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Research Article

Global Behavior of a Discrete Anticompetitive System in the Plane

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The main goal of this paper is to investigate the global asymptotic behavior of the difference system $x_{n+1} = \gamma_1 y_n/(A_1 + x_n)$, $y_{n+1} = \beta_2 x_n/(B_2 + y_n)$, $n = 0, 1, 2, \ldots$. with $\gamma_1, \beta_2, A_1, B_2 \in (0, \infty)$ and the initial condition $(x_0, y_0) \in [0, \infty) \times [0, \infty)$. We obtain some global attractivity results of this system for different values of the parameters, which answer the open problem proposed in "Rational systems in the plane, J. Difference Equ. Appl. 15 (2009), 303-323".

1. Introduction and Preliminaries

Difference equations or systems have been attracting more and more attention by authors since these models display some complicated character comparing with its analogue differential equations; see [1–11]. In [12], the authors gave a discussion on the following rational system in the plane :

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_1 y_n}{A_2 + A_2 x_n + C_2 y_n}, \\ & n = 0, 1, 2, \cdots . \end{aligned} \tag{1}$$

System (1) contains, as special cases, a large number of equations whose dynamics have not been thoroughly understood yet and there still exists research aspect to be further studied. Several global asymptotic results for some special cases of (1) have been obtained in [13–18]. As a special anticompetitive system of (1), the difference system

$$\begin{aligned} x_{n+1} &= \frac{\gamma_1 y_n}{A_1 + x_n}, \\ y_{n+1} &= \frac{\beta_2 x_n}{A_2 + y_n}, \\ & n = 0, 1, 2, \ldots, \end{aligned} \tag{2}$$

is considered in this paper, where the parameters $\beta_2, \gamma_1, A_1, A_2 \in (0, \infty)$ and the initial value $(x_0, y_0) \in \mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$.

In [12], system is labeled as (16,16) and is concerned as an open problem (Open Problem 7) which asked for determining the boundedness of its solutions, the local stability of its equilibrium, the existence of prime period-two solutions, and the global character of system (2). To answer the open problem, the main goal of this paper is to study the dynamic behavior of system (2).

As far the definition of stability and the method of linearized stability, one can see [19–21].

2. Linearized Stability

The equilibrium $(\overline{x}, \overline{y})$ of system (2) is the intersection of the two following curves:

$$\begin{aligned} C_1 &: x = \frac{\gamma_1 y}{A_1 + x}, \\ C_2 &: y = \frac{\beta_2 x}{A_2 + y}. \end{aligned} \tag{3}$$

The slopes of the tangent line of the two curves at the origin $(0, 0)$ are

$$k_{C_1} = \frac{A_1}{\gamma_1},$$

$$k_{C_2} = \frac{\beta_2}{A_2}. \tag{4}$$

When $\beta_2\gamma_1 \le A_1A_2$, $k_{C_1} \ge k_{C_2}$, the two curves C_1 and C_2 have a unique intersection in $\mathbb{R}_+^2$, that is $(0,0)$. When $\beta_2\gamma_1 > A_1A_2$, $k_{C_1} < k_{C_2}$, the curves C_1 and C_2 have also another intersection $E_+ = (\overline{x}, \overline{y})$ locating in the first quadrant, satisfying $(A_1 + \overline{x})(A_2 + \overline{y}) = \beta_2\gamma_1$. So we have the following lemma.

Lemma 1. *If $\beta_2\gamma_1 \le A_1A_2$, then system (2) has a unique fixed point $E_0 = (0,0)$. If $\beta_2\gamma_1 > A_1A_2$, then system (2) has another positive fixed point $E_+ = (\overline{x}, \overline{y})$ locating in the first quadrant, satisfying $(A_1 + \overline{x})(A_2 + \overline{y}) = \beta_2\gamma_1$.*

The linearized system of (2) about E_0 is

$$\begin{aligned} x_{n+1} &= \frac{\gamma_1}{A_1} y_n, \\ y_{n+1} &= \frac{\beta_2}{A_2} x_n, \end{aligned} \tag{5}$$

and its characteristic equation is $\lambda^2 - \beta_2\gamma_1/A_1A_2 = 0$. So $\lambda_{1,2} = \pm\sqrt{\beta_2\gamma_1/A_1A_2}$.

The linearized system of (2) about E_+ is

$$\begin{aligned} x_{n+1} &= -\frac{\overline{x}}{A_1 + \overline{x}} x_n + \frac{\gamma_1}{A_1 + \overline{x}} y_n, \\ y_{n+1} &= \frac{\beta_2}{A_2 + \overline{y}} x_n - \frac{\overline{y}}{A_2 + \overline{y}} y_n. \end{aligned} \tag{6}$$

Noticing the fact that $(A_1 + \overline{x})(A_2 + \overline{y}) = \beta_2\gamma_1$, the characteristic equation associated with E_+ can be written as

$$\lambda^2 + p\lambda + q = 0, \tag{7}$$

where $p = (A_1\overline{y} + A_2\overline{x} + 2\overline{x}\,\overline{y})/\beta_2\gamma_1 = 1 + (\overline{x}\,\overline{y} - A_1A_2)/\beta_2\gamma_1 > 0$, $q = (\overline{x}\,\overline{y} - \beta_2\gamma_1)/\beta_2\gamma_1 < 0$.

A simple calculation shows that

$$\begin{aligned} \Delta = p^2 - 4q &= \left(1 + \frac{\overline{x}\,\overline{y} - A_1A_2}{\beta_2\gamma_1}\right)^2 - 4\frac{\overline{x}\,\overline{y} - \beta_2\gamma_1}{\beta_2\gamma_1} \\ &> \left(1 + \frac{\overline{x}\,\overline{y} - A_1A_2}{\beta_2\gamma_1}\right)^2 - 4\frac{\overline{x}\,\overline{y} - A_1A_2}{\beta_2\gamma_1} \\ &= \left(1 - \frac{\overline{x}\,\overline{y} - A_1A_2}{\beta_2\gamma_1}\right)^2 > 0. \end{aligned} \tag{8}$$

Since $1 + (\overline{x}\,\overline{y} - A_1A_2)/\beta_2\gamma_1 > 0$, we get $(\overline{x}\,\overline{y} - A_1A_2)/\beta_2\gamma_1 > -1$, and hence $1 \pm (\overline{x}\,\overline{y} - A_1A_2)/\beta_2\gamma_1 < \sqrt{\Delta} < 3 + (\overline{x}\,\overline{y} - A_1A_2)/\beta_2\gamma_1$. Hence,

$$\begin{aligned} 0 < \lambda_1 &= \frac{1}{2}\left(-1 - \frac{\overline{x}\,\overline{y} - A_1A_2}{\beta_2\gamma_1} + \sqrt{\Delta}\right) < 1, \\ \lambda_2 &= \frac{1}{2}\left(-1 - \frac{\overline{x}\,\overline{y} - A_1A_2}{\beta_2\gamma_1} - \sqrt{\Delta}\right) < -1. \end{aligned} \tag{9}$$

Therefore, we have the following results.

Theorem 2. *(i) Assume that $\beta_2\gamma_1 \le A_1A_2$. Then E_0 is the only equilibrium of system (2), and it is locally stable while $\beta_2\gamma_1 < A_1A_2$ and is nonhyperbolic while $\beta_2\gamma_1 = A_1A_2$.*

(ii) Assume that $\beta_2\gamma_1 > A_1A_2$. Then E_0 and E_+ are equilibria of system (2) and they are all unstable. In fact, E_0 is a repeller, and E_+ is a saddle point.

3. Periodic Character

Let $a = \gamma_1/A_1$, $b = \beta_2/A_2$. Then the following statements are true.

Lemma 3. *(i) The initiation value $(x_0, 0)$ generates the solution*

$$\begin{gathered} (x_0, 0), (0, bx_0), (abx_0, 0), \ldots, \left(0, b^{2n-1}a^{2n}x_0\right), \\ \left((ab)^{2n} x_0, 0\right), \ldots, \end{gathered} \tag{10}$$

(ii) The initiation value $(0, y_0)$ generates the solution

$$\begin{gathered} (0, y_0), (ay_0, 0), (0, aby_0), \ldots, \left(a^{2n}b^{2n-1} y_0, 0\right), \\ \left(0, (ab)^{2n} y_0\right), \ldots. \end{gathered} \tag{11}$$

(iii) The initial value (x_0, y_0) with $x_0y_0 > 0$ generates the solution $\{(x_n, y_n)\}$ with $x_ny_n > 0$ for $n \ge 0$.

Theorem 4. *System (2) has positive prime period-two solution if and only if $\beta_2\gamma_1 = A_1A_2$. Further, the prime period-two solution possesses the form such that*

(i) $\cdots, (x_0, 0), (0, (\beta_2/A_2)x_0), (x_0, 0), (0, (\beta_2/A_2)x_0), \cdots$ with $x_0 > 0$;

(ii) $\cdots, (0, y_0), ((\gamma_1/A_1)y_0, 0), (0, y_0), ((\gamma_1/A_1)y_0, 0), \cdots$ with $y_0 > 0$.

Proof. Applying Lemma 3 (i)-(ii) and the condition that $\beta_2\gamma_1 = A_1A_2$, it is easy to see that the solution of system (2) which starting on either axis are all of period-two. This establishes the sufficient condition. Moreover, it is also a necessary and vital condition, as may be seen by the following argument.

Let (m, l), (M, L) be a period-two solution of system (2). Then they should satisfy the following equation:

$$\begin{aligned} M &= \frac{\gamma_1 l}{A_1 + m}, \\ m &= \frac{\gamma_1 L}{A_1 + M}, \\ L &= \frac{\beta_2 m}{A_2 + l}, \\ l &= \frac{\beta_2 M}{A_2 + L}, \end{aligned} \tag{12}$$

from which it follows that

$$\begin{aligned} A_1(M - m) &= \gamma_1(l - L), \\ A_2(L - l) &= \beta_2(m - M). \end{aligned} \tag{13}$$

Obviously, $m = M$ yields $l = L$, or vice versa, this is a contradiction. Thus $m \neq M$ and $l \neq L$, and (13) yields $\beta_2\gamma_1 = A_1A_2$.

Equation (12) also yields

$$mM = \frac{\gamma_1^2 lL}{(A_1+m)(A_1+M)},$$
$$lL = \frac{\beta_2^2 mM}{(A_2+l)(A_2+L)}, \tag{14}$$

from which it follows that

$$mMlL\left[(A_1+m)(A_1+M)(A_2+l)(A_2+L) - \beta_2^2\gamma_1^2\right] = 0. \tag{15}$$

Since $\beta_2\gamma_1 = A_1A_2$ and $m+M>0$, $l+L>0$, we obtain

$$(A_1+m)(A_1+M)(A_2+l)(A_2+L) - \beta_2^2\gamma_1^2 > A_1A_2(m+M)(l+L) > 0, \tag{16}$$

and so $mMlL = 0$.

If $m = 0$, then $L = 0$, or vice versa. In this case, we claim that $M \neq 0$. Otherwise, there would be the fact that $m = M = l = L = 0$ holds, a contradiction. So system (2) possesses a period-two solution

$$\ldots,(M,0),\left(0,\frac{\beta_2}{A_2}M\right),(M,0),\ldots, \tag{17}$$

where $M = x_0 > 0$.

Similarly, if $M = 0$, then $l = 0$, or vice versa, and system (2) possesses a period-two solution

$$\ldots,(0,L),\left(\frac{\gamma_1}{A_1}L,0\right),(0,L),\ldots, \tag{18}$$

where $L = y_0 > 0$.

The proof is complete. □

4. Global Attractivity

4.1. The Case $\beta_2\gamma_1 \leq A_1A_2$

Theorem 5. *Assume that $\beta_2\gamma_1 < A_1A_2$. Then the unique equilibrium E_0 is globally asymptotically stable.*

Proof. Let $\{(x_n, y_n)\}$ be a solution of system (2). Clearly, $\beta_2\gamma_1 < A_1A_2$ implies that $ab < 1$. Notice that for $n > 0$ yield

$$\begin{pmatrix} x_{2n} \\ y_{2n} \end{pmatrix} \leq \begin{pmatrix} ay_{2n-1} \\ bx_{2n-1} \end{pmatrix} \leq ab\begin{pmatrix} x_{2n-2} \\ y_{2n-2} \end{pmatrix} \leq \cdots \leq (ab)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{19}$$

and

$$\begin{pmatrix} x_{2n+1} \\ y_{2n+1} \end{pmatrix} \leq \begin{pmatrix} ay_{2n} \\ bx_{2n} \end{pmatrix} \leq ab\begin{pmatrix} x_{2n-1} \\ y_{2n-1} \end{pmatrix} \leq \cdots \leq (ab)^n \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{20}$$

So $(x_n, y_n) \longrightarrow (0,0)$. Consequently, E_0 is a global attractor and hence by Theorem 2 (i), the conclusion follows. □

Consider the map F on $\mathbb{R}^2$ associated with system (2); that is,

$$F(x,y) = (f(x,y), g(x,y)) = \left(\frac{\gamma_1 y}{A_1+x}, \frac{\beta_2 x}{A_2+y}\right), \quad x, y \geq 0. \tag{21}$$

Set

$$\Delta_x = f^2(x,y) - x = x\left(\frac{\beta_2\gamma_1(A_1+x)}{(A_2+y)(A_1(A_1+x)+\gamma_1 y)} - 1\right),$$
$$\Delta_y = g^2(x,y) - y = y\left(\frac{\beta_2\gamma_1(A_2+y)}{(A_1+x)(A_2(A_2+y)+\beta_2 x)} - 1\right). \tag{22}$$

Lemma 6. *Assume that $\beta_2\gamma_1 = A_1A_2$. Then $\Delta_x < 0$, $\Delta_y < 0$ with $x, y > 0$.*

Proof. Using (22) and the condition that $\beta_2\gamma_1 = A_1A_2$, $x, y > 0$, we have

$$\Delta_x = -xy\frac{A_2\gamma_1 + A_1^2 + A_1x + \gamma_1 y}{(A_2+y)[A_1(A_1+x)+\gamma_1 y]} < 0 \tag{23}$$

and

$$\Delta_y = -xy\frac{A_1\beta_2 + A_2^2 + A_2y + \beta_2 x}{(A_1+x)[A_2(A_2+y)+\beta_2 x]} < 0. \tag{24}$$

The proof is complete. □

Theorem 7. *Assume that $\beta_2\gamma_1 = A_1A_2$. Then*

(i) every solution of system (2) starting on either axis is of period-two;

(ii) every solution of system (2) with $x_0y_0 > 0$ converges to E_0.

Proof. (i) It is a direct consequence of Theorem 4.

(ii) Let $\{(x_n, y_n)\}$ be a solution of system (2) with initial value $x_0y_0 > 0$. Then by Lemma 6, the subsequence $\{x_{2n}\}$, $\{x_{2n+1}\}$ and $\{y_{2n}\}$, $\{y_{2n+1}\}$ are all strictly decreasing and bounded below by zero. Notice the form of period-two solution of system (2) mentioned by Theorem 4; then there is only one result; that is, $\lim_{n\longrightarrow\infty}(x_n, y_n) = (0,0)$. This completes the proof. □

4.2. The Case $\beta_2\gamma_1 > A_1A_2$. Let Q_i for $i = 1, \ldots, 4$ be the usual four quadrants at E_+ and numbered in a counterclockwise direction, e.g., $Q_1 = \{(x, y) \in \mathbb{R}_+^2 : x \geq \overline{x},\ y \geq \overline{y}\}$. Then we have the following results.

Lemma 8. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then $F(Q_2) \subseteq Q_4$, $F(Q_4) \subseteq Q_2$. Further, the regions Q_2 and Q_4 are invariant under the iteration by F^2.*

Proof. Let $(x, y) \in Q_2$. Then $x \leq \overline{x}$ and $y \geq \overline{y}$, so

$$f(x, y) = \frac{\gamma_1 y}{A_1 + x} \geq \frac{\gamma_1 \overline{y}}{A_1 + \overline{x}} = \overline{x},$$
$$g(x, y) = \frac{\beta_2 x}{A_2 + y} \leq \frac{\beta_2 \overline{x}}{A_2 + \overline{y}} = \overline{y}, \quad (25)$$

from which it follows that $(F(x, y) = (f(x, y), g(x, y)) \in Q_4$.

Further,

$$f^2(x, y) = \frac{\gamma_1 g(x, y)}{A_1 + f(x, y)} \leq \frac{\gamma_1 \overline{y}}{A_1 + \overline{x}} = \overline{x},$$
$$g^2(x, y) = \frac{\beta_2 f(x, y)}{A_2 + g(x, y)} \geq \frac{\beta_2 \overline{x}}{A_2 + \overline{y}} = \overline{y}, \quad (26)$$

from which it follows that $F^2(x, y) \in Q_2$.

The proof for $F(Q_4) \subseteq Q_2$ is similar and is omitted, finishing the proof. □

To obtain the global attractivity of the positive equilibrium E_+, we further divide $\mathbb{R}_+^2$ into four distinct regions, which are described as follows:

(i) $R_1 = \{(x, y) \mid (\overline{x}/\gamma_1)(A_1 + x) < y < (\beta_2/\overline{y})x - A_2,\ x > \overline{x}\}$,

(ii) $R_2 = \{(x, y) \mid y \geq \max\{(\overline{x}/\gamma_1)(A_1 + x),\ (\beta_2/\overline{y})x - A_2\},\ x > 0\}\backslash\{E_+\}$,

(iii) $R_3 = \{(x, y) \mid \max\{0, (\beta_2/\overline{y})x - A_2\} < y < (\overline{x}/\gamma_1)(A_1 + x),\ 0 < x < \overline{x}\}$,

(iv) $R_4 = \{(x, y) \mid 0 < y \leq \min\{(\overline{x}/\gamma_1)(A_1 + x),\ (\beta_2/\overline{y})x - A_2\},\ x > A_2\overline{y}/\beta_2\}\backslash\{E_+\}$.

Lemma 9. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then*
(i) $F(R_1) \subseteq Q_1$, $F(R_3) \subseteq Q_3$;
(ii) $F(R_2) \subseteq Q_4$, $F(R_4) \subseteq Q_2$.

Proof. (i) Let $(x, y) \in R_1$. Then $(\overline{x}/\gamma_1)(A_1 + x) < y < (\beta_2/\overline{y})x - A_2$ and $x > \overline{x}$; thus

$$f(x, y) = \frac{\gamma_1 y}{A_1 + x} > \frac{\gamma_1 ((\overline{x}/\gamma_1)(A_1 + x))}{A_1 + x} = \overline{x},$$
$$g(x, y) = \frac{\beta_2 x}{A_2 + y} > \frac{\beta_2 x}{A_2 + ((\beta_2/\overline{y})x - A_2)} = \overline{y}, \quad (27)$$

from which it follows that $(f(x, y), g(x, y)) \in Q_1$.

The proof of $F(R_3) \subseteq Q_3$ is similar and is omitted.

(ii) Here we only prove that $F(R_2) \subseteq Q_4$; the proof for $F(R_4) \subseteq Q_2$ is the same and is omitted.

Set $(x, y) \in R_2$. Then yield $y \geq (\overline{x}/\gamma_1)(A_1 + x) > (\beta_2/\overline{y})x - A_2$ if $x \leq \overline{x}$, and $y \geq (\beta_2/\overline{y})x - A_2 > (\overline{x}/\gamma_1)(A_1 + x)$ if $x > \overline{x}$. Hence

$$f(x, y) = \frac{\gamma_1 y}{A_1 + x} \geq \frac{\gamma_1 ((\overline{x}/\gamma_1)(A_1 + x))}{A_1 + x} = \overline{x},$$
$$g(x, y) = \frac{\beta_2 x}{A_2 + y} \leq \frac{\beta_2 x}{A_2 + ((\beta_2/\overline{y})x - A_2)} = \overline{y}, \quad (28)$$

from which it follows that $F(x, y) = (f(x, y), g(x, y)) \in Q_4$.

The proof is complete. □

Lemma 10. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then*
(i) $\Delta_x < 0$, $\Delta_y > 0$ with $(x, y) \in Q_2$;
(ii) $\Delta_x > 0$, $\Delta_y < 0$ with $(x, y) \in Q_4$.

Proof. Using the equalities that $(A_1 + \overline{x})(A_2 + \overline{y}) = \beta_2\gamma_1$, $\beta_2\gamma_1(A_1 + \overline{x})/(A_2 + \overline{y})[A_1(A_1 + \overline{x}) + \gamma_1\overline{y}] = 1$, and $\beta_2\gamma_1(A_2 + \overline{y})/(A_1 + \overline{x})[A_2(A_2 + \overline{y}) + \beta_2\overline{x}] = 1$, Δ_x and Δ_y can be rewritten as follows:

$$\Delta_x = x\left(\frac{\beta_2\gamma_1(A_1 + x)}{(A_2 + y)[A_1(A_1 + x) + \gamma_1 y]} - \frac{\beta_2\gamma_1(A_1 + \overline{x})}{(A_2 + \overline{y})[A_1(A_1 + \overline{x}) + \gamma_1\overline{y}]}\right)$$
$$= \beta_2\gamma_1 x \frac{\overline{x}(A_2 + \overline{y})(x - \overline{x}) + [A_1(A_1 + x) + \gamma_1(A_2 + \overline{y}) + \gamma_1 y](\overline{y} - y)}{(A_2 + y)[A_1(A_1 + x) + \gamma_1 y]}, \quad (29)$$

and

$$\Delta_y = y\left(\frac{\beta_2\gamma_1(A_2 + y)}{(A_1 + x)[A_2(A_2 + y) + \beta_2 x]} - \frac{\beta_2\gamma_1(A_2 + \overline{y})}{(A_1 + \overline{x})[A_2(A_2 + \overline{y}) + \beta_2\overline{x}]}\right)$$
$$= \beta_2\gamma_1 y \frac{[A_2(A_2 + y) + \beta_2(A_1 + \overline{x}) + \beta_2 x](\overline{x} - x) + \overline{y}(A_1 + \overline{x})(y - \overline{y})}{(A_1 + x)[A_2(A_2 + y) + \beta_2 x]}. \quad (30)$$

(i) In the region Q_2, yield $x \le \overline{x}$ and $y \ge \overline{y}$; hence $\Delta_x < 0$, $\Delta_y > 0$.

(ii) In the region Q_4, yield $x \ge \overline{x}$ and $y \le \overline{y}$; hence $\Delta_x > 0$, $\Delta_y < 0$.

The proof is complete. □

Lemma 11. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then*

(i) $\Delta_x < 0$, $\Delta_y < 0$ with $(x, y) \in R_1$;
(ii) $\Delta_x > 0$, $\Delta_y > 0$ with $(x, y) \in R_3$.

Proof. Here, we only prove (i), the proof of (ii) is similar and is omitted.

(i) Let $(x, y) \in R_1$. Then we have

$$\frac{\overline{x}}{\gamma_1}(A_1 + x) < y < \frac{\beta_2}{\overline{y}}x - A_2 \tag{31}$$

and $x > \overline{x}$, $y > \overline{y}$. So

$$\begin{aligned}
\Delta_x &= x\frac{\beta_2\gamma_1(A_1 + x) - A_1(A_1 + x)(A_2 + y) - \gamma_1 y(A_2 + y)}{(A_2 + y)[A_1(A_1 + x) + \gamma_1 y]} \\
&< x\frac{\beta_2\gamma_1(A_1 + x) - A_1(A_1 + x)(A_2 + \overline{y}) - \gamma_1(\overline{x}/\gamma_1)(A_1 + x)(A_2 + \overline{y})}{(A_2 + y)[(A_1(A_1 + x) + \gamma_1 y]} \\
&= x(A_1 + x)\frac{\beta_2\gamma_1 - A_1(A_2 + \overline{y}) - \overline{x}(A_2 + \overline{y})}{(A_2 + y)[A_1(A_1 + x) + \gamma_1 y]} = 0.
\end{aligned} \tag{32}$$

From (31), we can get that $x > (\overline{y}/\beta_2)(A_2 + y)$, and so

$$\begin{aligned}
\Delta_y &= y\frac{\beta_2\gamma_1(A_2 + y) - A_2(A_1 + x)(A_2 + y) - \beta_2 x(A_1 + x)}{(A_1 + x)[A_2(A_2 + y) + \beta_2 x]} \\
&< y\frac{\beta_2\gamma_1(A_2 + y) - A_2(A_1 + \overline{x})(A_2 + y) - \beta_2(\overline{y}/\beta_2)(A_2 + y)(A_1 + \overline{x})}{(A_1 + x)[A_2(A_2 + y) + \beta_2 x]} \\
&= y(A_2 + y)\frac{\beta_2\gamma_1 - A_2(A_1 + \overline{x}) - \overline{y}(A_1 + \overline{x})}{(A_1 + x)[A_2(A_2 + y) + \beta_2 x]} = 0.
\end{aligned} \tag{33}$$

The proof is complete. □

Theorem 12. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then every solution with the initial value $(x_0, y_0) \in R_2$ has the following character:*

(i) *(x_{2n}, y_{2n}) eventually enters the region Q_2 and satisfies $(x_{2n}, y_{2n}) \longrightarrow (0, \infty)$;*

(ii) *(x_{2n+1}, y_{2n+1}) eventually enters the region Q_4 and satisfies $(x_{2n+1}, y_{2n+1}) \longrightarrow (\infty, 0)$.*

Proof. By Lemmas 9(ii) and 8, it easy to obtain that $(x_{2n}, y_{2n}) \in Q_2$ and $(x_{2n+1}, y_{2n+1}) \in Q_4$ for $n \ge 0$ with $(x_0, y_0) \in R_2$.

In view of Lemma 10 (i), we find that in the region Q_2 the sequence $\{x_{2n}\}$ is strictly decreasing and bounded below by 0, and the sequence $\{y_{2n}\}$ is strictly increasing. Notice that, in this case, system (2) has no prime period-two solution and the other equilibrium point; hence $(x_{2n}, y_{2n}) \longrightarrow (0, \infty)$.

From Lemma 10 (ii), we find that in the region Q_4 the sequence $\{x_{2n+1}\}$ is strictly increasing, the sequence $\{y_{2n+1}\}$ is strictly decreasing and bounded below by 0. Hence $y_{2n+1} \longrightarrow 0$, and $x_{2n+1} \longrightarrow \infty$, and the proof is complete. □

Similarly, we have the following result.

Theorem 13. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then every solution with the initial value $(x_0, y_0) \in R_4$ has the following character:*

(i) *(x_{2n}, y_{2n}) eventually enters the region Q_4 and satisfies $(x_{2n}, y_{2n}) \longrightarrow (\infty, 0)$;*

(ii) *(x_{2n+1}, y_{2n+1}) eventually enters the region Q_2 and satisfies $(x_{2n+1}, y_{2n+1}) \longrightarrow (0, \infty)$.*

Theorem 14. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then every solution with the initial value $(x_0, y_0) \in R_1$ has stability trichotomy; that is, exactly one of the following three cases holds:*

(i) *(x_{2n}, y_{2n}) eventually enters the region Q_2 satisfying $(x_{2n}, y_{2n}) \longrightarrow (0, \infty)$ and (x_{2n+1}, y_{2n+1}) eventually enters the region Q_4 satisfying $(x_{2n+1}, y_{2n+1}) \longrightarrow (\infty, 0)$;*

(ii) *(x_{2n}, y_{2n}) eventually enters the region Q_4 satisfying $(x_{2n}, y_{2n}) \longrightarrow (\infty, 0)$ and (x_{2n+1}, y_{2n+1}) eventually enters the region Q_2 satisfying $(x_{2n+1}, y_{2n+1}) \longrightarrow (0, \infty)$;*

(iii) *it remains in the region R_1 forever and satisfies $(x_n, y_n) \longrightarrow E_+$.*

Proof. Let $(x_0, y_0) \in R_1$. Then Lemma 8 implies that $(x_1, y_1) \in Q_1$. Hence $(x_1, y_1) \in Q_1 \cap R_2 \subset R_2$, $(x_1, y_1) \in Q_1 \cap R_4 \subset R_4$, or $(x_1, y_1) \in R_1$.

Case (a). If $(x_1, y_1) \in R_2$, then Theorem 12 implies that $(x_{2n+1}, y_{2n+1}) \in Q_2$, $(x_{2n}, y_{2n}) \in Q_4$ for $n \geq 1$. Further, $(x_{2n+1}, y_{2n+1}) \longrightarrow (0, \infty)$, $(x_{2n}, y_{2n}) \longrightarrow (\infty, 0)$.

Case (b). If $(x_1, y_1) \in R_4$, then Theorem 13 implies that $(x_{2n+1}, y_{2n+1}) \in Q_4$, $(x_{2n}, y_{2n}) \in Q_2$ for $n \geq 1$. Further, $(x_{2n+1}, y_{2n+1}) \longrightarrow (\infty, 0)$, $(x_{2n}, y_{2n}) \longrightarrow (0, \infty)$.

Case (c). If $(x_1, y_1) \in R_1$ and there exists an integer N_1 such that $(x_{N_1}, y_{N_1}) \in R_2$, then by *Case (a)*, we have $(x_{N_1+2n}, y_{N_1+2n}) \in Q_2$, $(x_{N_1+2n+1}, y_{N_1+2n+1}) \in Q_4$ for $n \geq 0$, and $(x_{N_1+2n}, y_{N_1+2n}) \longrightarrow (0, \infty)$, $(x_{N_1+2n+1}, y_{N_1+2n+1}) \longrightarrow (\infty, 0)$.

If $(x_1, y_1) \in R_1$ and there exists an integer N_2 such that $(x_{N_2}, y_{N_2}) \in R_4$, then by *Case (b)*, we have $(x_{N_2+2n}, y_{N_2+2n}) \in Q_4$, $(x_{N_2+2n+1}, y_{N_2+2n+1}) \in Q_2$ for $n \geq 0$. Furthermore, $(x_{N_2+2n}, y_{N_2+2n}) \longrightarrow (\infty, 0)$, $(x_{N_2+2n+1}, y_{N_2+2n+1)} \longrightarrow (0, \infty)$.

If $(x_1, y_1) \in R_1$, and $(x_n, y_n) \in R_1$ for all $n \geq 1$, then by Lemma 11 (i), we have that the sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$ and $\{y_{2n}\}$, $\{y_{2n+1}\}$ are all strictly decreasing, and bounded below by $\overline{x}$, and $\overline{y}$, respectively. Hence they are all convergent; moreover, $(x_n, y_n) \longrightarrow E_+$.

The proof is complete. □

Similarly, we can show that the orbits starting in R_3 have the following character.

Theorem 15. *Assume that $\beta_2\gamma_1 > A_1A_2$. Then every solution with the initial value $(x_0, y_0) \in R_3$ has stability trichotomy; that is, exactly one of the following three cases holds:*

(i) *(x_{2n}, y_{2n}) eventually enters the region Q_2 satisfying $(x_{2n}, y_{2n}) \longrightarrow (0, \infty)$ and (x_{2n+1}, y_{2n+1}) eventually enters the region Q_4 satisfying $(x_{2n+1}, y_{2n+1}) \longrightarrow (\infty, 0)$;*

(ii) *(x_{2n}, y_{2n}) eventually enters the region Q_4 satisfying $(x_{2n}, y_{2n}) \longrightarrow (\infty, 0)$ and (x_{2n+1}, y_{2n+1}) eventually enters the region Q_2 satisfying $(x_{2n+1}, y_{2n+1}) \longrightarrow (0, \infty)$;*

(iii) *it remains in the region R_3 forever and satisfies $(x_n, y_n) \longrightarrow E_+$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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References

[1] Z. Balanov, C. Garca-Azpeitia, and W. Krawcewicz, "On variational and topological methods in nonlinear difference equations," *Communications on Pure and Applied Analysis*, vol. 17, no. 6, pp. 2813–2844, 2018.

[2] J. M. Cushing, S. Levarge, N. Chitnis, and S. M. Henson, "Some discrete competition models and the competitive exclusion principle," *Journal of Difference Equations and Applications*, vol. 10, no. 13–15, pp. 1139–1151, 2004.

[3] D. Clark and M. R. S. Kulenović, "A coupled system of rational difference equations," *Computers & Mathematics with Applications*, vol. 43, no. 6-7, pp. 849–867, 2002.

[4] Z. M. Guo and J. S. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China Series A: Mathematics*, vol. 46, no. 4, pp. 506–515, 2003.

[5] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society*, vol. 68, no. 02, pp. 419–430, 2003.

[6] M. W. Hirsch and H. Smith, "Monotone maps: a review," *Journal of Difference Equations and Applications*, vol. 11, no. 4-5, pp. 379–398, 2005.

[7] L.-X. Hu, W.-T. Li, and S. Stević, "Global asymptotic stability of a second order rational difference equation," *Journal of Difference Equations and Applications*, vol. 14, no. 8, pp. 779–797, 2008.

[8] L.-X. Hu, W.-S. He, and H.-M. Xia, "Global asymptotic behavior of a rational difference equation," *Applied Mathematics and Computation*, vol. 218, no. 15, pp. 7818–7828, 2012.

[9] L. Erbe, B. Jia, and Q. Zhang, "Homoclinic solutions of discrete nonlinear systems via variational method," *Journal of Applied Analysis and Computation*, vol. 9, no. 1, pp. 271–294, 2019.

[10] H. L. Smith, "Planar competitive and cooperative difference equations," *Journal of Difference Equations and Applications*, vol. 3, no. 5-6, pp. 335–357, 1998.

[11] Q. Zhang, "Homoclinic orbits for discrete Hamiltonian systems with local super-quadratic conditions," *Communications on Pure and Applied Analysis*, vol. 18, no. 1, pp. 425–434, 2019.

[12] E. Camouzis, M. R. S. Kulenović, G. Ladas, and O. Merino, "Open problem and Conjetures," *Journal of Difference Equations and Applications*, vol. 15, no. 3, pp. 303–323, 2009.

[13] A. S. Clark and J. Weiss, "On the geometry of a four-parameter rational planar system of difference equations," *Journal of Difference Equations and Applications*, vol. 18, no. 3, pp. 509–524, 2012.

[14] Lin-Xia Hu and Xiu-Mei Jia, "Global Asymptotic Stability of a Rational System," *Abstract and Applied Analysis*, vol. 2014, Article ID 286375, 6 pages, 2014.

[15] M. R. S. Kulenović and O. Merino, "Competitive-exclusion versus competitive-coexistence for systems in the plane," *Discrete and Continuous Dynamical Systems - Series B (DCDS-B)*, vol. 6, no. 5, pp. 1141–1156, 2006.

[16] M. R. S. Kulenović and O. Merino, "Global bifurcation for discrete competitive systems in the plane," *Discrete and Continuous Dynamical Systems - Series B*, vol. 12, no. 1, pp. 133–149, 2009.

[17] M. R. Kulenović and M. Nurkanović, "Global behavior of a two-dimensional competitive system of difference equations with stocking," *Mathematical and Computer Modelling*, vol. 55, no. 7-8, pp. 1998–2011, 2012.

[18] S. Kalabušić and M. R. Kulenović, “Dynamics of certain anti-competitive systems of rational difference equations in the plane,” *Journal of Difference Equations and Applications*, vol. 17, no. 11, pp. 1599–1615, 2011.

[19] S. Elydi, *An Introduction to Difference Equations*, Springer Science + Business Media, Inc, 3rd edition, 2005.

[20] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Application*, Kluwer Academic, Dordrecht, The Netherlands, 1993.

[21] M. R. S. Kulenovic and G. Ladas, *Dynamics of Second Order Rational Difference Equations*, Chapman & Hall, CRC, 2002.