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Research Article

A New Global Stable Conclusion on a Diffusive Leslie-Gower Predator-Prey System with Additive Allee Effect

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In this work, a diffusive Leslie-Gower predator-prey model with additive Allee effect on prey under a homogeneous Neumann boundary condition is reconsidered. We establish new sufficient conditions for the global stability of the unique positive equilibrium point of the system by using the comparison method rather than the Lyapunov function method. It is shown that our result supplements and complements one of the main results of Yang and Zhong, 2015. Furthermore, numerical simulations are performed to consolidate the analytic finding.

1. Introduction

Taking into account the inhomogeneous distribution of the predators and their preys in different spatial locations, the authors [1] established the following diffusive Leslie-Gower predator-prey model with additive Allee effect:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(r_1 - a_1 u) - \dfrac{mu}{u+b} - c_1 uv, & t > 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + v\left[r_2 - \dfrac{c_2 v}{u+k}\right], & t > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & t > 0,\ x \in \partial\Omega, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ denote the densities of prey and predator at time t and position x, respectively. m and b are constants that indicate the severity of the Allee effect that has been modeled. In this model, the Allee effect is induced by predation. In such a case, the predator rate consumption is conveniently modeled by a monotonic function $p(u) = \alpha u$, corresponding to a Holling I-type functional response. Δ is the Laplacian operator, $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, and n is the outward unit normal vector of the boundary $\partial\Omega$. d_1 and d_2 are the diffusion coefficients of prey and predator, and initial data $u_0(x), v_0(x)$ are non-negative continuous functions due to its biological sense.

For the global stability of the diffusive system (1), the authors established the sufficient conditions with the Lyapunov function method in [1], which was used in most studies [1–4]. In this paper, we will obtain a new global stability conclusion by the comparison method, which was used in [5, 6].

2. Globally Asymptotical Stability

Obviously, if $r_1 - (m/b) > (kc_1 r_2/c_2)$, then system (1) has a unique positive equilibrium point (u^*, v^*) (coexistence of prey and predator). Firstly, we recall the following results from [1].

Theorem 1. *If $r_1 - (m/b) > (c_1 r_2 (r_1 + a_1 k)/a_1 c_2)$, then system (1) is permanent.*

Theorem 2. *The positive constant steady state (u^*, v^*) of (1) is globally asymptotically stable if*

$$(H_1)\colon r_1 - \frac{m}{b} > \frac{c_1 r_2 (r_1 + a_1 k)}{a_1 c_2},$$
$$(H_2)\colon a_1 - \frac{m}{b(u^* + b)} - \frac{c_2 v^*}{2k(u^* + k)} - \frac{c_1}{2} > 0, \tag{2}$$
$$(H_3)\colon \frac{c_2}{c_1 + k} - \frac{c_1}{2} - \frac{c_2 v^*}{2k(u^* + k)} > 0.$$

In [1], Theorem 2 was proved by using a Lyapunov function. In this section, we will prove the global stability under some new conditions. Thus, our conclusion significantly improves and supplements the one given in [1]. Our proof is based on the upper and lower solution method in [5, 6]. Now, we give the result of global stability of the unique positive equilibrium position (u^*, v^*) of (1). Biologically, however, quickly or slowly the two species diffuse, they will be spatially homogeneously distributed as time converges to infinity.

Theorem 3. *Suppose that all parameters are positive constants and*

$$\text{(I)}\ \frac{a_1}{c_1} > \frac{r_2}{c_2},$$
$$\text{(II)}\ r_1 - \frac{m}{b} > \max \frac{c_1 r_2}{c_2} \left\{ \left(k + \frac{r_1}{a_1} \right), k - b + \frac{m c_2}{b(a_1 c_2 - c_1 r_2)} \right\}. \tag{3}$$

Then, the positive equilibrium point (u^*, v^*) of system (1) is globally asymptotically stable; that is, for any nonnegative initial value $(u_0(x), v_0(x))$, the solution of system (1) has the property that

$$\lim_{t \longrightarrow \infty} u(x,t) = u^*, \qquad \lim_{t \longrightarrow \infty} v(x,t) = v^* \tag{4}$$

uniformly for $x \in \overline{\Omega}$.

Proof. From the proof of Theorem 1 (i.e., Theorem 2 in [1]), if condition (II) holds, then system (1) is permanent and has a unique positive equilibrium position and there exist positive constants $\overline{c}_1, \underline{c}_1, \overline{c}_2, \underline{c}_2$ so that

$$\underline{c}_1 \le u(x,t) \le \overline{c}_1, \qquad \underline{c}_2 \le v(x,t) \le \overline{c}_2, \tag{5}$$

for t sufficiently large, and $\underline{c}_1, \underline{c}_2, \overline{c}_1, \overline{c}_2$ satisfy

$$\begin{gathered}
(r_1 - a_1 \overline{c}_1) - \frac{m}{\overline{c}_1 + b} - c_1 \underline{c}_2 \le 0, \\
r_2 - \frac{c_2 \overline{c}_2}{\overline{c}_1 + k} \le 0, \\
(r_1 - a_1 \underline{c}_1) - \frac{m}{\underline{c}_1 + b} - c_1 \overline{c}_2 \ge 0, \\
r_2 - \frac{h \underline{c}_2}{\underline{c}_1 + k} \ge 0.
\end{gathered} \tag{6}$$

The inequalities (6) show that $(\overline{c}_1, \overline{c}_2)$ and $(\underline{c}_1, \underline{c}_2)$ are a pair of coupled upper and lower solutions of system (1) as in the definition [7, 8], as the nonlinearities in (1) are mixed quasimonotone. It is clear that there exists $K > 0$ such that

$$\begin{gathered}
\left| u_1 (r_1 - a_1 u_1) - \frac{m u_1}{u_1 + b} - c_1 u_1 v_1 - u_2 (r_1 - a_1 u_2) + \frac{m u_2}{u_2 + b} + c_2 u_2 v_2 \right| \le K\left(|u_1 - u_2| + |v_1 - v_2| \right), \\
\left| v_1 \left(r_2 - \frac{c_2 v_1}{u_1 + k} \right) - v_2 \left(r_2 - \frac{c_2 v_2}{u_2 + k} \right) \right| \le K\left(|u_1 - u_2| + |v_1 - v_2| \right).
\end{gathered} \tag{7}$$

We define two iteration sequences $(\overline{c}_1^{(n)}, \overline{c}_2^{(n)})$ and $(\underline{c}_1^{(n)}, \underline{c}_2^{(n)})$ as follows: for $n \ge 1$

$$\begin{gathered}
\overline{c}_1^{(n)} = \overline{c}_1^{(n-1)} + \frac{1}{K} \overline{c}_1^{(n-1)} \left[\left(r_1 - a_1 \overline{c}_1^{(n-1)} \right) - \frac{m}{\overline{c}_1^{(n-1)} + b} - c_1 \underline{c}_2^{(n-1)} \right], \\
\overline{c}_2^{(n)} = \overline{c}_2^{(n-1)} + \frac{1}{K} \overline{c}_2^{(n-1)} \left(r_2 - \frac{c_2 \overline{c}_2^{(n-1)}}{\overline{c}_1^{(n-1)} + k} \right), \\
\underline{c}_1^{(n)} = \underline{c}_1^{(n-1)} + \frac{1}{K} \underline{c}_1^{(n-1)} \left[\left(r_1 - \underline{c}_1^{(n-1)} \right) - \frac{m}{\underline{c}_1^{(n-1)} + b} - c_1 \overline{c}_2^{(n-1)} \right], \\
\underline{c}_2^{(n)} = \underline{c}_2^{(n-1)} + \frac{1}{K} \underline{c}_2^{(n-1)} \left(r_2 - \frac{c_2 \underline{c}_2^{(n-1)}}{\underline{c}_1^{(n-1)} + k} \right),
\end{gathered} \tag{8}$$

and we denote $(\underline{c}_1^{(0)}, \underline{c}_2^{(0)}) = (\underline{c}_1, \underline{c}_2)$ and $(\overline{c}_1^{(0)}, \overline{c}_2^{(0)}) = (\overline{c}_1, \overline{c}_2)$. Then, for $n \ge 1$, we can get that

$$\begin{aligned}
(\underline{c}_1, \underline{c}_2) \le \left(\underline{c}_1^{(n)}, \underline{c}_2^{(n)} \right) \le \left(\underline{c}_1^{(n+1)}, \underline{c}_2^{(n+1)} \right) \le \left(\overline{c}_1^{(n+1)}, \overline{c}_2^{(n+1)} \right) \\
\le \left(\overline{c}_1^{(n)}, \overline{c}_2^{(n)} \right) \le (\overline{c}_1, \overline{c}_2),
\end{aligned} \tag{9}$$

and there exist $(\tilde{c}_1, \tilde{c}_2)$ and $(\check{c}_1, \check{c}_2)$ such that

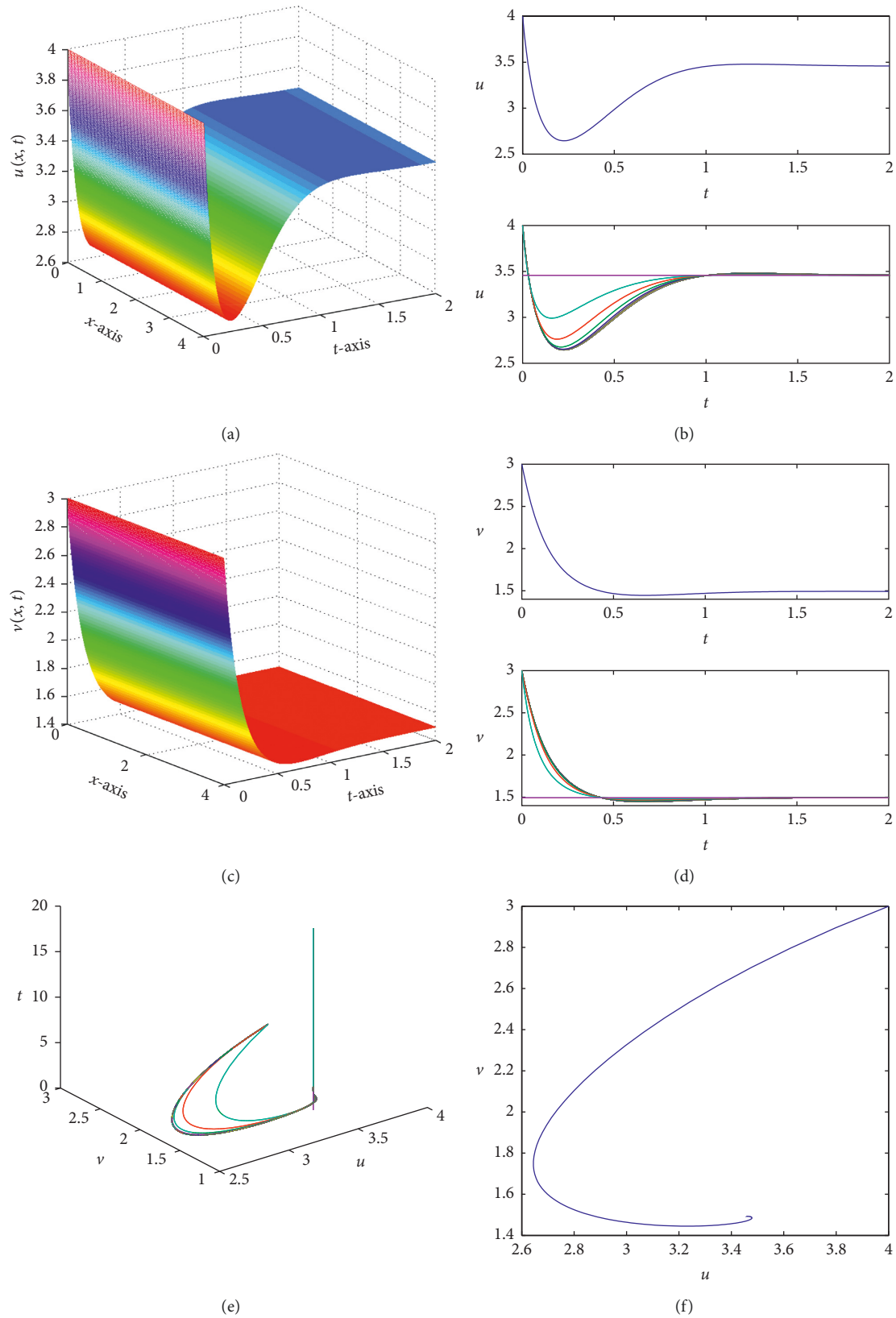

Figure 1: Global stability of the positive equilibrium point $(3.4581, 1.4916)$ of system (1) with the initial value $(u(0,x), v(0,x)) = (4,3)$. (a) The numerical solution surface of $u(x,t)$, (b) the numerical solution curve of $u(t,2)$ and $u(t)$, (c) the numerical solution surface of $v(t,x)$, (d) the numerical solution curve of $v(t,2)$ and $v(t)$, (e) the numerical solution surface of $u(t)$ and $v(t)$, and (f) the numerical solution curve of $u(t,2)$ and $v(t,2)$.

$$\begin{aligned}\lim_{t\longrightarrow\infty}\overline{c}_1^{(n)}&=\widetilde{c}_1,\\ \lim_{t\longrightarrow\infty}\underline{c}_1^{(n)}&=\check{c}_1,\\ \lim_{t\longrightarrow\infty}\overline{c}_2^{(n)}&=\widetilde{c}_2,\\ \lim_{t\longrightarrow\infty}\underline{c}_2^{(n)}&=\check{c}_2,\end{aligned}\tag{10}$$

and then

$$(\underline{c}_1,\underline{c}_2)\le(\check{c}_1,\check{c}_2)\le(\widetilde{c}_1,\widetilde{c}_2)\le(\overline{c}_1,\overline{c}_2).\tag{11}$$

Therefore, we can obtain that

$$\begin{aligned}(r_1-a_1\widetilde{c}_1)-\frac{m}{\widetilde{c}_1+b}-c_1\check{c}_2&=0,\\ r_2-\frac{c_2\widetilde{c}_2}{\widetilde{c}_1+k}&=0,\\ (r_1-a_1\check{c}_1)-\frac{m}{\check{c}_1+b}-c_1\widetilde{c}_2&=0,\\ r_2-\frac{c_2\check{c}_2}{\check{c}_1+k}&=0.\end{aligned}\tag{12}$$

Simplifying (12), we get

$$\begin{aligned}(r_1-a_1\widetilde{c}_1)-\frac{m}{\widetilde{c}_1+b}&=\frac{c_1r_2}{c_2}(\check{c}_1+k),\\ (r_1-a_1\check{c}_1)-\frac{m}{\check{c}_1+b}&=\frac{c_1r_2}{c_2}(\widetilde{c}_1+k).\end{aligned}\tag{13}$$

Subtracting the first equation of (13) from the second equation of (13), we have

$$(\widetilde{c}_1-\check{c}_1)\left[\frac{c_1r_2}{c_2}+\frac{m}{(\widetilde{c}_1+b)(\check{c}_1+b)}-a_1\right]=0.\tag{14}$$

If we assume that $\widetilde{c}_1\neq\check{c}_1$, then

$$(\widetilde{c}_1+b)(\check{c}_1+b)=\frac{mc_2}{a_1c_2-r_2c_1}\triangleq p>0.\tag{15}$$

Substituting (15) into (13), we have

$$\begin{aligned}(r_1-a_1\widetilde{c}_1)-\frac{m}{\widetilde{c}_1+b}&=\frac{c_1r_2}{c_2}\left(\frac{p}{\widetilde{c}_1+b}-b+k\right),\\ (r_1-a_1\check{c}_1)-\frac{m}{\check{c}_1+b}&=\frac{c_1r_2}{c_2}\left(\frac{p}{\check{c}_1+b}-b+k\right).\end{aligned}\tag{16}$$

Hence, we get the following equation:

$$(r_1-a_1x)-\frac{m}{x+b}=\frac{c_1r_2}{c_2}\left[\frac{p}{x+b}-b+k\right],\tag{17}$$

which has two positive roots $\widetilde{c}_1,\check{c}_1$. Equation (17) can be rewritten as follows:

$$\begin{aligned}a_1x^2&-\left[r_1-a_1b-\frac{c_1r_2}{c_2}(k-b)\right]x\\ &-\left[r_1b-m-\frac{c_1r_2}{c_2}\left(p-b^2+bk\right)\right]=0.\end{aligned}\tag{18}$$

Since $r_1-(m/b)>(c_1r_2/c_2)((p/b)-b+k)$, we can easily get that (18) cannot have two positive roots. Hence, $\widetilde{c}_1=\check{c}_1$, and consequently, $\widetilde{c}_2=\check{c}_2$. And then by the results in [7, 8], the solution $(u(x,t),v(x,t))$ of system (1) satisfies $\lim_{t\longrightarrow\infty}u(x,t)=u^*$, $\lim_{t\longrightarrow\infty}v(x,t)=v^*$ uniformly for $x\in\overline{\Omega}$. Then, the constant equilibrium (u^*,v^*) is globally asymptotic stable for system (1). Thus, the whole proof is completed. □

Remark 1. Obviously, the parameter region in Theorem 3 is not contained in the set given by Theorem 2. That is, if the conditions of Theorem 2 hold, then the ones of Theorem 3 may not hold. Then, our global stable conclusion complements the one in [1]. Moreover, it is inconvenient and unnecessary to utilize the positive equilibrium point (u^*,v^*) to conclude the global stability. In addition, we depend only on the parameter value in Theorem 3 to come to conclusion. Therefore, it is more reasonable.

3. Numerical Simulations

In this section, we give the numerical simulation to consolidate our theoretical finding.

Example 1. In system (1), suppose $r_1=8$, $a_1=1$, $m=0.5$, $b=4$, $c_1=3$, $r_2=4$, $c_2=20$, $k=4$, $d_1=0.05$, $d_2=0.05$ and initial value $(u(0,x),v(0,x))=(4,3)$ for all $x\in\Omega=[0,4]$. Obviously, $r_1-(m/b)-(c_1r_2/c_2)=5.475>0$. Then, system (1) exists a unique positive equilibrium point $(u^*,v^*)=(3.4581,1.4916)$. Straightforward calculation shows that $(c_2/r_2)-(c_1/a_1)=2>0$,

$$\begin{aligned}r_1-\frac{m}{b}-\max\frac{c_1r_2}{c_2}&\left\{\left(k+\frac{r_1}{a_1}\right),k-b+\frac{mc_2}{b(a_1c_2-c_1r_2)}\right\}\\ &=2.7>0.\end{aligned}\tag{19}$$

Then, all conditions of Theorem 3 hold. Hence, by Theorem 3, we know that the positive constant equilibrium state (u^*,v^*) of system (1) is globally asymptotically stable. Figure 1 shows the dynamics behavior of system (1).

Remark 2. As $a_1-(c_1/2)=1-(3/2)<0$, then the condition H_2 of Theorem 2 is not satisfied, so we cannot judge the global stability of system (1) with the above parameters by using Theorem 2 (i.e., Theorem 2 in [1]).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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