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Research Article

Oscillation for Forced Second-Order Impulsive Nonlinear Dynamic Equations on Time Scales

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As the unification and development of impulsive differential equations and difference equations, impulsive dynamic equations on time scales are a powerful tool to simulate the natural and social phenomena. In this paper, we study the interval oscillation of a type of forced second-order nonlinear impulsive dynamic equations with changing signs coefficients. By using the Riccati transformation technique, we obtain some new interval oscillation criteria, based only on information of a sequence of subintervals of positive axis. In addition, we provide an example to illustrate the use of our oscillatory results.

1. Introduction

To unify continuous and discrete dynamics in a synthetical theory, Hilger introduced the theory of time scales in 1990 [1], which provides a powerful tool to study some traditionally separated fields in an uniform model [2, 3]. As an adequate mathematical tool to model the natural and social phenomena observed in physics, chemistry, biology, economics, neural networks, and social sciences, the topic is in spotlight with significant implications and has been a hot area of research for several years [4–7]. For example, Anderson and Zafer [5] established interval oscillation criteria for second-order super-half-linear functional dynamic equations with delay and advance arguments. Next, Agarwal et al. [6] extended the delay dynamic equations in [5] to mixed nonlinearities. In addition, they give analogous results for related advance type equations, as well as extended delay and advance equations.

Impulsive dynamic (or differential) equation with forced term is a powerful mathematical tool to simulate external action on a system, for example, an external force acted on a physical system, or some new species introduced to a biological system. A nonoscillatory spring can switch to be oscillatory by being appended a series of appropriate external forces. It is very interesting and meaningful to study the impacts of impulse and forced term on the oscillation of the system and its formation mechanism, and impulse is an important mechanism generates oscillation for differential equations or dynamic equations [8–12]. For instance, Ozbekler and Zafer [10] derived new interval oscillation criteria for forced super-half-linear differential equations with impulse effects. Huang et al. [13] studied a general second-order nonlinear dynamic equations with impulses and proved in theory that suitable impulses would convert nonoscillatory solutions to be oscillatory. Based information only on a sequence of intervals on the half interval $[0, \infty)$, Wong [12] introduced interval oscillatory criteria to differential equation with forced term and proved external force can produce oscillation. Liu extended Wong's interval oscillation criteria to a super-linear second-order impulsive forced differential equations. Huang et al. [14] extended Wong's methods to dynamic equation on time scales with forced term.

To cover and develop the oscillatory criteria established by Wong and Liu, it is naturally to ask if we can extend the interval oscillation criteria to impulsive dynamic equations with forced term. In this paper, we study the oscillation of a forced second-order nonlinear impulsive dynamic equation on time scales.

To be clear and self-contained, we introduce some fundamental conceptions and symbols used in time scales [1–3].

Denote $\mathbb{T}$ a time scale with $0 \in \mathbb{T}$. Define the *forward* and *backward jump operators* by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \tag{1}$$

where $\inf \phi = \sup \mathbb{T}$, $\sup \phi = \inf \mathbb{T}$, and ϕ denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ or *right-scattered* if $\sigma(t) > t$. Similarly, define a nonminimal element $t \in \mathbb{T}$ being *left-dense* or *left-scattered* by reversing the inequality. Denote $\mu(t) = \sigma(t) - t$ as the *graininess*.

A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called *rd-continuous*, if it is continuous at right-dense points and its left-sided limits exist at left-dense points in $\mathbb{T}$. Let $C_{rd}(\mathbb{T}, \mathbb{R})$ be the set of rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$. A mapping $f : \mathbb{T} \longrightarrow \mathbb{X}$ is differentiable at $t \in \mathbb{T}$, if there is $b \in \mathbb{X}$ such that, for any $\varepsilon > 0$, there exists a neighborhood $\mathbf{U}$ of t satisfying $|[f(\sigma(t)) - f(s)] - b[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|$, for all $s \in \mathbf{U}$. If $f^{\Delta}(t)$ exist for all $t \in \mathbb{T}$, then f is *delta differentiable* (or in short: differentiable) on $\mathbb{T}$. The delta-derivative and forward jump operator σ are related by the formula

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2}$$

Consider the following forced second-order nonlinear impulsive dynamic equations:

$$\begin{aligned} &x^{\Delta\Delta}(t) + p(t) f(x^{\sigma}(t)) = e(t), \\ &\qquad t \in \mathbb{J}_{\mathbb{T}} := [0, \infty) \cap \mathbb{T},\ t \ne t_k,\ k = 1, 2, \ldots, \\ &\qquad x(t_k^+) = a_k x(t_k^-), \\ &\qquad x^{\Delta}(t_k^+) = b_k x^{\Delta}(t_k^-), \\ &\qquad\qquad k = 1, 2, \ldots, \\ &\qquad x(t_0) = x_0, \\ &\qquad x^{\Delta}(t_0) = x_0^{\Delta}, \end{aligned} \tag{3}$$

where $a_k \ge b_k > 0$, $k = 1, 2, \cdots$ are constants, $e, p \in C_{rd}(\mathbb{T}, \mathbb{R})$, $t_k \in \mathbb{T}$, and $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k \longrightarrow \infty} t_k = \infty$.

$$x(t_k^+) = \lim_{h \longrightarrow 0^+} x(t_k + h), \qquad x^{\Delta}(t_k^+) = \lim_{h \longrightarrow 0^+} x^{\Delta}(t_k + h), \tag{4}$$

which represent right limits of $x(t)$ and $x^{\Delta}(t)$ at $t = t_k$ in the sense of time scales. If t_k is right-scattered, then $x(t_k^+) = x(t_k)$, $x^{\Delta}(t_k^+) = x^{\Delta}(t_k)$. Similarly, we can define $x(t_k^-)$ and $x^{\Delta}(t_k^-)$. Assume that all the impulsive points t_k, $k = 1, 2, \cdots$ are right-dense. To define the solutions of (3), we introduce the following space.

$AC^i = \{x : \mathbb{J}_{\mathbb{T}} \longrightarrow \mathbb{R}$ is i–times Δ–differentiable, whose i–th delta-derivative $x^{\Delta^{(i)}}$ is absolutely continuous $\}$.

$PC = \{x : \mathbb{J}_{\mathbb{T}} \longrightarrow \mathbb{R}$ is rd-continuous except at the points $t_k, k = 1, 2, \cdots$, for which $x(t_k^-), x(t_k^+), x^{\Delta}(t_k^-)$ and $x^{\Delta}(t_k^+)$ exist with $x(t_k^-) = x(t_k), x^{\Delta}(t_k^-) = x^{\Delta}(t_k)\}$.

Definition 1. A function $x \in PC \cap AC^2(\mathbb{J}_{\mathbb{T}} \backslash \{t_1, t_2, \cdots\}, \mathbb{R})$ is said to be a solution of (3), if it satisfies $x^{\Delta\Delta}(t) + p(t) f(x^{\sigma}(t)) = e(t)$ a.e. on $\mathbb{J}_{\mathbb{T}} \backslash \{t_k\}, k = 1, 2, \cdots$, and for each $k = 1, 2, \ldots, x$ satisfies the impulsive condition $x(t_k^+) = a_k x(t_k^-), x^{\Delta}(t_k^+) = b_k x^{\Delta}(t_k^-)$ and the initial conditions $x(t_0^+) = x_0, x^{\Delta}(t_0^+) = x_0^{\Delta}$.

Definition 2. A solution x of (3) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (3) is called oscillatory if all solutions are oscillatory.

By using generalized Riccati transformation technique, we study the interval oscillation criteria of (3) with changing sign coefficients $p(t)$ and forced term $e(t)$ in Section 2. The results extend and improve those of the earlier publications. We provide an example to illustrate the use of our main oscillatory results in Section 3.

2. Main Results

In this section, we develop some interval oscillation criteria of (3), which base information on a couple of intervals $[c_1, d_1]$ and $[c_2, d_2]$. To model the real world more accurate, we consider the coefficients $p(t)$ and $e(t)$ of (3) being changing signs. Without loss of generality, assume that $p(t)$ and $e(t)$ satisfy *(C)*. $c_i,\ d_i \notin \{t_k\},\ i = 1, 2$ with $c_1 < d_1,\ c_2 < d_2$, and

$$p(t) \ge 0, \quad t \in [c_1, d_1] \cup [c_2, d_2], \tag{5}$$

$$e(t) \begin{cases} \le 0, & t \in [c_1, d_1], \\ \ge 0, & t \in [c_2, d_2]. \end{cases} \tag{6}$$

Denote $h(s) = \max\{i : t_0 < t_i < s\}$, $\sum_{\Delta_1}^{\Delta_2} = 0$, if $\Delta_2 < \Delta_1$, and

$$D(c_i, d_i) = \left\{u \in C_{rd}^1[c_i, d_i] : u(t) \not\equiv 0,\ u(c_i) = u(d_i) = 0\right\}, \quad i = 1, 2. \tag{7}$$

We formulate the well-known inequality for convenience, which will be used in the proof of our main results.

Lemma 3. *For any nonnegative A and B, we have*

$$A^{\lambda} - \lambda A B^{\lambda - 1} + (\lambda - 1) B^{\lambda} \ge 0, \quad \lambda > 1, \tag{8}$$

and the equality holds if and only if $A = B$.

Theorem 4. *Let $f(x)/x \ge \alpha > 0$ for $x \ne 0$. Suppose that for any $T \ge 0$, there exists constants $c_1, d_1, c_2, d_2 \in \mathbb{T}$ with $T \le c_1 < d_1$, $T \le c_2 < d_2$, and condition (C) holds. If there exists $u \in D(c_i, d_i)$ such that*

$$\begin{aligned} &\int_{c_i}^{d_i} \left\{ \alpha p(t) u^2(\sigma(t)) \right. \\ &\quad \left. - \frac{\mu(t) + t - t_{h(d_i)}}{4\left(t - t_{h(d_i)}\right)} \left[\frac{u(t) + u(\sigma(t))}{u(\sigma(t))} u^{\Delta}(t) \right]^2 \right\} \Delta t \\ &\ge H(u, c_i, d_i), \end{aligned} \tag{9}$$

where $H(u, c_i, d_i) = 0$ for $h(c_i) = h(d_i)$, and

$$H(u, c_i, d_i) = \frac{b_{h(c_i)+1} - a_{h(c_i)+1}}{a_{h(c_i)+1}\left(t_{h(c_i)+1} - c_i\right)} u^2\left(t_{h(c_i)+1}\right) + \sum_{j=h(c_i)+2}^{h(d_i)} \frac{b_i - a_i}{a_i\left(t_i - t_{i-1}\right)} u^2\left(t_i\right), \tag{10}$$

for $h(c_i) < h(d_i)$, $i = 1, 2$, then (3) is oscillatory.

Proof. We prove the oscillation of (3) by argument of contradiction. Suppose that (3) has a nonoscillatory solution x, which is eventually positive. Say $x(t) > 0$, $x^\sigma(t) > 0$ for $t \geq t_0 \geq 0$. Define the Riccati transformation $w(t) = -x^\Delta(t)/x(t)$ for $t \geq t_0$. The derivative of $w(t)$ along with the solutions of (3) takes the form

$$\begin{aligned} w^\Delta(t) &= -\frac{x^{\Delta\Delta}(t)}{x^\sigma(t)} + \frac{x(t)}{x^\sigma(t)}\left(\frac{x^\Delta(t)}{x(t)}\right)^2 \\ &= \frac{1}{1 + \mu(t)\left(x^\Delta(t)/x(t)\right)} w^2(t) \\ &\quad + p(t)\frac{f\left(x^\sigma(t)\right)}{x^\sigma(t)} - \frac{e(t)}{x^\sigma(t)}, \end{aligned} \tag{11}$$

with $t \neq t_k$, $k = 1, 2, \cdots$. By the assumption (C), we can select $c_1, d_1 \in \mathbb{T}$ with $d_1 > c_1 \geq t_0$ and $p(t) \geq 0$, $e(t) \leq 0$, $t \in [c_1, d_1]$, such that

$$\begin{aligned} &x^{\Delta\Delta}(t) = e(t) - p(t) f\left(x^\sigma(t)\right) \leq 0, \\ &t \in \left[c_1, t_{h(c_1)+1}\right] \cup \left(t_{h(c_1)+1}, t_{h(c_1)+2}\right] \cup \cdots \cup \left(t_{h(d_1)}, d_1\right], \end{aligned} \tag{12}$$

which gives $x^\Delta(t)$ nonincreasing on $t \in [c_1, t_{h(c_1)+1}] \cup (t_{h(c_1)+1}, t_{h(c_1)+2}] \cup \cdots \cup (t_{h(d_1)}, d_1]$.

For $t \in [c_1, t_{h(c_1)+1}]$, we have

$$x(t) \geq x(t) - x(c_1) = \int_{c_1}^{t} x^\Delta(s)\,\Delta s \geq x^\Delta(t)(t - c_1). \tag{13}$$

Hence

$$\frac{x^\Delta(t)}{x(t)} \leq \frac{1}{t - c_1}, \quad t \in \left(c_1, t_{h(c_1)+1}\right]. \tag{14}$$

Making similar analysis on the intervals $(t_k, t_{k+1}]$ and $(t_{h(d_1)}, d_1]$, $i = h(c_1) + 1, \ldots, h(d_1) - 1$, we obtain

$$\begin{aligned} &\frac{x^\Delta(t)}{x(t)} \leq \frac{1}{t - t_k}, \\ &\qquad t \in \left(t_k, t_{k+1}\right], \ k = h(c_1) + 1, \ldots, h(d_1) - 1, \end{aligned} \tag{15}$$

and

$$\frac{x^\Delta(t)}{x(t)} \leq \frac{1}{t - t_{h(d_1)}}, \quad t \in \left(t_{h(d_1)}, d_1\right]. \tag{16}$$

By using (14), (15), (16), and the condition $f(x)/x \geq \alpha$, (11) yields

$$\begin{aligned} w^\Delta(t) &\geq \frac{t - t_{h(d_1)}}{\mu(t) + t - t_{h(d_1)}} w^2(t) + \alpha p(t) \\ &= \lambda_1(t) w^2(t) + \alpha p(t), \quad t \neq t_k, \ k = 1, 2, \ldots, \end{aligned} \tag{17}$$

where $\lambda_1(t) = (t - t_{h(d_1)})/(\mu(t) + t - t_{h(d_1)})$. For the impulsive moments $t = t_k$, $k = 1, 2, \cdots$, it follows from (3) that

$$w\left(t_k^+\right) = \frac{b_k}{a_k} w\left(t_k\right). \tag{18}$$

For the case $h(c_1) < h(d_1)$, all the impulsive moments in $[c_1, d_1]$ are $t_{h(c_1)+1}, t_{h(c_1)+2}, \ldots, t_{h(d_1)}$. Let $u(t) \in D(c_1, d_1)$ be a function satisfying the assumption of the theory. Multiply both sides of (17) by $u^2(\sigma(t))$ and integrate it from c_1 to d_1, using the integration by parts formula

$$\int_a^b f^\Delta(t) g(t)\,\Delta t = f(t) g(t)\Big|_a^b - \int_a^b f^\sigma(t) g^\Delta(t)\,\Delta t, \tag{19}$$

for any constants a, b, and differential functions f, g, we obtain

$$\begin{aligned} &\sum_{k=h(c_1)+1}^{h(d_1)} u^2(t_i)\left[w(t_k) - w\left(t_k^+\right)\right] \geq \int_{c_1}^{d_1} \alpha p(t) \\ &\quad \cdot u^2(\sigma(t))\,\Delta t + \int_{c_1}^{t_{h(c_1)+1}} \left[\lambda_1(t) w^2(t) u^2(\sigma(t)) \right. \\ &\quad \left. + \left(u(t) + u^\sigma(t)\right) u^\Delta(t) w(t)\right]\Delta t \\ &\quad + \sum_{k=h(c_1)+1}^{h(d_1)-1} \int_{t_k}^{t_{k+1}} \left[\lambda_1(t) w^2(t) u^2(\sigma(t)) \right. \\ &\quad \left. + \left(u(t) + u^\sigma(t)\right) u^\Delta(t) w(t)\right]\Delta t \\ &\quad + \int_{t_{h(d_1)}}^{d_1} \left[\lambda_1(t) w^2(t) u^2(\sigma(t)) + \left(u(t) + u^\sigma(t)\right) \right. \\ &\quad \left. \cdot u^\Delta(t) w(t)\right]\Delta t = \int_{c_1}^{d_1} \left[\alpha p(t) u^2(\sigma(t)) \right. \\ &\quad \left. - \frac{1}{4\lambda_1(t)}\left(\frac{u(t) + u^\sigma(t)}{u^\sigma(t)} u^\Delta(t)\right)^2\right]\Delta t \\ &\quad + \int_{c_1}^{t_{h(c_1)+1}} \left[\sqrt{\lambda_1(t)} w(t) u^\sigma(t) + \frac{u(t) + u^\sigma(t)}{2\sqrt{\lambda_1(t)} u^\sigma(t)} \right. \\ &\quad \left. \cdot u^\Delta(t)\right]^2 \Delta t \\ &\quad + \sum_{k=h(c_1)+1}^{h(d_1)-1} \int_{t_k}^{t_{k+1}} \left[\sqrt{\lambda_1(t)} w(t) u^\sigma(t) \right. \end{aligned}$$

$$+\frac{u(t)+u^{\sigma}(t)}{2\sqrt{\lambda_1(t)}u^{\sigma}(t)}u^{\Delta}(t)\Bigg]^2\Delta t$$
$$+\int_{t_{h(d_1)}}^{d_1}\Bigg[\sqrt{\lambda_1(t)}w(t)u^{\sigma}(t)+\frac{u(t)+u^{\sigma}(t)}{2\sqrt{\lambda_1(t)}u^{\sigma}(t)}$$
$$\cdot u^{\Delta}(t)\Bigg]^2\Delta t>\int_{c_1}^{d_1}\Bigg[\alpha p(t)u^2(\sigma(t))$$
$$-\frac{1}{4\lambda_1(t)}\left(\frac{u(t)+u^{\sigma}(t)}{u^{\sigma}(t)}u^{\Delta}(t)\right)^2\Bigg]\Delta t. \tag{20}$$

Using (18), we get

$$\sum_{k=h(c_1)+1}^{h(d_1)}\frac{a_k-b_k}{a_k}u^2(t_k)w(t_k)>\int_{c_1}^{d_1}\Bigg[\alpha p(t)u^2(\sigma(t))$$
$$-\frac{1}{4\lambda_1(t)}\left(\frac{u(t)+u^{\sigma}(t)}{u^{\sigma}(t)}u^{\Delta}(t)\right)^2\Bigg]\Delta t. \tag{21}$$

It follows from (14)-(16) that

$$w\left(t_{h(c_1)+1}\right)=-\frac{x^{\Delta}\left(t_{h(c_1)+1}\right)}{x\left(t_{h(c_1)+1}\right)}\geq-\frac{1}{t_{h(c_1)+1}-c_1}, \tag{22}$$

and

$$w(t_k)\geq-\frac{1}{t_k-t_{k-1}},\quad k=h(c_1)+2,\ldots,h(d_1). \tag{23}$$

The conditions $b_k\geq a_k>0, k=1,2,\cdots$, (22), and (23) yield

$$\sum_{k=h(c_1)+1}^{h(d_1)}\frac{b_k-a_k}{a_k}u^2(t_k)w(t_k)$$
$$\geq-\frac{b_{h(c_1)+1}-a_{h(c_1)+1}}{a_{h(c_1)+1}\left(t_{h(c_1)+1}-c_1\right)}u^2\left(t_{h(c_1)+1}\right)$$
$$-\sum_{k=h(c_1)+2}^{h(d_1)}\frac{b_k-a_k}{a_k\left(t_k-t_{k-1}\right)}u^2(t_k). \tag{24}$$

Thus

$$\sum_{k=h(c_1)+1}^{h(d_1)}\frac{a_k-b_k}{a_k}u^2(t_k)w(t_k)$$
$$\leq\frac{b_{h(c_1)+1}-a_{h(c_1)+1}}{a_{h(c_1)+1}\left(t_{h(c_1)+1}-c_1\right)}u^2\left(t_{h(c_1)+1}\right)$$
$$+\sum_{k=h(c_1)+2}^{h(d_1)}\frac{b_k-a_k}{a_k\left(t_k-t_{k-1}\right)}u^2(t_k)=H(u,c_1,d_1). \tag{25}$$

According to (21), we have

$$\int_{c_1}^{d_1}\Bigg[\alpha p(t)u^2(\sigma(t))$$
$$-\frac{1}{4\lambda_1(t)}\left(\frac{u(t)+u^{\sigma}(t)}{u^{\sigma}(t)}u^{\Delta}(t)\right)^2\Bigg]\Delta t$$
$$<H(u,c_1,d_1), \tag{26}$$

which is contradicted with (9).

For the case $h(c_1)=h(d_1)$, $H(u,c_1,d_1)=0$, and there are no impulsive points in $[c_1,d_1]$. Similar to the proof of (21), we obtain

$$\int_{c_1}^{d_1}\Bigg[\alpha p(t)u^2(\sigma(t))$$
$$-\frac{1}{4\lambda_1(t)}\left(\frac{u(t)+u^{\sigma}(t)}{u^{\sigma}(t)}u^{\Delta}(t)\right)^2\Bigg]\Delta t<0, \tag{27}$$

which is contradicted with (9) again.

For the case $x(t)<0, t\geq t_0\geq 0$, by letting $y=-x$, then y is positive solution of the dynamic equations

$$\begin{aligned}
&x^{\Delta\Delta}(t)+p(t)f\left(x^{\sigma}(t)\right)=-e(t),\\
&\qquad t\in\mathbb{J}_{\mathbb{T}}:=[0,\infty)\cap\mathbb{T},\ t\neq t_k,\ k=1,2,\ldots,\\
&\qquad x\left(t_k^{+}\right)=a_kx\left(t_k^{-}\right),\\
&\qquad x^{\Delta}\left(t_k^{+}\right)=b_kx^{\Delta}\left(t_k^{-}\right),\\
&\qquad\qquad k=1,2,\ldots,\\
&\qquad x(t_0)=x_0,\\
&\qquad x^{\Delta}(t_0)=x_0^{\Delta}.
\end{aligned} \tag{28}$$

Repeating the above procedure on the interval $[c_2,d_2]$, we will obtain a contradiction again. This completes the proof of Theorem 4. □

Theorem 5. *Let $xf(x)>0$ for $x\neq 0$ and $|f(x)|\geq|x|^r$ for $r>1$. Suppose that for any $T\geq 0$, there exists constants $c_1,d_1,c_2,d_2\in\mathbb{T}$ with $T\leq c_1<d_1$, $T\leq c_2<d_2$, and (C) holds. If there exists $u\in D(c_i,d_i)$ such that*

$$\int_{c_i}^{d_i}\Bigg\{r(r-1)^{(1-r)/r}p^{1/r}(t)|e(t)|^{(r-1)/r}u^2(\sigma(t))$$
$$-\frac{\mu(t)+t-t_{h(d_i)}}{4\left(t-t_{h(d_i)}\right)}\left[\frac{u(t)+u^{\sigma}(t)}{u^{\sigma}(t)}u^{\Delta}(t)\right]^2\Bigg\}\Delta t$$
$$\geq H(u,c_i,d_i), \tag{29}$$

where H is given by (10), then (3) is oscillatory.

Proof. Assume that $x(t)>0$, $x^{\sigma}(t)>0$, $t\geq t_0\geq 0$, is a nonoscillatory solution of (3). Let $A=p^{1/r}(t)x^{\sigma}(t)$,

$B = (-e(t)/(r-1))^{1/r}$, and choose $c_1, d_1 \in \mathbb{T}$ with $d_1 > c_1 \geq t_0 \geq 0$ and $p(t) \geq 0$, $e(t) \leq 0$ for $t \in [c_1, d_1]$. Hence, $A > 0$, $B > 0$ for $r > 1$. Lemma 3 gives

$$\begin{aligned} & p(t)\, x^r(\sigma(t)) - e(t) \\ & \quad \geq r(r-1)^{(1-r)/r} p^{1/r}(t) |e(t)|^{(r-1)/r} x(\sigma(t)) \\ & \quad = \lambda_2 p^{1/r}(t) |e(t)|^{(r-1)/r} x(\sigma(t)), \end{aligned} \tag{30}$$

where $\lambda_2 = r(r-1)^{(1-r)/r}$ is a positive constant. By (3) and (30), we obtain

$$x^{\Delta\Delta}(t) + \lambda_2 p^{1/r}(t) |e(t)|^{(r-1)/r} x(\sigma(t)) \leq 0. \tag{31}$$

Let $w(t) = x^{\Delta}(t)/x(t)$. By using (14)-(16), the formulas (2) and

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}, \tag{32}$$

we have

$$\begin{aligned} w^{\Delta}(t) &= \frac{x^{\Delta\Delta}(t)}{x^{\sigma}(t)} - \frac{x(t)}{x^{\sigma}(t)} \left[\frac{x^{\Delta}(t)}{x(t)}\right]^2 \\ &\leq -\lambda_2 p^{1/r}(t) |e(t)|^{(r-1)/r} - \lambda_1(t) w^2(t), \end{aligned} \tag{33}$$

where $t \neq t_k, k = 1, 2, \cdots$ and $\lambda_1(t) = (t - t_{h(d_1)})/(\mu(t) + t - t_{h(d_1)})$.

In the case $h(c_1) < h(d_1)$, all the impulsive moments in $[c_1, d_1]$ are $t_{h(c_1)+1}, t_{h(c_1)+2}, \ldots, t_{h(d_1)}$. Let $u(t) \in D(c_1, d_1)$ be given as in the theorem. Multiplying both sides of (33) by $u^2(\sigma(t))$ and integrating it from c_1 to d_1 give

$$\begin{aligned} & \sum_{k=h(c_1)+1}^{h(d_1)} u^2(t_k)\left[w(t_k) - w(t_k^+)\right] \leq -\int_{c_1}^{d_1} \lambda_2 p^{1/r}(t) \\ & \quad \cdot |e(t)|^{(r-1)/r} u^2(\sigma(t))\, \Delta t - \int_{c_1}^{t_{h(c_1)+1}} \Big[\lambda_1(t) w^2(t) \\ & \quad \cdot u^2(\sigma(t)) + \left(u(t) + u^{\sigma}(t)\right) u^{\Delta}(t) w(t)\Big] \Delta t \\ & \quad - \sum_{k=h(c_1)+1}^{h(d_1)-1} \int_{t_i}^{t_{k+1}} \Big[\lambda_1(t) w^2(t) u^2(\sigma(t)) \\ & \quad + \left(u(t) + u^{\sigma}(t)\right) u^{\Delta}(t) w(t)\Big] \Delta t \\ & \quad - \int_{t_{h(d_1)}}^{d_1} \Big[\lambda_1(t) w^2(t) u^2(\sigma(t)) + \left(u(t) + u^{\sigma}(t)\right) \\ & \quad \cdot u^{\Delta}(t) w(t)\Big] \Delta t = -\int_{c_1}^{d_1} \Big[\lambda_2 p^{1/r}(t) \\ & \quad \cdot |e(t)|^{(r-1)/r} u^2(\sigma(t)) \\ & \quad - \frac{1}{4\lambda_1(t)} \left(\frac{u(t) + u^{\sigma}(t)}{u^{\sigma}(t)} u^{\Delta}(t)\right)^2\Big] \Delta t \\ & \quad - \int_{c_1}^{t_{h(c_1)+1}} \Big[\sqrt{\lambda_1(t)} w(t) u^{\sigma}(t) + \frac{u(t) + u^{\sigma}(t)}{2\sqrt{\lambda_1(t)} u^{\sigma}(t)} \\ & \quad \cdot u^{\Delta}(t)\Big]^2 \Delta t \\ & \quad - \sum_{k=h(c_1)+1}^{h(d_1)-1} \int_{t_k}^{t_{k+1}} \Big[\sqrt{\lambda_1(t)} w(t) u^{\sigma}(t) \\ & \quad + \frac{u(t) + u^{\sigma}(t)}{2\sqrt{\lambda_1(t)} u^{\sigma}(t)} u^{\Delta}(t)\Big]^2 \Delta t \\ & \quad - \int_{t_{h(d_1)}}^{d_1} \Big[\sqrt{\lambda_1(t)} w(t) u^{\sigma}(t) + \frac{u(t) + u^{\sigma}(t)}{2\sqrt{\lambda_1(t)} u^{\sigma}(t)} \\ & \quad \cdot u^{\Delta}(t)\Big]^2 \Delta t < -\int_{c_1}^{d_1} \Big[\lambda_2 p^{1/r}(t) |e(t)|^{(r-1)/r} \\ & \quad \cdot u^2(\sigma(t)) \\ & \quad - \frac{1}{4\lambda_1(t)} \left(\frac{u(t) + u^{\sigma}(t)}{u^{\sigma}(t)} u^{\Delta}(t)\right)^2\Big] \Delta t. \end{aligned} \tag{34}$$

Similar to the proof of Theorem 4, by using (18), we have

$$\begin{aligned} & \sum_{k=h(c_1)+1}^{h(d_1)} u^2(t_k)\left[w(t_k) - w(t_k^+)\right] \\ & \quad = \sum_{k=h(c_1)+1}^{h(d_1)} \frac{a_k - b_k}{a_k} u^2(t_k) w(t_k) \geq -H(u, c_1, d_1). \end{aligned} \tag{35}$$

Thus, (34) yields

$$\begin{aligned} & \int_{c_i}^{d_i} \Big[\lambda_2 p^{1/r}(t) |e(t)|^{(r-1)/r} u^2(\sigma(t)) \\ & \quad - \frac{1}{4\lambda_1} \left(\frac{u(t) + u^{\sigma}(t)}{u^{\sigma}(t)} u^{\Delta}(t)\right)^2\Big] \Delta t < H(u, c_1, \\ & \quad d_1), \end{aligned} \tag{36}$$

which is contradicted with (29).

In the case $h(c_1) = h(d_1)$, we have $H(u, c_1, d_1) = 0$ and there are no impulsive moments in $[c_1, d_1]$. Similar to the proof of Theorem 4, we get a contradiction again.

If $x(t) < 0$, $t \geq t_0 \geq 0$, we can derive a similar contradiction on $[c_2, d_2]$. This completes the proof of Theorem 5. □

3. Example

Since the time scale $\mathbb{P}_{a,b} = \bigcup_{n=0}^{\infty} [n(a+b), n(a+b) + a]$ is an important and useful tool used extensively to simulate biological population, electric circuit, and so on, we give an example in $\mathbb{P}_{a,b}$ to show the application of our oscillatory criteria.

Example 6. Consider the following system:

$$\begin{aligned}
&x^{\Delta\Delta}(t) + m\sin t x(\sigma(t)) = \cos t,\\
&t \in \mathbb{P}_{\pi,\pi} = \bigcup_{n=0}^{\infty} [2n\pi, (2n+1)\pi],\\
&\qquad t \neq \left(2k + \frac{1}{2}\right)\pi \pm \frac{3}{8}\pi,\ k = 0, 1, 2, \ldots,\\
&x(t_k^+) = a_k x(t_k^-),\\
&x^{\Delta}(t_k^+) = b_k x^{\Delta}(t_k^-),\\
&\qquad t_k = \left(2k + \frac{1}{2}\right)\pi \pm \frac{3}{8}\pi,\ k = 0, 1, 2, \ldots,\\
&x(t_0) = x_0,\\
&x^{\Delta}(t_0) = x_0^{\Delta},
\end{aligned} \tag{37}$$

with the transition condition

$$x(2n\pi) = x((2n-1)\pi), \quad n \geq 1, \tag{38}$$

where $b_k \geq a_k > 0$, $m > 0$ are constants.

Hence $p(t) = m\sin t$, $e(t) = \cos t$, and $f(x(\sigma(t))) = x(\sigma(t))$. For any $T \geq 0$, choose $c_1 = 2n\pi + 3\pi/4$, $d_1 = 2n\pi + \pi$, $c_2 = 2n\pi$, $d_2 = 2n\pi + \pi/4$, ($c_i, d_i \in \mathbb{P}_{\pi,\pi}$, $i = 1, 2$), such that $c_i \geq T$ for sufficiently large n, $i = 1, 2$. Hence $p(t) \geq 0$ for $t \in [c_1, d_1] \cup [c_2, d_2]$, and $e(t) \leq 0$ for $t \in [c_1, d_1]$, and $e(t) \geq 0$ for $t \in [c_2, d_2]$. Let $u(t) = \sin 2t \cos 2t$, then $u(t) \in D(c_i, d_i)$, $i = 1, 2$. Furthermore, we have $\sigma(t) = t$, $\mu(t) = 0$ for $t \in [c_i, d_i]$, $i = 1, 2$. By some simple calculation, for $i = 1, 2$, we have

$$\begin{aligned}
&\int_{c_i}^{d_i} \left\{ kp(t)u^2(\sigma(t)) - \frac{\mu(t) + t - t_{h(d_i)}}{4\left(t - t_{h(d_i)}\right)} \left[\frac{u(t) + u(\sigma(t))}{u(\sigma(t))} \cdot u^{\Delta}(t) \right]^2 \right\} \Delta t = \int_{c_i}^{d_i} \left[p(t)u^2(t) - \left(u^{\Delta}(t)\right)^2 \right] dt = \int_{c_i}^{d_i} \left[m\sin t \sin^2 2t \cos^2 2t - 4\cos^2 4t \right] dt,\\
&\int_{c_i}^{d_i} 4\cos^2 4t dt = \frac{\pi}{2},
\end{aligned} \tag{39}$$

and

$$\int_{c_i}^{d_i} m\sin t \sin^2 2t \cos^2 2t dt = \frac{8}{63}\left(1 - \frac{\sqrt{2}}{2}\right) m. \tag{40}$$

Therefore,

$$\int_{c_i}^{d_i} \left\{ kp(t)u^2(\sigma(t)) - \frac{\mu(t) + t - t_{h(d_i)}}{4\left(t - t_{h(d_i)}\right)} \left[\frac{u(t) + u(\sigma(t))}{u(\sigma(t))} \cdot u^{\Delta}(t) \right]^2 \right\} \Delta t \tag{41}$$

$$= \frac{8}{63}\left(1 - \frac{\sqrt{2}}{2}\right) m - \frac{\pi}{2}. \tag{42}$$

The impulsive points in $[c_1, d_1] = [2n\pi + 3\pi/4, 2n\pi + \pi]$ are $2n\pi + (7/8)\pi$ and in $[c_2, d_2] = [2n\pi, 2n\pi + \pi/4]$ are $2n\pi + (1/8)\pi$. Thus

$$\begin{aligned}
H(u, c_i, d_i) &= \frac{b_{h(c_i)+1} - a_{h(c_i)+1}}{a_{h(c_i)+1}\left(t_{h(c_i)+1} - c_i\right)} u^2\left(t_{h(c_i)+1}\right)\\
&= \frac{2}{\pi} \frac{b_{h(c_i)+1} - a_{h(c_i)+1}}{a_{h(c_i)+1}}.
\end{aligned} \tag{43}$$

By Theorems 4, (3) is oscillatory if

$$\frac{8}{63}\left(1 - \frac{\sqrt{2}}{2}\right) m - \frac{\pi}{2} \geq \frac{2}{\pi} \frac{b_{h(c_i)+1} - a_{h(c_i)+1}}{a_{h(c_i)+1}}. \tag{44}$$

For example, letting $a_k = 1$, $b_k = (k+1)/k$ yields

$$\frac{8}{63}\left(1 - \frac{\sqrt{2}}{2}\right) m \geq \frac{\pi}{2} + \frac{2}{(2n + 7/8)\pi^2}, \tag{45}$$

for sufficiently large m, and so (38) holds. It follows from Theorem 5 that all the solutions of (37)-(38) are oscillatory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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