q^*} U[(X^u(T), Y^u(T))] < \infty$, where Q^* is the chosen model to describe the worst case, $E_{t,x,y,l}[\cdot]$ is the condition expectation given $X^u(t) = x, Y^u(t) = y, L(t) = l$. Let Π be the set of all admissible strategies.

Next, define a process $\{\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t)) \mid t \in [0, T]\}$ satisfying that

- (i) $\theta(t)$ is F_t -measurable, $\forall t \in [0, T]$;
- (ii) $E[\exp\{(1/2) \int_0^T \|\theta(t)\|^2 dt\}] < \infty$, where $\|\theta(t)\|^2 = \sum_{i=0}^2 \theta_i^2(t)$. We denote Θ for the space of all such processes θ .

For $\forall \theta \in \Theta$, we define a real-valued process $\{\Lambda^\theta(t) \mid t \in [0, T]\}$ on (Ω, F, P) by

$$\Lambda^\theta(t) = \left\{ -\int_0^t \theta(u) dW(u) - \frac{1}{2} \int_0^t \|\theta(u)\|^2 du \right\}, \quad (27)$$

where $W(t) = (W_0(t), W_1(t), W_2(t))'$. By Ito's differentiation rule,

$$d\Lambda^\theta(t) = \Lambda^\theta(t) [-\theta(t) dW(t)]. \quad (28)$$

Thus $\Lambda^\theta(t)$ is a P -martingale. Hence, $E[\Lambda^\theta(t)] = 1$. For $\forall \theta \in \Theta$, a new real-world probability measure Q absolutely continuous to P on F_T is defined by

$$\left. \frac{dQ}{dP} \right|_{F_T} := \Lambda^\theta(t). \quad (29)$$

So far, we have constructed a family of real-world probability measures Q parameterized by $\theta \in \Theta$. Applying Girsanov's theorem, we can see that

$$dW^Q(t) = dW(t) + \theta(t)' dt, \quad (30)$$

under the alternative measure Q , is a standard three-dimension Brownian motion, where

$$W^Q(t) = (W_0^Q(t), W_1^Q(t), W_2^Q(t))'. \quad (31)$$

Note that the alternative models in class Q only differ in the drift terms. Thus, the risky asset's price (12) under Q is

$$\begin{aligned} dS(t) &= S(t) \left[(r + \alpha L(t) - \theta_1(t) \sqrt{L(t)}) dt \right. \\ & \left. + \sqrt{L(t)} dW_1^Q(t) \right], \\ dL(t) &= \left[k(\delta - L(t)) - \sigma \rho \theta_1(t) \sqrt{L(t)} \right. \\ & \left. - \sigma \theta_2(t) \sqrt{L(t)} \sqrt{1 - \rho^2} \right] dt + \sigma \rho \sqrt{L(t)} dW_1^Q(t) \\ & \left. + \sigma \sqrt{L(t)} \sqrt{1 - \rho^2} dW_2^Q(t) \right]. \end{aligned} \quad (32)$$

And the wealth process (20) under Q is rewritten as

$$\begin{aligned} dX^u(t) &= [AX^u(t) + c_1 + c_2 + \bar{a}_1 Y^u(t) + a_2 Z^u(t) \\ & + \alpha \pi L(t)] dt - \left(\pi \theta_1(t) \sqrt{L(t)} \right. \\ & \left. + \theta_0(t) \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} \right) dt \\ & + \pi \sqrt{L(t)} dW_1^Q(t) \\ & \left. + \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} dW_0^Q(t) \right]. \end{aligned} \quad (33)$$

Inspired by [37, 38], we formulate the following robust control problem to modify problem (25), i.e.,

$$\begin{aligned} J(t, x, y, l) &= \sup_{u \in \bar{\Pi}} \left\{ \inf_{Q \in \Theta} E_{t,x,y,l}^Q \left[U(X^u(T), Y^u(T)) \right. \right. \\ & \left. \left. + \int_t^T \Psi(s, X^u(s), Y^u(s), \theta(s)) ds \right] \right\}, \end{aligned} \quad (34)$$

where

$$\Psi(t, X^u(t), Y^u(t), \theta(t)) = \frac{\|\theta(t)\|^2}{2\phi(t, X^u(t), Y^u(t))}, \quad (35)$$

and $E_{t,x,y,l}^Q$ is calculated under Q . In (34), the deviations from the reference model are penalized by the second term in the expectation. In fact, this penalty term depends on the relative entropy arising from diffusion risk. In addition, the parameter ϕ in (35) represents the strength of the preference for robustness. For analytical tractability, suppose that the parameter ϕ in (35) is state-dependent. In particular, following [37, 39], we set

$$\phi(t, x, y) = \frac{-m}{vJ(t, x, y, l)} \geq 0, \quad (36)$$

where m (≥ 0) stands for the ambiguity-aversion coefficient and describes the AAI's attitude to the diffusion risk.

For convenience, we first provide some notations. Let $O_0 \subset R \times R \times R^+$ be an open set and $O = [0, T] \times O_0$. Let $C^{1,2,1,2}$ be the space of $J(t, x, y, l)$ such that J and its partial derivatives $J_t, J_x, J_y, J_l, J_{xx}, J_{xl}, J_{ll}$ are continuous on O . To make the problem (34) solvable, by dynamic programming principle, the robust Hamilton-Jacobi-Bellman (HJB) equation for (34) can be derived as (see [39, 40]):

$$\begin{aligned} \sup_{u \in \Pi} \inf_{Q \in \mathcal{Q}} \{ & \mathbf{A}^{\theta, u} J(t, X^u(t), Y^u(t), l) \\ & + \Psi(t, X^u(t), Y^u(t), \theta(t)) \} = 0, \end{aligned} \quad (37)$$

for $t < T$ with boundary condition

$$J(T, x, y, l) = U(x, y), \quad (38)$$

where $\mathbf{A}^{\theta, u}$ is the generator of (33) under Q and is defined as

$$\begin{aligned} \mathbf{A}^{\theta, u} J = & J_t + \left[Ax + c_1 + c_2 + \bar{a}_1 y + a_2 z + \pi \alpha l \right. \\ & \left. - \pi \theta_1 \sqrt{l} - \theta_0 \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} \right] J_x \\ & + \frac{1}{2} \left[\pi^2 l + \gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2) \right] J_{xx} + (x \\ & - \beta y - e^{-\beta h} z) J_y + \left[k(\delta - l) - \sigma \rho \theta_1 \sqrt{l} \right. \\ & \left. - \sigma \theta_2 \sqrt{l} \sqrt{1 - \rho^2} \right] J_l + \frac{1}{2} l \sigma^2 J_{ll} + \pi l \sigma \rho J_{xl}. \end{aligned} \quad (39)$$

Here, J is a short notation for $J(t, x, y, l)$

3. Robust Optimal Results with Delay

The aim of this section is to find the robust optimal control strategy for problem (34) under the exponential utility. As mentioned above, in general, the delayed control problem is infinite-dimensional. In order to make this problem be finite-dimensional and solvable, according to [23], we assume the parameters satisfies the following conditions:

$$a_2 = \varepsilon e^{-\beta h}, \quad (40)$$

$$\bar{a}_1 = (\beta + A + \varepsilon) \varepsilon. \quad (41)$$

It is noted that the above two conditions are the sufficient conditions for the optimal control problem with delay, which guarantee that the HJB equation (37) has a closed-form solution. Furthermore, they help us explore the implication of the problems with delay and without delay.

In what follows, we try to solve the HJB equation (37) with boundary condition (38). We conjecture that the value function has the following form:

$$\begin{aligned} J(t, x, y, l) \\ = -\frac{1}{v} \exp \{ -v [H(t)(x + \varepsilon y) + K(t) + G(t, l)] \}, \end{aligned} \quad (42)$$

with $H(T) = 1, K(T) = 0, G(T, l) = 0$. Let G_t, G_l, G_{ll} be the partial derivatives of $G(t, l)$. From (42), we get

$$\begin{aligned} J_t &= -v \left[(x + \varepsilon y) H' + K' + G_t \right] J, \\ J_x &= -v H J, \\ J_y &= -v \varepsilon H J, \\ J_l &= -v G_l J, \\ J_{xx} &= v^2 H^2 J, \\ J_{xl} &= v^2 H G_l J, \\ J_{ll} &= (v^2 G_l^2 - v G_{ll}) J. \end{aligned} \quad (43)$$

Step 1. Putting (43) into (37) and rearranging terms, since $J < 0$, we get

$$\begin{aligned} \inf_{u \in \Pi} \sup_{Q \in \mathcal{Q}} \{ & - \left((x + \varepsilon y) H' + K' + G_t \right) \\ & - H \left[Ax + c_1 + c_2 + \bar{a}_1 y + a_2 z + \pi \alpha l \right. \\ & \left. - \pi \theta_1 \sqrt{l} - \theta_0 \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} \right] \\ & + \frac{v H^2}{2} \left[\pi^2 l + \gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2) \right] \\ & - \varepsilon H (x - \beta y - e^{-\beta h} z) - G_l \\ & \left[k(\delta - l) - \sigma \rho \theta_1 \sqrt{l} - \sigma \theta_2 \sqrt{l} \sqrt{1 - \rho^2} \right] \\ & \left. + \frac{l \sigma^2}{2} (v G_l^2 - G_{ll}) + \pi l \sigma \rho v H G_l - \frac{\|\theta(t)\|^2}{2m} \right\} \\ = & 0. \end{aligned} \quad (44)$$

Step 2. According to (44), fixing u and maximizing over θ yields the first-order condition for the minimum point θ^* as follows:

$$\begin{aligned}\theta_0^* &= mH\sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)}, \\ \theta_1^* &= m[\pi H\sqrt{l} + \sigma\rho\sqrt{l}G_l], \\ \theta_2^* &= m\sigma\sqrt{l}\sqrt{1 - \rho^2}G_l.\end{aligned}\tag{45}$$

Replacing (45) back into (44) leads to

$$\begin{aligned}& -((x + \varepsilon y)H' + K' + G_t) - [Ax + M + \bar{a}_1 y + a_2 z] \\ & \cdot H - \varepsilon H \cdot (x - \beta y - e^{-\beta h} z) - k(\delta - l)G_l \\ & + \frac{m + \nu}{2}l\sigma^2 G_l^2 - \frac{l\sigma^2}{2}G_{ll} + \inf_{\pi} \left\{ \frac{m + \nu}{2}H^2 l\pi^2 \right. \\ & \left. + Hl\pi((m + \nu)\sigma\rho G_l - \alpha) \right\} \\ & + \inf_{m_1, m_2} \left\{ \frac{m + \nu}{2}(\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2))H \right. \\ & \left. - \xi_1(\lambda_1 + \lambda)g_X(m_1) - \xi_2(\lambda_2 + \lambda)g_Y(m_2) \right\} H \\ & = 0,\end{aligned}\tag{46}$$

where

$$M = (\lambda_1 + \lambda)(\eta_1 - \xi_1)\mu_X + (\lambda_2 + \lambda)(\eta_2 - \xi_2)\mu_Y.\tag{47}$$

By the first-order condition for π , we get

$$\pi^*(t) = \frac{\alpha - \sigma\rho(m + \nu)G_l}{(m + \nu)H}.\tag{48}$$

Plugging (48) into (46) derives

$$\begin{aligned}& -((x + \varepsilon y)H' + K' + G_t) - [Ax + M + \bar{a}_1 y + a_2 z] \\ & \cdot H - \varepsilon(x - \beta y - e^{-\beta h} z)H - k(\delta - l)G_l + \frac{m + \nu}{2} \\ & \cdot l\sigma^2 G_l^2 - \frac{l\sigma^2}{2}G_{ll} - \frac{l(\alpha - (m + \nu)\sigma\rho G_l)^2}{2(m + \nu)} \\ & + \inf_{m_1, m_2} \left\{ \frac{m + \nu}{2}(\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2))H \right. \\ & \left. - \xi_1(\lambda_1 + \lambda)g_X(m_1) - \xi_2(\lambda_2 + \lambda)g_Y(m_2) \right\} H \\ & = 0.\end{aligned}\tag{49}$$

Let

$$\begin{aligned}f(m_1, m_2, t) &= \left\{ \frac{m + \nu}{2}(\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2))H \right. \\ & \left. - \xi_1(\lambda_1 + \lambda)g_X(m_1) - \xi_2(\lambda_2 + \lambda)g_Y(m_2) \right\} H.\end{aligned}\tag{50}$$

To find the minimizer $m_1^*(t)$ and $m_2^*(t)$ of f , we need to take the first and the second derivatives of f with respect to m_1 and m_2 . It is assumed that $F_X(x)$ and $F_Y(y)$ are continuous and differentiable, and $F_X'(x) = f_X'(x)$, $F_Y'(y) = f_Y'(y)$. For any $t \in [0, T]$, differentiating f with respect to m_1, m_2 yields

$$\begin{aligned}\frac{\partial f}{\partial m_1} &= (\lambda_1 + \lambda)H\bar{F}_X(m_1) \\ & \cdot \left[\left(m_1 + \frac{\lambda}{\lambda_1 + \lambda}g_Y(m_2) \right) (m + \nu)H - \xi_1 \right], \\ \frac{\partial f}{\partial m_2} &= (\lambda_2 + \lambda)H\bar{F}_Y(m_2) \\ & \cdot \left[\left(m_2 + \frac{\lambda}{\lambda_2 + \lambda}g_X(m_1) \right) (m + \nu)H - \xi_2 \right].\end{aligned}\tag{51}$$

We first consider that $m_i^*(t) \in [0, D_i)$, then $0 < \bar{F}_X(m_1^*(t)) \leq 1$ and $0 < \bar{F}_Y(m_2^*(t)) \leq 1$. Moreover, suppose that there exists at least one point (m_1^*, m_2^*) satisfying the following equation:

$$\begin{aligned}m_1 + \frac{\lambda}{\lambda_1 + \lambda}g_Y(m_2) - \frac{\xi_1}{(m + \nu)H} &= 0, \\ m_2 + \frac{\lambda}{\lambda_2 + \lambda}g_X(m_1) - \frac{\xi_2}{(m + \nu)H} &= 0.\end{aligned}\tag{52}$$

Taking (m_1^*, m_2^*) into the second derivatives of f , we will get the following Hessian matrix:

$$\begin{aligned}\begin{vmatrix} \frac{\partial^2 f}{\partial m_1^2} & \frac{\partial^2 f}{\partial m_1 \partial m_2} \\ \frac{\partial^2 f}{\partial m_2 \partial m_1} & \frac{\partial^2 f}{\partial m_2^2} \end{vmatrix}_{(m_1^*, m_2^*)} &= H^4 \bar{F}_X(m_1^*) \bar{F}_Y(m_2^*) \\ & \cdot [(\lambda_1 + \lambda)(\lambda_2 + \lambda) - \lambda^2 \bar{F}_X(m_1^*) \bar{F}_Y(m_2^*)] > 0,\end{aligned}\tag{53}$$

which means this Hessian matrix is positive definite at the point (m_1^*, m_2^*) . Therefore, if we can find (m_1^*, m_2^*) such that (52) holds, then the point (m_1^*, m_2^*) is the minimizer of f .

Step 3. Inserting (m_1^*, m_2^*) into (49) yields

$$\begin{aligned}(x + \varepsilon y)H' + K' + G_t &+ [(A + \varepsilon)x + M + (\bar{a}_1 - \beta\varepsilon)y + (a_2 - \varepsilon e^{-\beta h})z]H \\ &+ (k(\delta - l) - \sigma\rho l\alpha)G_l - \frac{m + \nu}{2}l\sigma^2(1 - \rho^2)G_l^2 \\ &+ \frac{l\sigma^2}{2}G_{ll} + \frac{l\alpha^2}{2(m + \nu)} - f(m_1^*, m_2^*, t) = 0.\end{aligned}\tag{54}$$

With the assumptions (40) and (41), we have $(a_2 - \varepsilon e^{-\beta h})_Z = 0$ and $\bar{a}_1 - \beta \varepsilon = (A + \varepsilon)\varepsilon$, and then (54) can be transformed into

$$\begin{aligned} & (x + \varepsilon y) \left[H'(t) + (A + \varepsilon) H(t) \right] + K' + MH + G_t \\ & + (k(\delta - l) - \sigma \rho \alpha) G_l - \frac{m + \nu}{2} l \sigma^2 (1 - \rho^2) G_l^2 \\ & + \frac{l \sigma^2}{2} G_{ll} + \frac{l \alpha^2}{2(m + \nu)} - f(m_1^*, m_2^*, t) = 0. \end{aligned} \quad (55)$$

According to the arbitrariness of x and y , (55) is equivalent to the two following equations:

$$H'(t) + (A + \varepsilon) H(t) = 0, \quad (56)$$

$$\begin{aligned} & K' + MH - f(m_1^*, m_2^*, t) + G_t \\ & + (k(\delta - l) - \sigma \rho \alpha) G_l - \frac{m + \nu}{2} l \sigma^2 (1 - \rho^2) G_l^2 \\ & + \frac{l \sigma^2}{2} G_{ll} + \frac{l \alpha^2}{2(m + \nu)} = 0. \end{aligned} \quad (57)$$

For (56), taking the boundary condition $H(T) = 1$ into account yields

$$H(t) = e^{(A+\varepsilon)(T-t)}, \quad 0 \leq t \leq T. \quad (58)$$

In order to determine the point (m_1^*, m_2^*) clearly, (52) is transformed into

$$\frac{m_1 + (\lambda/(\lambda_1 + \lambda)) g_Y(m_2)}{m_2 + (\lambda/(\lambda_2 + \lambda)) g_X(m_1)} = \frac{\xi_1}{\xi_2}, \quad (59)$$

or equivalently,

$$\xi_2 m_1 - \frac{\lambda \xi_1}{\lambda_2 + \lambda} g_X(m_1) = \xi_1 m_2 - \frac{\lambda \xi_2}{\lambda_1 + \lambda} g_Y(m_2). \quad (60)$$

Next, we need to define three following auxiliary functions:

$$\begin{aligned} l_X(x) &= \xi_2 x - \frac{\lambda \xi_1}{\lambda_2 + \lambda} g_X(x), \\ l_Y(x) &= \xi_1 x - \frac{\lambda \xi_2}{\lambda_1 + \lambda} g_Y(x), \\ k(x) &= \xi_2 \left(\frac{x}{\xi_1} - \frac{e^{-(A+\varepsilon)(T-t)}}{m + \nu} \right). \end{aligned} \quad (61)$$

For convenience, we assume that $\xi_1 \geq \xi_2$. It is easy to verify that both $l_Y(x)$ and $k(x)$ are strictly increasing functions for $x \geq 0$, so their inverse functions $l_Y^{-1}(x)$ and $k^{-1}(x)$ exist. From (60), we get $l_X(m_1) = l_Y(m_2)$, and then

$$m_2 = l_Y^{-1}(l_X(m_1)). \quad (62)$$

Plugging (62) and (58) into the second equation of (52), we obtain

$$l_Y^{-1}(l_X(m_1)) + \frac{\lambda}{\lambda_2 + \lambda} g_X(m_1) = \frac{\xi_2}{m + \nu} e^{-(A+\varepsilon)(T-t)}. \quad (63)$$

Therefore, let

$$\begin{aligned} h(x) &= l_Y^{-1}(l_X(x)) + \frac{\lambda}{\lambda_2 + \lambda} g_X(x) \\ &\quad - \frac{\xi_2}{m + \nu} e^{-(A+\varepsilon)(T-t)}. \end{aligned} \quad (64)$$

If the equation $h(x) = 0$ has a solution on $[0, D_1]$, the solution is indeed m_1^* we try to derive, and as a result, m_2^* will be easily determined. Then, the robust optimal reinsurance strategy can be derived and summarized in the following theorem.

Theorem 2. Assume that $\xi_1 \geq \xi_2$, and let

$$\begin{aligned} a_l &= \sup \{x \geq 0, l_X(x) = 0\}, \\ a_k &= k^{-1}(0) = \frac{\xi_1}{m + \nu} e^{-(A+\varepsilon)(T-t)}. \end{aligned} \quad (65)$$

$$t_0 = T - \frac{1}{A + \varepsilon} \ln \frac{\xi_1}{(m + \nu) a_l}. \quad (66)$$

For the robust optimal control problem (34), the optimal excess-of-loss reinsurance strategy is given as follows.

(i) If $\xi_1/(m + \nu) a_l \geq 1$ or $a_l = 0$, we have $t_0 \leq T$. For $t_0 \leq t \leq T$, the optimal excess-of-loss reinsurance strategy is $(m_1^*(t), m_2^*(t)) = (\tilde{m}_1^*(t), \tilde{m}_2^*(t))$, where

$$\begin{aligned} \tilde{m}_1^*(t) &= x_0(t) I \{h(D_1) \geq 0\} + D_1 I \{h(D_1) < 0\}, \\ \tilde{m}_2^*(t) &= l_Y^{-1}(l_X(x_0(t))) I \{h(D_1) \geq 0, l_X(x_0(t)) \\ &\leq l_Y(D_2)\} + D_2 I \{h(D_1) \geq 0, l_X(x_0(t)) \\ &> l_Y(D_2)\} + \left(\frac{\xi_2}{m + \nu} e^{-(A+\varepsilon)(T-t)} \right. \\ &\quad \left. - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \right) \cdot I \left\{ h(D_1) \right. \\ &< 0, \frac{\xi_2}{m + \nu} e^{-(A+\varepsilon)(T-t)} - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) < D_2 \left. \right\} \\ &\quad + D_2 I \left\{ h(D_1) < 0, \frac{\xi_2}{m + \nu} e^{-(A+\varepsilon)(T-t)} \right. \\ &\quad \left. - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \geq D_2 \right\}, \end{aligned} \quad (67)$$

and $x_0(t) \in [a_l, D_1]$ is the unique solution to $h(x) = 0$ if $h(D_1) \geq 0$ holds.

For $0 \leq t < t_0$, if the solution to $h(x) = 0$ exists, the optimal excess-of-loss reinsurance strategy is $(m_1^*(t), m_2^*(t)) = (\tilde{m}_1^*(t), \tilde{m}_2^*(t))$, where

$$\begin{aligned} \tilde{m}_1^*(t) &= \frac{\xi_1}{m + \nu} e^{-(A+\varepsilon)(T-t)} I \left\{ \frac{\xi_1}{m + \nu} e^{-(A+\varepsilon)(T-t)} < D_1 \right\} \\ &\quad + D_1 I \left\{ \frac{\xi_1}{m + \nu} e^{-(A+\varepsilon)(T-t)} \geq D_1 \right\}, \\ \tilde{m}_2^*(t) &= 0. \end{aligned} \quad (68)$$

(ii) If $\xi_1/(m + \nu)a_l < 1$, we have $t_0 > T$. For $0 \leq t \leq T < t_0$, the expression of optimal excess-of-loss reinsurance strategy is $(m_1^*(t), m_2^*(t)) = (\widehat{m}_1^*(t), \widehat{m}_2^*(t))$.

Proof. (i) If $a_l = 0$ or $\xi_1/(m + \nu)a_l \geq 1$, then we have $t_0 \leq T$.

For $t_0 \leq t \leq T$, we have $0 < a_l \leq a_k$. Since $l_Y(x)$ and $k(x)$ are increasing functions on $[a_l, +\infty)$, it is easy to verify that $h(x)$ increases on $[a_l, +\infty)$. What is more, we can get $h(a_l) = k(a_l) \leq k(a_k) = 0$ by the fact that $l_X(a_l) = 0$.

If $h(D_1) \geq 0$, the equation $h(x) = 0$ admits a unique solution $x_0(t) \in [a_l, D_1]$, and thus the optimal excess-of-loss strategy is

$$m_1^*(t) = x_0(t). \tag{69}$$

Due to $m_2(t) \leq D_1$, we obtain

$$\begin{aligned} m_2^*(t) &= l_Y^{-1}(l_X(x_0(t))) \\ &\cdot I \{h(D_1) \geq 0, l_X(x_0(t)) \leq l_Y(D_2)\} \\ &+ D_2 I \{h(D_1) \geq 0, l_X(x_0(t)) > l_Y(D_2)\} \end{aligned} \tag{70}$$

If $h(D_1) < 0$, it implies that the equation $h(x) = 0$ has a unique solution on $(D_1, +\infty)$. Due to $m_1(t) \leq D_1$, we choose $m_1^*(t) = D_1$. At this time, substituting $m_1^*(t) = D_1$ back into $f(m_1, m_2, t)$ in (50) and taking the first derivative of $f(D_1, m_2, t)$ with respect to m_2 derives the minimizer of $f(D_1, m_2, t)$ is

$$m_2^*(t) = \frac{\xi_2}{m + \nu} e^{-(A+\varepsilon)(T-t)} - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \tag{71}$$

Consequently, from (68), (69), and (70), $m_1^*(t)$ and $m_2^*(t)$ can be expressed as $\widehat{m}_1^*(t)$ and $\widehat{m}_2^*(t)$ in (67) and (68), respectively.

For $0 \leq t < t_0$, $a_l > a_k$ holds; then we have $h(a_l) = k(a_l) > k(a_k) = 0$. Meanwhile, $h(x)$ strictly increases on $[a_l, +\infty)$ and $h(x) > 0$, $x \in [a_l, +\infty)$. As a result, the equation $h(x) = 0$ has no solution on $[a_l, +\infty)$. In other words, there does not exist the solution (m_1^*, m_2^*) satisfying (52) when $m_1 \in [a_l, +\infty)$ and $m_2 \in [0, +\infty)$. However, it does not mean that (52) has no solution on $m_1 \in [0, a_l)$ and $m_2 \in [0, +\infty)$. It is not difficult to prove that $l_X(x)$ is a convex function on $[0, a_l)$. So for $0 \leq t < t_0$, if the solution of the equation $h(x) = 0$ exists, it will only be obtained on $x \in [0, a_l)$ which is indeed $m_1^*(t)$ we try to derive.

Because $l_X(0) = l_X(a_l) = 0$, and $l_X(x)$ is a convex function on $[0, a_l)$, we can derive that $l_X(m_1) \leq 0$ on $m_1 \in [0, a_l)$. What is more, $l_Y^{-1}(x)$ is strictly increasing and $l_Y^{-1}(0) = 0$, so we obtain from (62) that $m_2 = l_Y^{-1}(l_X(m_1)) \leq 0$. Due to $m_2 \geq 0$, we get $m_2^*(t) = 0$. Plugging $m_2^*(t) = 0$ into $f(m_1, m_2, t)$ in (50) and taking the first derivative of $f(m_1, 0, t)$ with respect to m_1 derives the minimizer of $f(m_1, 0, t)$ is

$$m_1^*(t) = \frac{\xi_1}{m + \nu} e^{-(A+\varepsilon)(T-t)} = a_k \leq a_l. \tag{72}$$

To sum up, for the case of $0 \leq t < t_0$, we derive (68).

(ii) If $\xi_1/(m + \nu)a_l < 1$, we have $t_0 > T$. Thus for $0 \leq t \leq T < t_0$, the inequality $a_l \geq a_k$ holds. The optimal problem is similar to that of $0 \leq t < t_0$ in case (i). At this time, we obtain the optimal reinsurance strategy for $0 \leq t \leq T$ is (68). This ends the proof of Theorem 2. \square

In order to get the expressions for π^* and the value function $J(t, x, y, l)$, we have to derive the expressions of $K(t)$ and $G(t, l)$ in (57). By (58), we can rewrite (57) as

$$\begin{aligned} K' + Me^{(A+\varepsilon)(T-t)} - f(m_1^*, m_2^*, t) + G_t \\ + (k(\delta - l) - \sigma\rho l\alpha)G_l - \frac{m + \nu}{2}l\sigma^2(1 - \rho^2)G_l^2 \\ + \frac{l\sigma^2}{2}G_{ll} + \frac{l\alpha^2}{2(m + \nu)} = 0. \end{aligned} \tag{73}$$

Now we discuss this problem in two cases as follows.

Case 1. If $\xi_1/(m + \nu)a_l \geq 1$, for $t_0 \leq t \leq T$, the optimal reinsurance strategy for the problem (34) is (67). Denoting by G_1 the function G in (73), we have

$$\begin{aligned} K' + Me^{(A+\varepsilon)(T-t)} - f(\widehat{m}_1^*, \widehat{m}_2^*, t) + G_{1t} \\ + (k(\delta - l) - \sigma\rho l\alpha)G_{1l} - \frac{m + \nu}{2}l\sigma^2(1 - \rho^2)G_{1l}^2 \\ + \frac{l\sigma^2}{2}G_{1ll} + \frac{l\alpha^2}{2(m + \nu)} = 0. \end{aligned} \tag{74}$$

For equation (74), we conjecture a solution of the following form

$$G_1(t, l) = g_1(t)l + \bar{g}_1(t), \tag{75}$$

with $g_1(T) = 0$, $\bar{g}_1(T) = 0$. Then (74) is transformed into

$$\begin{aligned} (\bar{g}_1 + K)' + Me^{(A+\varepsilon)(T-t)} - f(\widehat{m}_1^*, \widehat{m}_2^*, t) + k\delta g_1 \\ + l \left(g_1' - (k + \sigma\rho\alpha)g_1 - \frac{m + \nu}{2}\sigma^2(1 - \rho^2)g_1^2 \right. \\ \left. + \frac{\alpha^2}{2(m + \nu)} \right) = 0. \end{aligned} \tag{76}$$

That is

$$\begin{aligned} \bar{g}_1' + Me^{(A+\varepsilon)(T-t)} - f(\widehat{m}_1^*, \widehat{m}_2^*, t) + k\delta g_1 + l \left(g_1' \right. \\ \left. - (k + \sigma\rho\alpha)g_1 - \frac{m + \nu}{2}\sigma^2(1 - \rho^2)g_1^2 \right. \\ \left. + \frac{\alpha^2}{2(m + \nu)} \right) = 0. \end{aligned} \tag{77}$$

where

$$\bar{g}_1(t) = \bar{g}_1(t) + K(t) \tag{78}$$

with the boundary condition $\bar{g}_1(T) = 0$.

According to the arbitrariness of l , decomposing (77) into

$$g_1' - (k + \sigma\rho\alpha)g_1 - \frac{m + \nu}{2}\sigma^2(1 - \rho^2)g_1^2 + \frac{\alpha^2}{2(m + \nu)} = 0 \tag{79}$$

and

$$\bar{g}_1' + Me^{(A+\varepsilon)(T-t)} - f(\bar{m}_1^*, \bar{m}_2^*, t) + k\delta g_1 = 0. \tag{80}$$

According to [41], taking the boundary condition $g_1(T) = 0$, $\bar{g}_1(T) = 0$ into account yields

$$\hat{g}_1(t) = \begin{cases} l_2 k \delta (T-t) - 2k\delta (\sigma^2(m + \nu)(1 - \rho^2))^{-1} \ln \left| \frac{l_1 - l_2}{l_1 - l_2 e^{(1/2)\sigma^2(l_1 - l_2)(m + \nu)(1 - \rho^2)(T-t)}} \right|, & \rho \neq \pm 1, \\ \frac{\alpha^2 k \delta}{2(m + \nu)(k + \sigma\alpha)} \left(T - t - \frac{1}{k + \sigma\alpha} (1 - e^{-(k + \sigma\alpha)(T-t)}) \right), & \rho = 1, \\ \frac{\alpha^2 k \delta}{2(m + \nu)(k - \sigma\alpha)} \left(T - t - \frac{1}{k - \sigma\alpha} (1 - e^{-(k - \sigma\alpha)(T-t)}) \right), & \rho = -1, k \neq \sigma\alpha, \\ \frac{\alpha^2 k \delta}{4(m + \nu)} (T-t)^2, & \rho = -1, k = \sigma\alpha, \end{cases} \tag{83}$$

and

$$l_{1,2} = \frac{-(k + \sigma\rho\alpha) \pm \sqrt{(k + \sigma\rho\alpha)^2 + \alpha^2\sigma^2(1 - \rho^2)}}{\sigma^2(m + \nu)(1 - \rho^2)}. \tag{84}$$

Similarly, for $0 \leq t < t_0$, the optimal excess-of-loss strategy is (68). Denote the function G by G_2 in (73), and let

$$G_2(t, l) = g_2(t)l + \bar{g}_2(t). \tag{85}$$

Similar to (77)—(82), we obtain

$$g_2(t) = \begin{cases} \frac{l_1 - l_2 C_1 e^{-(1/2)\sigma^2(l_1 - l_2)(m + \nu)(1 - \rho^2)t}}{1 - C_1 e^{-(1/2)\sigma^2(l_1 - l_2)(m + \nu)(1 - \rho^2)(T-t)}}, & \rho \neq \pm 1, \\ \frac{\alpha^2}{2(m + \nu)(k + \sigma\alpha)} + C_1 e^{(k + \sigma\nu)t}, & \rho = 1, \\ \frac{\alpha^2}{2(m + \nu)(k - \sigma\alpha)} + C_1 e^{(k - \sigma\alpha)t}, & \rho = -1, k \neq \sigma\alpha, \\ -\frac{\alpha^2}{2(m + \nu)}t + C_1, & \rho = -1, k = \sigma\alpha, \end{cases} \tag{86}$$

$g_1(t)$

$$= \begin{cases} \frac{l_1 l_2 - l_1 l_2 e^{-(1/2)\sigma^2(l_1 - l_2)(m + \nu)(1 - \rho^2)(T-t)}}{l_2 - l_1 e^{-(1/2)\sigma^2(l_1 - l_2)(m + \nu)(1 - \rho^2)(T-t)}}, & \rho \neq \pm 1, \\ \frac{\alpha^2}{2(m + \nu)(k + \sigma\alpha)} (1 - e^{-(k + \sigma\alpha)(T-t)}), & \rho = 1, \\ \frac{\alpha^2}{2(m + \nu)(k - \sigma\alpha)} (1 - e^{-(k - \sigma\alpha)(T-t)}), & \rho = -1, k \neq \sigma\alpha, \\ \frac{\alpha^2}{2(m + \nu)} (T-t), & \rho = -1, k = \sigma\alpha, \end{cases} \tag{81}$$

and

$$\bar{g}_1(t) = \hat{g}_1(t) + \frac{M}{A + \varepsilon} (e^{(A+\varepsilon)(T-t)} - 1) - \int_t^T f(\bar{m}_1^*(s), \bar{m}_2^*(s), s) ds, \tag{82}$$

where $\hat{g}_1(t) = k\delta \int_t^T g_1(s) ds$, that is,

and

$$\bar{g}_2(t) = \tilde{g}_2(t) + K(t) = -k\delta \int_0^t g_2(s) ds + \frac{M}{A + \varepsilon} e^{(A+\varepsilon)(T-t)} + \int_0^t f(\bar{m}_1^*(s), 0, s) ds + C_2. \tag{87}$$

Hence for $\xi_1/(m + \nu)a_l \geq 1$, we have

$J(t, x, y, l)$

$$= \begin{cases} -\frac{1}{v} \exp \{-v[H(t)(x + \varepsilon y) + g_1(t)l + \bar{g}_1(t)]\}, & t_0 \leq t \leq T, \\ -\frac{1}{v} \exp \{-v[H(t)(x + \varepsilon y) + g_2(t)l + \bar{g}_2(t)]\}, & 0 \leq t < t_0. \end{cases} \tag{88}$$

Since $J(t, x, y, l)$ is continuous at $t = t_0$, we have

$$\begin{aligned} g_1(t_0) &= g_2(t_0), \\ \bar{g}_1(t_0) &= \bar{g}_2(t_0), \end{aligned} \tag{89}$$

and we derive

$$C_1 = \begin{cases} \frac{l_1}{l_2} e^{-(1/2)\sigma^2(l_1-l_2)(m+\nu)(1-\rho^2)T}, & \rho \neq \pm 1, \\ -\frac{\alpha^2}{2(m+\nu)(k+\sigma\alpha)} e^{-(k+\sigma\alpha)T}, & \rho = 1, \\ -\frac{\alpha^2}{2(m+\nu)(k-\sigma\alpha)} e^{-(k-\sigma\alpha)T}, & \rho = -1, k \neq \sigma\alpha, \\ \frac{\alpha^2}{2(m+\nu)} T, & \rho = -1, k = \sigma\alpha. \end{cases} \quad (90)$$

Based on (80), (81), (82), (83), (84), (85), (86), (87), (89), and (90), it is not difficult to find that $g_1(t) = g_2(t)$. Moreover, since $\bar{g}_1(t_0) = \bar{g}_2(t_0)$, we obtain

$$C_2 = k\delta \int_0^T g_1(s) ds - \frac{M}{A+\varepsilon} - \int_0^{t_0} f(\bar{m}_1^*(s), 0, s) ds - \int_{t_0}^T f(\bar{m}_1^*(s), \bar{m}_2^*(s), s) ds. \quad (91)$$

As a result, (87) can be rewritten as

$$\begin{aligned} \bar{g}_2(t) &= \hat{g}_1(t) + \frac{M}{A+\varepsilon} (e^{(A+\varepsilon)(T-t)} - 1) \\ &+ \int_0^t f(\bar{m}_1^*(s), 0, s) ds \\ &- \int_0^{t_0} f(\bar{m}_1^*(s), 0, s) ds \\ &- \int_{t_0}^T f(\bar{m}_1^*(s), \bar{m}_2^*(s), s) ds. \end{aligned} \quad (92)$$

By (48), (75), and (84), we can see that the optimal investment strategy π^* for $0 \leq t < t_0$ is the same as that for $t_0 \leq t \leq T$, which is

$$\pi^*(t) = \left(\frac{\alpha}{m+\nu} - \sigma\rho g_1(t) \right) e^{-(A+\varepsilon)(T-t)}. \quad (93)$$

Case 2. If $\xi_1/(m+\nu)a_1 < 1$, we have $t_0 > T$. For $0 \leq t \leq T < t_0$, the optimal excess-of-loss strategy is (68). Denoting by G_3 the function G in (73), we have

$$\begin{aligned} &K' + Me^{(A+\varepsilon)(T-t)} - f(\bar{m}_1^*, \bar{m}_2^*, t) + G_{3t} \\ &+ (k(\delta - l) - \sigma\rho\alpha) G_{3l} \\ &- \frac{m+\nu}{2} l\sigma^2 (1-\rho^2) G_{3l}^2 + \frac{l\sigma^2}{2} G_{3ll} + \frac{l\alpha^2}{2(m+\nu)} \\ &= 0, \end{aligned} \quad (94)$$

with boundary condition $K(T) = 0, G_3(T, l) = 0$. Similar to the analysis for $t_0 \leq t \leq T$ in Case 1, we conjecture a solution to (94) of the following form

$$G_3(t, l) = g_3(t)l + \bar{g}_3(t), \quad (95)$$

with $g_3(T) = 0, \bar{g}_3(T) = 0$. Then (94) is transformed into

$$\begin{aligned} &\bar{g}_3' + Me^{(A+\varepsilon)(T-t)} - f(\bar{m}_1^*, \bar{m}_2^*, t) + k\delta g_3 + l \left(g_3' \right. \\ &- (k + \sigma\rho\alpha) g_3 - \frac{m+\nu}{2} \sigma^2 (1-\rho^2) g_3^2 \\ &\left. + \frac{\alpha^2}{2(m+\nu)} \right) = 0. \end{aligned} \quad (96)$$

where $\bar{g}_3(t) = \bar{g}_3(t) + K(t)$ with the boundary condition $\bar{g}_3(T) = 0$.

Then employing the same method to solve the optimal problem as Case 1, a direct calculation yields $g_3(t) = g_1(t)$, and

$$\begin{aligned} \bar{g}_3(t) &= \hat{g}_1(t) + \frac{M}{A+\varepsilon} (e^{(A+\varepsilon)(T-t)} - 1) \\ &- \int_t^T f(\bar{m}_1^*(s), 0, s) ds. \end{aligned} \quad (97)$$

Thus, for $0 \leq t \leq T$, we can obtain the expression of optimal investment strategy which is the same as (93) and the corresponding value function.

The following theorem summarizes the above analysis.

Theorem 3. Recall functions $g_i(t)$ and $\bar{g}_i(t)$ ($i = 1, 2, 3$) defined in (81), (82), (86), (92), and (97), respectively. Then, for the problem (34), we have the robust optimal investment strategy

$$\pi^*(t) = \left(\frac{\alpha}{m+\nu} - \sigma\rho g_1(t) \right) e^{-(A+\varepsilon)(T-t)}, \quad 0 \leq t \leq T, \quad (98)$$

and (i) if $\xi_1/(m+\nu)a_1 \geq 1$ holds, the optimal value function is

$$J(t, x, y, l) = \begin{cases} -\frac{1}{\nu} \exp\{-\nu[(x+\varepsilon y)e^{(A+\varepsilon)(T-t)} + g_1(t)l + \bar{g}_1(t)]\}, & t_0 \leq t \leq T, \\ -\frac{1}{\nu} \exp\{-\nu[(x+\varepsilon y)e^{(A+\varepsilon)(T-t)} + g_1(t)l + \bar{g}_2(t)]\}, & 0 \leq t < t_0, \end{cases} \quad (99)$$

(ii) if $\xi_1/(m + \nu)a_1 < 1$ holds, the optimal value function is

$$J(t, x, y, l) = -\frac{1}{\nu} \cdot \exp \left\{ -\nu \left[(x + \varepsilon y) e^{(A+\varepsilon)(T-t)} + g_1(t)l + \bar{g}_3(t) \right] \right\}, \quad (100)$$

$$0 \leq t \leq T.$$

Remark 4. Form Theorems 2 and 3, we find that

(1) in the light of assumptions (40) and (41), the robust optimal strategy u^* depends on the parameters h, β, a_1 , and a_2 .

(2) the robust optimal value function $J(t, x, y, l)$ depends on parameters m, h, β, a_1, a_2 and the average past performance $Y^u(t)$.

4. Verification Theorem

In this section, we will verify the candidate optimal strategies $u^* := u^*(t) = (m_1^*(t), m_2^*(t), \pi^*(t))$ and θ^* given by (45), Theorems 2 and 3 are indeed optimal, and the candidate value function (42) is just the value function $J(t, x, y, l)$ defined in (34). The main theorem is summarized as follows.

Theorem 5. *If there exists a function $V(t, x, y, l) \in C^{1,2,1,2}$ and a control strategy $(\theta^*, u^*) \in \Theta \times \Pi$ such that*

(1) for $\forall \theta \in \Theta, A^{\theta, u^*} V(t, X^{u^*}(t), Y^{u^*}(t), L(t)) + \Psi(t, X^{u^*}(t), \theta(t)) \geq 0$;

(2) for $\forall u \in \Pi, A^{\theta^*, u} V(t, X^u(t), Y^u(t), L(t)) + \Psi(t, X^u(t), \theta^*(t)) \leq 0$;

(3) $A^{\theta^*, u^*} V(t, X^{u^*}(t), Y^{u^*}(t), L(t)) + \Psi(t, X^{u^*}(t), \theta^*(t)) = 0$;

(4) for $\forall (\theta, u) \in \Theta \times \Pi, \lim_{t \rightarrow T^-} V(t, X^u(t), Y^u(t), L(t)) = U(X^u(T), Y^u(T))$;

(5) for $\forall (\theta, u) \in \Theta \times \Pi, \{V(\tau, X^u(\tau), Y^u(\tau), L(\tau))\}_{\tau \in \Xi}$ and $\{\Psi(\tau, X^u(\tau), \theta(\tau))\}_{\tau \in \Xi}$ are uniformly integrable, where Ξ denotes the set of stopping times $\tau \leq T$.

Then $V(t, x, y, l) = J(t, x, y, l)$ and (θ^*, u^*) is an optimal control strategy.

Proof. It is similar to the proof of Theorem 3.2 in [42]. So we omit it here. \square

Next, conditions (1)–(5) in Theorem 5 will be checked. We first give two lemmas, which are useful for verifying Theorem 8.

Lemma 6. *If the parameters satisfy*

$$\frac{\alpha}{(m + \nu)^2} + \sigma^2 (1 - \rho^2) N^2 \leq \frac{k^2}{m^2 \sigma^2}, \quad (101)$$

then we have $E[\exp\{(1/2) \int_0^T \|\theta^*(t)\|^2 dt\}] < \infty$.

Proof. See Appendix. \square

Lemma 7. *For problem (34), $V(t, x, y, l)$ is the solution to equation (37) with boundary condition $V(T, x, y, l) = U(x, y)$, if the parameters satisfy condition (101) and*

$$\left(32\sigma^2 \rho^2 N^2 - 56 \frac{\alpha \sigma \rho}{m + \nu} N \right) \vee 0 + \frac{8\alpha^2 (4 - e^{-\varepsilon T})}{(m + \nu)^2} \leq \frac{k^2}{2\nu^2 \sigma^2}, \quad (102)$$

then

$$E^{Q^*} \left[\sup_{t \in [0, T]} \left| V(t, X^{u^*}(t), Y^{u^*}(t), L(t)) \right|^4 \right] < \infty, \quad (103)$$

$$E^{Q^*} \left[\sup_{t \in [0, T]} \left| \Psi(t, X^{u^*}(t), \theta^*(t)) \right|^2 \right] < \infty.$$

Proof. See Appendix. \square

Theorem 8. *For problem (34), if the parameters satisfy the conditions in Lemma 7 and $V(t, x, y, l)$ is the solution to (37) with boundary condition $V(T, x, y, l) = U(x, y)$, then the optimal value function is $J(t, x, y, l) = V(t, x, y, l)$, and the optimal strategy is $(m_1^*(t), m_2^*(t), \pi^*(t))$, which is given in Theorems 2 and 3.*

Proof. By the proofs of Theorems 2 and 3, conditions (1)–(4) in Theorem 5 hold for $V(t, x, y, l)$. By Lemma 7, we know that condition (5) in Theorem 5 also holds for $V(t, x, y, l)$. Then, according to Theorem 5, the results of Theorem 8 are obtained. \square

5. Special Cases

In this section, we consider the robust optimal problem (34) without insurance (investment-only case), with ambiguity-neutral insurer (ANI) and without delay, respectively. Since they are all special cases of (34), we only provide the results here without giving the proofs.

5.1. Investment-Only Case. If there is no reinsurance, i.e., $m_1(t) = D_1, m_2(t) = D_2$, then we have $X_i^m = X_i, Y_i^m = Y_i, \xi_1 = \xi_2 = 0$, and the wealth process (6) can be rewritten as

$$dX(t) = (\eta_1 (\lambda_1 + \lambda) \mu_X + \eta_2 (\lambda_2 + \lambda) \mu_Y) dt + \sqrt{(\lambda_1 + \lambda) \sigma_X^2 + (\lambda_2 + \lambda) \sigma_Y^2 + 2\lambda \mu_X \mu_Y} dW_0(t), \quad (104)$$

According to (33), the above wealth process with delay for an AAI under the probability Q is

$$\begin{aligned}
 dX^u(t) &= [AX^u(t) + \eta_1(\lambda_1 + \lambda)\mu_X + \eta_2(\lambda_2 + \lambda)\mu_Y + \bar{a}_1 Y^u(t) + a_2 Z^u(t) + \alpha \pi L(t)] dt \\
 &- \left(\pi \theta_1(t) \cdot \sqrt{L(t)} + \theta_0(t) \cdot \sqrt{(\lambda_1 + \lambda)\sigma_X^2 + (\lambda_2 + \lambda)\sigma_Y^2 + 2\lambda\mu_X\mu_Y} \right) dt \\
 &+ \pi \sqrt{L(t)} dW_1^Q(t) + \sqrt{(\lambda_1 + \lambda)\sigma_X^2 + (\lambda_2 + \lambda)\sigma_Y^2 + 2\lambda\mu_X\mu_Y} dW_0^Q(t),
 \end{aligned} \tag{105}$$

and the HJB equation becomes

$$\begin{aligned}
 \sup_{\pi \in R} \inf_{Q \in Q} \{ & \mathbf{A}^{\theta, u} \tilde{J}(t, X^u(t), Y^u(t), l) \\
 & + \Psi(t, X^u(t), Y^u(t), \theta(t)) \} = 0,
 \end{aligned} \tag{106}$$

where \tilde{J} is a short notation for $\tilde{J}(t, x, y, l)$, which represents the optimal value function of the investment-only problem with the boundary condition $\tilde{J}(T, x, y, l) = U(x, y)$, and

$$\begin{aligned}
 \mathbf{A}^{\theta, u} \tilde{J} &= \tilde{J}_t + \left[Ax + \eta_1(\lambda_1 + \lambda)\mu_X + \eta_2(\lambda_2 + \lambda)\mu_Y \right. \\
 &+ \bar{a}_1 y + a_2 z + \pi \alpha l - \pi \theta_1 \sqrt{l} \\
 &- \theta_0 \sqrt{(\lambda_1 + \lambda)\sigma_X^2 + (\lambda_2 + \lambda)\sigma_Y^2 + 2\lambda\mu_X\mu_Y} \left. \right] \tilde{J}_x \\
 &+ \frac{1}{2} \left[\pi^2 l + (\lambda_1 + \lambda)\sigma_X^2 + (\lambda_2 + \lambda)\sigma_Y^2 + 2\lambda\mu_X\mu_Y \right] \\
 &\cdot \tilde{J}_{xx} + (x - \beta y - e^{-\beta h} z) \tilde{J}_y + \left[k(\delta - l) - \sigma \rho \theta_1 \sqrt{l} \right. \\
 &- \left. \sigma \theta_2 \sqrt{l} \sqrt{1 - \rho^2} \right] \tilde{J}_l + \frac{1}{2} l \sigma^2 \tilde{J}_{ll} + \pi l \sigma \rho \tilde{J}_{xl}.
 \end{aligned} \tag{107}$$

Theorem 9. For the investment-only problem with delay, i.e., $m_1(t) = D_1$, $m_2(t) = D_2$, under the assumptions (23), (36), (40), and (41), the robust optimal investment strategy is

$$\pi_{01}^*(t) = \left(\frac{\alpha}{m + v} - \sigma \rho g_1(t) \right) e^{-(A+\varepsilon)(T-t)}, \tag{108}$$

and the optimal value function is

$$\begin{aligned}
 \tilde{J}(t, x, y, l) &= -\frac{1}{v} \exp \left\{ -v \left[(x + \varepsilon y) e^{(A+\varepsilon)(T-t)} + \bar{K}(t) \right. \right. \\
 &\left. \left. + g_1(t)l + \hat{g}_1(t) \right] \right\},
 \end{aligned} \tag{109}$$

for $0 \leq t \leq T$, where $g_1(t)$ and $\hat{g}_1(t)$ are given by (81) and (83), and

$$\begin{aligned}
 \bar{K}(t) &= \frac{\eta_1 \mu_X (\lambda_1 + \lambda) + \eta_2 \mu_Y (\lambda_2 + \lambda)}{A + \varepsilon} \left(e^{(A+\varepsilon)(T-t)} \right. \\
 &- 1 \left. \right) + \frac{m + v}{4(A + \varepsilon)} \left[(\lambda_1 + \lambda)\sigma_X^2 + (\lambda_2 + \lambda)\sigma_Y^2 \right. \\
 &\left. + 2\lambda\mu_X\mu_Y \right] \left(1 - e^{2(A+\varepsilon)(T-t)} \right).
 \end{aligned} \tag{110}$$

In addition, if $\lambda_2 = \lambda = 0$, $m_1(t) = D_1$, it means that we only consider one class of insurance business; in other words, there are no dependent risks in our model.

Corollary 10. For the investment-only problem with delay, $\lambda_2 = \lambda = 0$, $m_1(t) = D_1$, the robust optimal investment strategy is (108), and the optimal value function is

$$\begin{aligned}
 \tilde{J}_1(t, x, y, l) &= -\frac{1}{v} \exp \left\{ -v \left[(x + \varepsilon y) e^{(A+\varepsilon)(T-t)} \right. \right. \\
 &\left. \left. + \bar{K}_1(t) + g_1(t)l + \hat{g}_1(t) \right] \right\}, \quad 0 \leq t \leq T,
 \end{aligned} \tag{111}$$

where

$$\begin{aligned}
 \bar{K}_1(t) &= \frac{\lambda_1 \eta_1 \mu_X}{A + \varepsilon} \left(e^{(A+\varepsilon)(T-t)} - 1 \right) \\
 &+ \frac{\lambda_1 \sigma_X^2 (m + v)}{4(A + \varepsilon)} \left(1 - e^{2(A+\varepsilon)(T-t)} \right).
 \end{aligned} \tag{112}$$

Remark 11. According to Theorems 3 and 9 and Corollary 10, we find that the robust optimal investment strategy $\pi^*(t)$ of investment-only case is the same as that of reinsurance-investment case, which implies that the robust optimal reinsurance strategy and the robust optimal investment strategy can be separated.

In what follows, some special cases of the investment-only problem without delay are provided.

Corollary 12. For the investment-only problem without delay, i.e., $\beta = h = a_1 = a_2 = 0$,

(1) if $m_1(t) = D_1$, $m_2(t) = D_2$, the robust optimal investment strategy is

$$\pi_{02}^*(t) = \left(\frac{\alpha}{m + v} - \sigma \rho g_1(t) \right) e^{-r(T-t)}, \tag{113}$$

and the optimal value function is

$$\begin{aligned}
 \tilde{J}_2(t, x, y, l) &= -\frac{1}{v} \exp \left\{ -v \left[(x + \varepsilon y) e^{r(T-t)} + \bar{K}_2(t) \right. \right. \\
 &\left. \left. + g_1(t)l + \hat{g}_1(t) \right] \right\},
 \end{aligned} \tag{114}$$

for $0 \leq t \leq T$, where

$$\begin{aligned} \bar{K}_2(t) &= \frac{\eta_1 \mu_X (\lambda_1 + \lambda) + \eta_2 \mu_Y (\lambda_2 + \lambda)}{r} (e^{r(T-t)} - 1) \\ &+ \frac{m + \nu}{4r} [(\lambda_1 + \lambda) \sigma_X^2 + (\lambda_2 + \lambda) \sigma_Y^2 + 2\lambda \mu_X \mu_Y] \\ &\cdot (1 - e^{2r(T-t)}). \end{aligned} \quad (115)$$

(2) If $\lambda_2 = \lambda = 0$, $m_1(t) = D_1$, for $0 \leq t \leq T$, the robust optimal investment strategy is (113), and the optimal value function is

$$\begin{aligned} \bar{J}_3(t, x, y, l) &= -\frac{1}{\nu} \exp \left\{ -\nu \left[(x + \varepsilon y) e^{r(T-t)} + \bar{K}_3(t) \right. \right. \\ &\left. \left. + g_1(t)l + \bar{g}_1(t) \right] \right\}, \end{aligned} \quad (116)$$

where

$$\begin{aligned} \bar{K}_3(t) &= \frac{\lambda_1 \eta_1 \mu_X}{r} (e^{r(T-t)} - 1) \\ &+ \frac{\lambda_1 \sigma_X^2 (m + \nu)}{4r} (1 - e^{2r(T-t)}). \end{aligned} \quad (117)$$

5.2. ANI Case. If ambiguity-aversion coefficient $m = 0$, our model will reduce to an optimal control problem for an ANI. Then, the wealth process under probability measurers P is given by (20). Denote the optimal value function by

$$\hat{J}(t, x, y, l) = \sup_{\bar{u} \in \bar{\Pi}} E_{t,x,y,l} [U(X^{\bar{u}}(T), Y^{\bar{u}}(T))], \quad (118)$$

where $\bar{u} = \{(\bar{m}_1(t), \bar{m}_2(t)), \bar{\pi}(t)\}$, $t \in [0, T]$, and the corresponding HJB equation is

$$\begin{aligned} \sup_{\bar{u} \in \bar{\Pi}} A^{\bar{u}} \left\{ \hat{J}_t + [Ax + c_1 + c_2 + \bar{a}_1 y + a_2 z + \pi \alpha l] \hat{J}_x \right. \\ \left. + \frac{1}{2} [\pi^2 l + \gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)] \hat{J}_{xx} \right. \\ \left. + (x - \beta y - e^{-\beta h} z) \hat{J}_y + k(\delta - l) \hat{J}_l + \frac{1}{2} l \sigma^2 \hat{J}_{ll} \right. \\ \left. + \pi l \sigma \hat{J}_{xl} \right\} = 0. \end{aligned} \quad (119)$$

Here, \hat{J} is the short notation for $\hat{J}(t, x, y, l)$ with boundary condition $\hat{J}(T, x, y, l) = U(x, y)$. Let

$$\begin{aligned} \hat{k}(x) &= \xi_2 \left(\frac{x}{\xi_1} - \frac{e^{-(A+\varepsilon)(T-t)}}{\nu} \right), \\ \hat{a}_k &= \hat{k}^{-1}(0) = \frac{\xi_1}{\nu} e^{-(A+\varepsilon)(T-t)}, \end{aligned}$$

$$\hat{t}_0 = T - \frac{1}{A + \varepsilon} \ln \frac{\xi_1}{\nu a_k},$$

$$\hat{J}(t, x, y, l) = \begin{cases} -\frac{1}{\nu} \exp \left\{ -\nu \left[(x + \varepsilon y) e^{(A+\varepsilon)(T-t)} + \bar{K}_1(t) + \bar{g}_1(t)l + \bar{\tilde{g}}_1(t) \right] \right\}, & \hat{t}_0 \leq t \leq T, \\ -\frac{1}{\nu} \exp \left\{ -\nu \left[(x + \varepsilon y) e^{(A+\varepsilon)(T-t)} + \bar{K}_2(t) + \bar{g}_1(t)l + \bar{\tilde{g}}_1(t) \right] \right\}, & 0 \leq t < \hat{t}_0. \end{cases} \quad (124)$$

$$\hat{h}(x) = I_Y^{-1}(I_X(x)) + \frac{\lambda}{\lambda_2 + \lambda} g_X(x) - \frac{\xi_2}{\nu} e^{-(A+\varepsilon)(T-t)}. \quad (120)$$

Note that if $\xi_1/\nu a_k \geq 1$ and $\hat{h}(D_1) \geq 0$, according to the proof of Theorem 2, by the similar analysis that the equation $h(x) = 0$ in (64) has a unique solution $x_0(t) \in [a_k, D_1]$ when $h(D_1) \geq 0$, it is easy to prove that equation $\hat{h}(x) = 0$ has a unique solution $\hat{x}_0(t) \in [a_k, D_1]$.

Theorem 13. When $\xi_1 \geq \xi_2$, for the optimal control problem (118) of an ANI who ignores ambiguity with utility (23), we obtain

$$\pi_1^*(t) = \left(\frac{\alpha}{\nu} - \sigma \rho \bar{g}_1(t) \right) e^{-(A+\varepsilon)(T-t)}, \quad 0 \leq t \leq T. \quad (121)$$

(i) If $a_k = 0$ or $\xi_1/\nu a_k \geq 1$, $\hat{t}_0 \leq T$ holds. For $\hat{t}_0 \leq t \leq T$, we derive that the optimal reinsurance strategy $(\hat{m}_1^*(t), \hat{m}_2^*(t))$ is expressed by

$$\begin{aligned} \hat{m}_1^*(t) &= \hat{x}_0(t) I \{ \hat{h}(D_1) \geq 0 \} + D_1 I \{ \hat{h}(D_1) < 0 \}, \\ \hat{m}_2^*(t) &= I_Y^{-1}(I_X(\hat{x}_0(t))) I \{ \hat{h}(D_1) \geq 0, I_X(\hat{x}_0(t)) \\ &\leq I_Y(D_2) \} + D_2 I \{ \hat{h}(D_1) \geq 0, I_X(\hat{x}_0(t)) \\ &> I_Y(D_2) \} + \left(\frac{\xi_2}{\nu} e^{-(A+\varepsilon)(T-t)} - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \right) \\ &\cdot I \left\{ \hat{h}(D_1) < 0, \frac{\xi_2}{\nu} e^{-(A+\varepsilon)(T-t)} - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \right. \\ &< D_2 \} + D_2 I \left\{ \hat{h}(D_1) < 0, \frac{\xi_2}{\nu} e^{-(A+\varepsilon)(T-t)} \right. \\ &\left. - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \geq D_2 \right\}. \end{aligned} \quad (122)$$

For $0 \leq t < \hat{t}_0$, if the solution to $\hat{h}(x) = 0$ exists, we obtain the optimal reinsurance strategy $(\hat{m}_1^*(t), \hat{m}_2^*(t))$ is

$$\begin{aligned} \hat{m}_1^*(t) &= \frac{\xi_1}{\nu} e^{-(A+\varepsilon)(T-t)} I \left\{ \frac{\xi_1}{\nu} e^{-(A+\varepsilon)(T-t)} < D_1 \right\} \\ &+ D_1 I \left\{ \frac{\xi_1}{\nu} e^{-(A+\varepsilon)(T-t)} \geq D_1 \right\}, \end{aligned} \quad (123)$$

$$\hat{m}_2^*(t) = 0,$$

and the optimal value function is

(ii) If $\xi_1/va_1 < 1$, in this case $\widehat{t}_0 > T$. For $0 \leq t \leq T < \widehat{t}_0$, the optimal reinsurance strategy is expressed by (123), and optimal value function is given by

$$\widehat{J}(t, x, y, l) = -\frac{1}{v} \exp \left\{ -v \left[(x + \varepsilon y) e^{(A+\varepsilon)(T-t)} + \widehat{K}_3(t) + \widehat{g}_1(t)l + \widetilde{g}_1(t) \right] \right\}, \quad 0 \leq t \leq T, \tag{125}$$

where

$$\begin{aligned} \widehat{l}_{1,2} &= \frac{-(k + \sigma\rho\alpha) \pm \sqrt{(k + \sigma\rho\alpha)^2 + \alpha^2\sigma^2(1 - \rho^2)}}{v\sigma^2(1 - \rho^2)}, \\ \widehat{g}_1(t) &= \begin{cases} \frac{\widehat{l}_1\widehat{l}_2 - \widehat{l}_1\widehat{l}_2 e^{-(1/2)v\sigma^2(\widehat{l}_1 - \widehat{l}_2)(1 - \rho^2)(T-t)}}{\widehat{l}_2 - \widehat{l}_1 e^{-(1/2)v\sigma^2(\widehat{l}_1 - \widehat{l}_2)(1 - \rho^2)(T-t)}}, & \rho \neq \pm 1, \\ \frac{2v(k + \sigma\alpha)}{\alpha^2} (1 - e^{-(k+\sigma\alpha)(T-t)}), & \rho = 1, \\ \frac{2v(k - \sigma\alpha)}{\alpha^2} (1 - e^{-(k-\sigma\alpha)(T-t)}), & \rho = -1, k \neq \sigma\alpha, \\ \frac{\alpha^2}{2v} (T - t), & \rho = -1, k = \sigma\alpha, \end{cases} \\ \widetilde{g}_1(t) &= \begin{cases} \left[\widehat{l}_2 k \delta (T - t) - 2k\delta (v\sigma^2(1 - \rho^2))^{-1} \ln \left| \frac{\widehat{l}_1 - \widehat{l}_2}{\widehat{l}_1 - \widehat{l}_2 e^{(1/2)v\sigma^2(\widehat{l}_1 - \widehat{l}_2)(1 - \rho^2)(T-t)}} \right| \right], & \rho \neq \pm 1, \\ \frac{\alpha^2 k \delta}{2v(k + \sigma\alpha)} \left(T - t - \frac{1}{k + \sigma\alpha} (1 - e^{-(k+\sigma\alpha)(T-t)}) \right), & \rho = 1, \\ \frac{\alpha^2 k \delta}{2v(k - \sigma\alpha)} \left(T - t - \frac{1}{k - \sigma\alpha} (1 - e^{-(k-\sigma\alpha)(T-t)}) \right), & \rho = -1, k \neq \sigma\alpha, \\ \frac{\alpha^2 k \delta}{4v} (T - t)^2, & \rho = -1, k = \sigma\alpha, \end{cases} \end{aligned} \tag{126}$$

$$\widehat{f}(m_1, m_2, t) = \left\{ \frac{1}{2} v e^{(A+\varepsilon)(T-t)} (\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)) - \xi_1 (\lambda_1 + \lambda) g_X(m_1) - \xi_2 (\lambda_2 + \lambda) g_Y(m_2) \right\} e^{(A+\varepsilon)(T-t)},$$

$$\widehat{K}_1(t) = \frac{M}{A + \varepsilon} (e^{(A+\varepsilon)(T-t)} - 1) - \int_t^T \widehat{f}(\widehat{m}_1^*(s), \widehat{m}_2^*(s), s) ds,$$

$$\widehat{K}_2(t) = \frac{M}{A + \varepsilon} (e^{(A+\varepsilon)(T-t)} - 1) + \int_0^t \widehat{f}(\widetilde{m}_1^*(s), 0, s) ds - \int_0^{\widehat{t}_0} \widehat{f}(\widetilde{m}_1^*(s), 0, s) ds - \int_{\widehat{t}_0}^T \widehat{f}(\widetilde{m}_1^*(s), \widehat{m}_2^*(s), s) ds,$$

$$\widehat{K}_3(t) = \frac{M}{A + \varepsilon} (e^{(A+\varepsilon)(T-t)} - 1) - \int_t^T f(\widetilde{m}_1^*(s), 0, s) ds,$$

and M is given by (47).

Furthermore, if $\lambda_2 = \lambda = 0$ in Theorem 13, the optimal strategy here along with the optimal value function will coincide with Theorem 3.1 in [23]; i.e., our model extends the results in [23] to the case of robust optimal formulation under dependent risks.

5.3. No Delay Case. If $\beta = h = a_1 = a_2 = 0$, our model will reduce to a robust optimal control problem without delay. Then, the wealth process under probability measurer Q with strategy $\tilde{u} = \{\tilde{m}_1(t), \tilde{m}_2(t), \tilde{\pi}(t)\}_{t \in [0, T]}$ is

$$dX^{\tilde{u}}(t) = [rX^{\tilde{u}}(t) + c_1 + c_2 + \alpha\pi L(t)] dt$$

$$\begin{aligned} & - \left(\pi\theta_1(t) \sqrt{L(t)} \right. \\ & + \theta_0(t) \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} \Big) dt \\ & + \pi \sqrt{L(t)} dW_1^Q(t) \\ & + \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} dW_0^Q(t). \end{aligned} \tag{127}$$

Suppose the AAI has an exponential utility as follows:

$$U(x) = -\frac{1}{v} e^{-vx}, \quad v > 0. \tag{128}$$

Denote the value function by

$$\begin{aligned} \check{J}(t, x, l) = \sup_{\check{u} \in \Pi} \left\{ \inf_{Q \in \mathcal{Q}} E_{t,x,l}^Q \left[U(X^{\check{u}}(T)) \right. \right. \\ \left. \left. + \int_t^T \Psi(s, X^{\check{u}}(s), \theta(s)) ds \right] \right\}, \end{aligned} \quad (129)$$

where

$$\begin{aligned} \Psi(t, X^{\check{u}}(t), \theta(t)) &= \frac{\|\theta(t)\|^2}{2\phi(t, X^{\check{u}}(t))}, \\ \phi(t, x) &= \frac{-m}{v\check{V}(t, x, l)} \geq 0, \quad m > 0. \end{aligned} \quad (130)$$

Then, the corresponding HJB equation is given by

$$\begin{aligned} \sup_{\check{u} \in \Pi} \inf_{Q \in \mathcal{Q}} \left\{ \mathbf{A}^{\theta, \check{u}} \check{J}(t, X^{\check{u}}(t), l) + \Psi(t, X^{\check{u}}(t), \theta(t)) \right\} \\ = 0, \end{aligned} \quad (131)$$

where \check{J} is a short notation for $\check{J}(t, x, l)$ with boundary condition $\check{J}(T, x, l) = U(x)$, and $\mathbf{A}^{\theta, \check{u}}$, which represents the generator of the process (127) under Q , is defined by

$$\begin{aligned} \mathbf{A}^{\theta, \check{u}} &= \check{J}_t + \left[rx + c_1 + c_2 + \pi\alpha l - \pi\theta_1 \sqrt{l} \right. \\ &\quad \left. - \theta_0 \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)} \right] \check{J}_x + \frac{1}{2} \left[\pi^2 l \right. \\ &\quad \left. + \gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2) \right] \check{J}_{xx} + \frac{1}{2} l \sigma^2 \check{J}_{ll} \\ &\quad + \pi l \sigma \rho \check{J}_{xl} + \left[k(\delta - l) - \sigma \rho \theta_1 \sqrt{l} \right. \\ &\quad \left. - \sigma \theta_2 \sqrt{l} \sqrt{1 - \rho^2} \right] \check{J}_l. \end{aligned} \quad (132)$$

Let

$$\begin{aligned} \check{h}(x) &= l_Y^{-1}(l_X(x)) + \frac{\lambda}{\lambda_2 + \lambda} g_X(x) - \frac{\xi_2}{m + v} e^{-r(T-t)}, \\ \check{k}(x) &= \xi_2 \left(\frac{x}{\xi_1} - \frac{e^{-r(T-t)}}{m + v} \right), \\ \check{a}_k &= \check{k}^{-1}(0) = \frac{\xi_1}{m + v} e^{-r(T-t)}, \\ \check{t}_0 &= T - \frac{1}{r} \ln \frac{\xi_1}{(m + v) a_l}. \end{aligned} \quad (133)$$

Theorem 14. When $\xi_1 \geq \xi_2$, for problem (129) with the utility (128), we obtain the robust optimal investment strategy is

$$\pi_2^*(t) = \left(\frac{\alpha}{m + v} - \sigma \rho g_1(t) \right) e^{-r(T-t)}, \quad 0 \leq t \leq T, \quad (134)$$

and (i) if $a_l = 0$ or $\xi_1/(m + v)a_l \geq 1$ holds, for $\check{t}_0 \leq t \leq T$, the robust optimal excess-of-loss reinsurance strategy $(\check{m}_1^*(t), \check{m}_2^*(t))$ is

$$\begin{aligned} \check{m}_1^*(t) &= \check{x}_0(t) I \{ \check{h}(D_1) \geq 0 \} + D_1 I \{ \check{h}(D_1) < 0 \}, \\ \check{m}_2^*(t) &= l_Y^{-1}(l_X(\check{x}_0(t))) I \{ \check{h}(D_1) \geq 0, l_X(\check{x}_0(t)) \\ &\leq l_Y(D_2) \} + D_2 I \{ \check{h}(D_1) \geq 0, l_X(\check{x}_0(t)) \\ &> l_Y(D_2) \} + \left(\frac{\xi_2}{m + v} e^{-r(T-t)} - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \right) \\ &\cdot I \{ \check{h}(D_1) < 0, \frac{\xi_2}{m + v} e^{-r(T-t)} - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \\ &< D_2 \} + D_2 I \left\{ \check{h}(D_1) < 0, \frac{\xi_2}{m + v} e^{-r(T-t)} \right. \\ &\quad \left. - \frac{\lambda}{\lambda_2 + \lambda} g_X(D_1) \geq D_2 \right\}, \end{aligned} \quad (135)$$

where $\check{x}_0(t) \in [a_l, D_1]$ is the unique solution to the equation $\check{h}(x) = 0$ if $\check{h}(D_1) \geq 0$ holds.

For $0 \leq t < \check{t}_0$, if the solution of $\check{h}(x) = 0$ exists, the robust optimal excess-of-loss reinsurance strategy $(\check{m}_1^*(t), \check{m}_2^*(t))$ is

$$\begin{aligned} \check{m}_1^*(t) &= \frac{\xi_1}{m + v} e^{-r(T-t)} I \left\{ \frac{\xi_1}{m + v} e^{-r(T-t)} < D_1 \right\} \\ &\quad + D_1 I \left\{ \frac{\xi_1}{m + v} e^{-r(T-t)} \geq D_1 \right\}, \\ \check{m}_2^*(t) &= 0, \end{aligned} \quad (136)$$

and the corresponding optimal value function is

$$\check{J}(t, x, l) = \begin{cases} -\frac{1}{v} \exp \left\{ -v \left[x e^{r(T-t)} + \check{K}_1(t) + g_1(t) l + \hat{g}_1(t) \right] \right\}, & \check{t}_0 \leq t \leq T, \\ -\frac{1}{v} \exp \left\{ -v \left[x e^{r(T-t)} + \check{K}_2(t) + g_1(t) l + \hat{g}_1(t) \right] \right\}, & 0 \leq t < \check{t}_0, \end{cases} \quad (137)$$

(ii) if $\xi_1/(m+\nu)a_1 < 1$ holds, in this case $\check{t}_0 > T$. For $0 \leq t \leq T < \check{t}_0$, the robust optimal reinsurance strategy is (136), and the corresponding value function is

$$\begin{aligned} & \check{J}(t, x, l) \\ &= -\frac{1}{\nu} \exp \left\{ -\nu \left[x e^{r(T-t)} + \check{K}_3(t) + g_1(t)l + \hat{g}_1(t) \right] \right\}, \quad (138) \\ & \quad \quad \quad 0 \leq t \leq T, \end{aligned}$$

where $g_1(t)$ and $\hat{g}_1(t)$ are given by (81) and (83), respectively, and

$$\begin{aligned} & \check{f}(m_1, m_2, t) \\ &= \left\{ \frac{m+\nu}{2} e^{r(T-t)} (\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1) g_Y(m_2)) \right. \\ & \quad \left. - \xi_1(\lambda_1 + \lambda) g_X(m_1) - \xi_2(\lambda_2 + \lambda) g_Y(m_2) \right\} \\ & \quad \cdot e^{r(T-t)}, \\ & \check{K}_1(t) = \frac{M}{r} (e^{r(T-t)} - 1) \\ & \quad - \int_t^T \check{f}(\tilde{m}_1^*(s), \tilde{m}_2^*(s), s) ds, \\ & \check{K}_2(t) = \frac{M}{r} (e^{r(T-t)} - 1) + \int_0^t \check{f}(\tilde{m}_1^*(s), 0, s) ds \\ & \quad - \int_0^{\check{t}_0} \check{f}(\tilde{m}_1^*(s), 0, s) ds \\ & \quad - \int_{\check{t}_0}^T \check{f}(\tilde{m}_1^*(s), \tilde{m}_2^*(s), s) ds, \\ & \check{K}_3(t) = \frac{M}{r} (e^{r(T-t)} - 1) - \int_t^T \check{f}(\tilde{m}_1^*(s), 0, s) ds. \end{aligned} \quad (139)$$

Remark 15. Following Corollary 12 and Theorem 14, we find there is a similar conclusion to Remark 11 for the problem without delay.

6. Conclusion

For the optimal control problems in insurance, most papers only consider the control systems without delay, while this paper studies a robust optimal reinsurance-investment problem with delay and dependent risks when the risky asset's price is described by Heston model. To make the optimal control problem closer to reality, we furthermore consider some possible extensions of this paper. For example, we

can consider the robust equilibrium reinsurance-investment problem for a mean-variance insurer with other kinds of dependent risks, such as copulas, which is a very challenging problem.

Appendix

Lemma A. $g_1(t)$ is a decreasing function with respect to t and satisfies

$$0 \leq g_1(t) \leq N, \quad (A.1)$$

where

$$N = \begin{cases} \frac{l_1 l_2 - l_1 l_2 e^{-(1/2)\sigma^2(l_1-l_2)(m+\nu)(1-\rho^2)T}}{l_2 - l_1 e^{-(1/2)\sigma^2(l_1-l_2)(m+\nu)(1-\rho^2)T}}, & \rho \neq \pm 1, \\ \frac{\alpha^2}{2(m+\nu)(k+\sigma\alpha)} (1 - e^{-(k+\sigma\alpha)T}), & \rho = 1, \\ \frac{\alpha^2}{2(m+\nu)(k-\sigma\alpha)} (1 - e^{-(k-\sigma\alpha)T}), & \rho = -1, k \neq \sigma\alpha, \\ \frac{\alpha^2}{2(m+\nu)} T, & \rho = -1, k = \sigma\alpha. \end{cases} \quad (A.2)$$

Proof. For $\rho \neq \pm 1$, differentiating (81) with respect to t yields

$$\begin{aligned} g_1'(t) &= -\frac{1}{2} l_1 l_2 (l_1 - l_2)^2 \sigma^2 (m + \nu) (1 - \rho^2) \\ & \quad \cdot \frac{e^{-\sigma^2(l_1-l_2)(m+\nu)(1-\rho^2)(T-t)}}{(l_2 - l_1 e^{-(1/2)\sigma^2(l_1-l_2)(m+\nu)(1-\rho^2)(T-t)})^2}. \end{aligned} \quad (A.3)$$

According to (84), it is true that $g_1'(t) < 0$ for $\rho \neq \pm 1$. In addition, for $\rho = 1$,

$$g_1'(t) = -\frac{\alpha^2}{2(m+\nu)} e^{-(k+\sigma\alpha)(T-t)} < 0; \quad (A.4)$$

For $\rho = -1, k \neq \sigma\alpha$,

$$g_1'(t) = -\frac{\alpha^2}{2(m+\nu)} e^{-(k-\sigma\alpha)(T-t)} < 0; \quad (A.5)$$

for $\rho = -1, k = \sigma\alpha$,

$$g_1'(t) = -\frac{\alpha^2}{2(m+\nu)} < 0. \quad (A.6)$$

As a result, $g_1(t)$ decreases with respect to t . Then plugging 0 and T into (81) yields (A.1). \square

Proof of Lemma 6. Due to (A.1), with an appropriate constant M_1 , we get

$$E \left[\exp \left\{ \frac{1}{2} \int_0^T \|\theta^*(t)\|^2 dt \right\} \right]$$

$$\begin{aligned}
&\leq M_1 E \left[\exp \left\{ \frac{m^2}{2} \int_0^T \left(\frac{\alpha}{(m+v)^2} - 2\sigma\rho g_1(t) + \sigma^2(1-\rho^2)g_1^2(t) \right) L(t) dt \right\} \right] \\
&\leq M_1 E \left[\exp \left\{ \frac{k^2}{2\sigma^2} \int_0^T L(t) dt \right\} \right] \\
&< \infty.
\end{aligned} \tag{A.7}$$

Since $\theta_0^*(t)$ is deterministic and bounded on $[0, T]$, the first estimate in (A.7) is valid. The second inequality holds according to condition (101), and by Theorem 5.1 in [43], the last estimate is easily derived. \square

Proof of Lemma 7. Substituting $\theta^*(t)$ and $u^*(t)$ into (33) derives

$$\begin{aligned}
X^{u^*}(t) &= x_0 e^{At} \\
&+ \int_0^t e^{A(t-s)} \left(c_1 \right. \\
&+ c_2 + \bar{a}_1 Y^{u^*}(s) + a_2 Z^{u^*}(s) \Big) ds \\
&+ \int_0^t e^{A(t-s)} \\
&\cdot \frac{v\alpha}{m+v} \pi^*(s) L(s) ds \\
&- \int_0^t e^{A(t-s)} \theta_0^*(s) \\
&\cdot \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1^*(s)) g_Y(m_2^*(s))} ds \\
&+ \int_0^t e^{A(t-s)} \pi^*(s) \sqrt{L(s)} dW_1^{Q^*}(s) \\
&+ \int_0^t e^{A(t-s)} \\
&\cdot \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1^*(s)) g_Y(m_2^*(s))} dW_0^{Q^*}(s).
\end{aligned} \tag{A.8}$$

Inserting (A.8) into (42), we obtain the following estimate with appropriate constants $M_2 < M_3$,

$$\begin{aligned}
&\left| V(t, X^{u^*}(t), Y^{u^*}(t), L(t)) \right|^4 \\
&= \frac{1}{v^4} \exp \left\{ -4v \left[H(t) \right. \right. \\
&\cdot \left. \left. \left(X^{u^*}(t) + \varepsilon Y^{u^*}(t) \right) + g_1(t) L(t) + \widehat{K}(t) \right] \right\} \\
&\leq M_2 \exp \left\{ -4v H(t) X^{u^*}(t) \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq M_3 \exp \left\{ -4v e^{(A+\varepsilon)(T-t)} \int_0^t e^{A(t-s)} \frac{v\alpha}{m+v} \pi^*(s) L(s) ds \right\} \\
&\cdot \exp \left\{ -4v e^{(A+\varepsilon)(T-t)} \int_0^t e^{A(t-s)} \right. \\
&\cdot \pi^*(s) \sqrt{L(s)} dW_1^{Q^*}(s) \Big\} \\
&\cdot \exp \left\{ -4v e^{(A+\varepsilon)(T-t)} \int_0^t e^{A(t-s)} \right. \\
&\cdot \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1^*(s)) g_Y(m_2^*(s))} dW_0^{Q^*}(s) \Big\} \\
&\leq M_3 \exp \left\{ -4v \int_0^t e^{A(T-s)} \frac{v\alpha}{m+v} \pi^*(s) L(s) ds \right\} \\
&\cdot \exp \left\{ -4v \int_0^t e^{A(T-s)} \right. \\
&\cdot \pi^*(s) \sqrt{L(s)} dW_1^{Q^*}(s) \Big\} \\
&\cdot \exp \left\{ -4v \int_0^t e^{A(T-s)} \right. \\
&\cdot \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1^*(s)) g_Y(m_2^*(s))} dW_0^{Q^*}(s) \Big\} \\
&= M_3 \prod_{i=1}^4 \exp \{ E_i(t) \},
\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}
\widehat{K}(t) &= \bar{g}_1(t) I \left\{ \frac{\xi_1}{(m+v)a_1} \geq 1, t_0 \leq t \leq T \right\} \\
&+ \bar{g}_2(t) I \left\{ \frac{\xi_1}{(m+v)a_1} \geq 1, 0 \leq t \leq t_0 \right\} \\
&+ \bar{g}_3(t) I \left\{ \frac{\xi_1}{(m+v)a_1} < 1 \right\},
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
E_1(t) &= 8v^2 \int_0^t e^{2A(T-s)} \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1^*(s)) g_Y(m_2^*(s))} ds,
\end{aligned}$$

$$\begin{aligned}
 E_2(t) &= -4v \int_0^t e^{A(T-s)} \\
 &\quad \cdot \sqrt{\gamma_1^2 + \gamma_2^2 + 2\lambda g_X(m_1^*(s)) g_Y(m_2^*(s))} dW_0^{Q^*}(s) \\
 &\quad - E_1(t), \\
 E_3(t) &= -4v \int_0^t e^{A(T-s)} \pi^*(s) \sqrt{L(s)} dW_1^{Q^*}(s) \\
 &\quad - 16v^2 \int_0^t e^{2A(T-s)} \pi^{*2}(s) L(s) ds, \\
 E_4(t) &= 16v^2 \int_0^t e^{2A(T-s)} \pi^{*2}(s) L(s) ds \\
 &\quad - 4v \int_0^t e^{A(T-s)} \frac{\alpha v}{m+v} \pi^*(s) L(s) ds.
 \end{aligned} \tag{A.11}$$

Because $g_1(t)$ and $\widehat{K}(t)$ are deterministic and bounded on $[0, T]$, we obtain that the first estimate in (A.9) is valid. Since $Y^{u^*}(t)$, $Z^{u^*}(t)$, $m_1^*(t)$, $m_2^*(t)$, and $\theta_0^*(t)$ are deterministic and bounded on $[0, T]$, the second inequality holds. The third inequality follows the fact that $e^{\varepsilon(T-t)}$ is bounded.

Now we firstly consider the integrals $E_1(t)$ and $E_2(t)$. Note that $m_1^*(t)$ and $m_2^*(t)$ are bounded on $[0, T]$, and then we get $E^{Q^*}[\exp\{E_1(t)\}] < \infty$. It is not difficult to verify that $\exp\{E_2(t)\}$ is a martingale under Q^* , and thus $E^{Q^*}[\exp\{E_2(t)\}] < \infty$.

Secondly, for $\forall t \in [0, T]$, note that $\pi^*(t)$ is deterministic and bounded; then by the Lemma 4.3 in [43], it is easily seen that $\exp\{2E_3(t)\}$ is a martingale under Q^* , and, consequently,

$$E^{Q^*}[\exp\{2E_3(t)\}] < \infty. \tag{A.12}$$

Further, following Theorem 5.1 in [43], a sufficient condition for

$$E^{Q^*}[\exp\{2E_4(t)\}] < \infty, \tag{A.13}$$

is obtained, which is

$$\begin{aligned}
 &32e^{-2\varepsilon(T-t)} \left(\frac{\alpha}{m+v} - \sigma\rho g_1(t) \right)^2 \\
 &\quad - \frac{8\alpha}{m+v} \left(\frac{\alpha}{m+v} - \sigma\rho g_1(t) \right) e^{-\varepsilon(T-t)} \\
 &\leq \frac{k^2}{2v^2\sigma^2}, \quad \forall t \in [0, T].
 \end{aligned} \tag{A.14}$$

According to (A.1) and condition (102), we get

$$\begin{aligned}
 &32 \left(\frac{\alpha}{m+v} - \sigma\rho g_1(t) \right)^2 + \frac{8\alpha\sigma\rho}{m+v} g_1(t) \\
 &\quad - \frac{8\alpha^2}{(m+v)^2} e^{-\varepsilon T} \leq \frac{k^2}{2v^2\sigma^2}, \quad \forall t \in [0, T].
 \end{aligned} \tag{A.15}$$

Consequently, (A.14) holds.

Applying $E^{Q^*}[\exp\{E_i(t)\}] < \infty$ and $E^{Q^*}[\exp\{E_{i+2}(t)\}] < \infty$, $i = 1, 2$, and by (A.9), we arrive at

$$\begin{aligned}
 &E^{Q^*} \left| V \left(t, X^{u^*}(t), Y^{u^*}(t), L(t) \right) \right|^4 \\
 &\leq M_3 \prod_{i=1}^2 E^{Q^*}[\exp\{E_i(t)\}] \\
 &\quad \cdot E^{Q^*}[\exp\{E_3(t)\}] \cdot \exp\{E_4(t)\} \\
 &\leq M_4 \left[E^{Q^*}[\exp\{E_3(t)\}]^2 \cdot E^{Q^*}[\exp\{E_4(t)\}]^2 \right]^{1/2} \\
 &< \infty.
 \end{aligned} \tag{A.16}$$

Hence, we have proved the first part of this lemma.

Plugging $\theta^*(t)$ and $u^*(t)$ into (35) leads to

$$\begin{aligned}
 &E^{Q^*} \left| \Psi \left(t, X^{u^*}(t), Y^{u^*}(t), L(t) \right) \right|^2 \\
 &= E^{Q^*} \left[\frac{v^2}{m^2} \left| V \left(t, X^{u^*}(t), Y^{u^*}(t), L(t) \right) \right|^2 \right. \\
 &\quad \cdot \left. \left(\frac{1}{2} \|\theta^*(t)\|^2 \right)^2 \right] \\
 &\leq \frac{v^2}{m^2} \left[E^{Q^*} \left| V \left(t, X^{u^*}(t), Y^{u^*}(t), L(t) \right) \right|^4 \right. \\
 &\quad \cdot \left. E^{Q^*} \left(\frac{1}{2} \|\theta^*(t)\|^2 \right)^4 \right]^{1/2} < \infty.
 \end{aligned} \tag{A.17}$$

According to Cauchy-Schwarz inequality, we obtain the first estimate and the last one can be derived by (A.16) and Lemma 6 under condition (101). Thus, the second part of this lemma holds. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

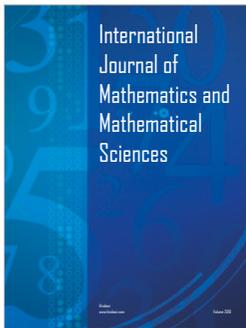
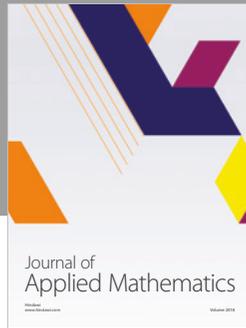
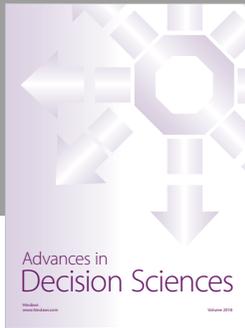
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