Research Article

Feedback Arc Number and Feedback Vertex Number of Cartesian Product of Directed Cycles

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For a digraph $D$, the feedback vertex number $\tau(D)$, (resp. the feedback arc number $\tau'(D)$) is the minimum number of vertices, (resp. arcs) whose removal leaves the resultant digraph free of directed cycles. In this note, we determine $\tau(D)$ and $\tau'(D)$ for the Cartesian product of directed cycles $D = \overline{C}_n^+ \square \overline{C}_m^+ \square \ldots \square \overline{C}_k^+$. Actually, it is shown that $\tau'(D) = n_1 n_2 \ldots n_k \sum_{i=1}^{k} 1/n_i$ and if $n_k \geq \ldots \geq n_1 \geq 3$ then $\tau(D) = n_2 \ldots n_k$.

1. Introduction

Let $G = (V, E)$ be an undirected graph. A set $S \subseteq V(G)$ is called a feedback vertex set of $G$ if $G - S$ contains no cycle. The feedback vertex number of $G$, denoted by $\tau(G)$, is the cardinality of a minimum feedback vertex set of $G$. In general, it is NP-hard to determine the feedback vertex number of a graph $G$ [1]. However, it becomes polynomial for specific families of graphs such as interval graphs [2], permutation graphs [3], graphs with maximum degree 3 [4], and $k$-trees. The readers are referred to [5, 6] for a review of some earlier results and open problems, and [7–9] for some recent results on the feedback vertex number of graphs. Some bounds or exact values are established for various families of graph, for instance, outerplanar graphs [10], grids and butterflies [11], cubic graphs [12, 13], bipartite graphs [14], generalized Petersen graphs [15], regular graphs [16, 17]. Bau et al. [18] investigated the feedback vertex number of grid graphs.

Apart from its graph-theoretical importance, the feedback vertex problem has many applications, such as operating system [19, 20], artificial intelligence [21], synchronous distributed systems [22, 23], optical networks [24]. The feedback vertex set and the feedback vertex number are also known as decycling set and the decycling number, respectively, see [25].

In 2005, Pike and Zou [26] determined the feedback vertex number of the Cartesian products of two cycles as follows:

$$\tau(C_m \square C_n) = \begin{cases} \left\lfloor \frac{3n}{2} \right\rfloor, & \text{if } m = 4 \\
\left\lfloor \frac{3m}{2} \right\rfloor, & \text{if } n = 4 \\
\left\lfloor \frac{mn + 2}{3} \right\rfloor, & \text{otherwise.} \end{cases}$$ (1)

Our main concern in this note is the directed version of the feedback vertex number. A directed graph $D$ is said to be acyclic if it does not contain any directed cycle. A feedback vertex set in a digraph $D$ is a set $S$ of vertices such that $D - S$ is acyclic, and the feedback vertex number of $D$ is the minimum size of such a set is denoted by $\tau(D)$. We denote by $\nu(D)$ the number of vertex-disjoint cycles of $D$. Clearly, $\tau(D) \geq \nu(D)$ for any digraph $D$. A feedback arc set of a digraph $D$ is a set $S$ of arcs such that $D - S$ is acyclic. The feedback arc number of $D$, denoted by $\tau'(D)$, is the cardinality of a minimum feedback arc set of $D$. We denote by $\nu'(D)$ the number of arc-disjoint cycles of $D$. Clearly, $\tau'(D) \geq \nu'(D)$ for any digraph $D$.

Not much works were known for the feedback vertex number or the feedback arc number of directed graphs. Lien et al. [27] gave an upper bound for the feedback vertex number of generalized Kautz digraphs. Figueroa et al. [28] investigated the relation for the relationship between the minimum feedback arc set and the acyclic disconnection of a digraph. Even et al. [29] gave an $O(\log n \log \log n)$-approximation algorithm for the feedback vertex problem for a digraph of order $n$. For planar digraphs, the approximation ratio is not greater than $9/4$.
Theorem 1. For any $k \geq 2$ integers $n_1, \ldots, n_k$ with $n_i \geq 3$ for each $i \in \{1, \ldots, k\}$,
\[
\tau'(C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}) = n_1 n_2 \cdots n_k \sum_{i=1}^{k-1} \frac{1}{n_i}.
\]

Proof. First, we show that $\tau'(D) \geq n_1 n_2 \cdots n_k \sum_{i=1}^{k-1} \frac{1}{n_i}$ by showing that
\[
\nu'(D) \geq n_1 n_2 \cdots n_k \sum_{i=1}^{k-1} \frac{1}{n_i}.
\]

We proceed with induction on $k$. Let $k = 2$. By our notation, $D_1 = C_{n_1}$, for each $i \in \{0, 1, \ldots, n_i - 1\}$. Note that $D_1$ and $D_2$ are vertex-disjoint (and thus arc-disjoint). Moreover, since $D(\nu'(A)) = n_1 + n_2$. Now assume that $k \geq 3$. Since $D_i \equiv C_{n_i} \square C_{n_i} \square \cdots \square C_{n_i}$, for each $i \in \{0, 1, \ldots, n_i - 1\}$, by the induction hypothesis,
\[
\nu'(D) \geq n_1 n_2 \cdots n_{k-1} \sum_{i=1}^{k-1} \frac{1}{n_i}.
\]
\[ v(D) \geq n_2 \ldots n_k. \tag{9} \]

We proceed with induction on \( k \). For the case when \( k = 2 \), \( D_2 \equiv C_n \) for each \( i \in \{0, 1, \ldots, n_2 - 1\} \). It follows that \( D \) contains \( n_2 \) vertex-disjoint copies of \( C_n \), and thus \( v(D) \geq n_2 \). Now assume that \( k \geq 3 \). For every integer \( i \in \{0, 1, \ldots, n_k - 1\} \), \( D_i \equiv C_{n_i} \sqcap C_{n_i} \sqcap \ldots \sqcap C_{n_i} \), and hence, by the induction hypothesis,

\[ v(D_i) \geq n_i \ldots n_k. \tag{10} \]

Moreover, since \( D_i \) and \( D_j \) are vertex-disjoint for any \( 0 \leq i < j \leq n_k - 1 \), we have

\[ v(D) \geq n_2 \ldots n_k. \tag{11} \]

As an example for the case when \( n_1 = 3 \) and \( n_2 = 4 \), we have

\[ D = C_4 \sqcap C_5 \] and \( S_0 = \{(1,0),(0,1),(2,2),(1,3)\} \), see Figure 2 for an illustration. Clearly, \( S_D \) is a feedback vertex set of \( D \) with \( |S_D| = 4 \).

For any \( k \geq 2 \) and \( j \in \{0, 1, \ldots, n_k - 1\} \), let

\[ S_k := \{ (x_1, x_2, \ldots, x_{k-1}, x_k) : x_1 + x_2 + \ldots + x_k \equiv 1 \mod n_j, 0 \leq x_i \leq n_i - 1 \} \] \tag{12}

Since for any given value of \( (x_2, \ldots, x_{k-1}, x_k) \) with \( 0 \leq x_i \leq n_i - 1 \), there exists unique value of \( x_1 \) with \( 0 \leq x_1 \leq n_1 - 1 \) satisfying \( x_1 + x_2 + \ldots + x_k \equiv 1 \mod n_j \), implying that \( |S_k| = n_1 n_2 \ldots n_k \).

To show that \( S_k \) is a feedback vertex set of \( D \), we consider any directed cycle \( C = v_1 v_2 \ldots v_t \) of \( D \), where \( v_i = (x_{i-1}^1, x_{i-1}^2, \ldots, x_{i-1}^{t-1}) \) for each \( i \in \{1, \ldots, t\} \). Since for each \( i \), \( v_i v_{i+1} \in A(D) \), we have

\[ x_{i-1}^1 + x_{i-1}^2 + \ldots + x_{i-1}^{t-1} \equiv x_{i-1} + x_i + x_{i+1} + \ldots + x_t \equiv 1 \mod n_i. \tag{13} \]

Moreover, since \( t \geq n_1 = \min\{n_1, n_2, \ldots, n_k\} \), there exists an integer \( j \in \{1 \ldots, t\} \) such that

\[ x_1^j + x_2^j + \ldots + x_t^j \equiv 1 \mod n_j, \tag{14} \]

implying that \( v_j = (x_j^1, x_j^2, \ldots, x_j^j) \in S_k \), and thus \( S_k \) is a feedback vertex set of \( D \). This proves

\[ \tau(D) \leq n_1 n_2 \ldots n_k. \tag{15} \]

3. Conclusion

In this note, we determined the two important parameters \( \tau(D) \) and \( \tau'(D) \) for the Cartesian product of directed cycles \( D = C_{n_1} \sqcap C_{n_2} \sqcap \ldots \sqcap C_{n_k} \). Actually, it is shown that

\[ \tau'(D) = n_1 n_2 \ldots n_k \sum_{i=1}^k 1/n_i \] and if \( n_k \geq \ldots \geq n_1 \geq 3 \), then

\[ \tau(D) = n_1 n_2 \ldots n_k. \]

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in conducting this research work and writing this paper.

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