Research Article

A New Global Stable Conclusion on a Diffusive Leslie-Gower Predator-Prey System with Additive Allee Effect

Liu Yang 1,2

1College of Mathematics and Physics, Hunan Province Cooperative Innovation Center for the Construction and Development of Dongting Lake Ecological Economic Zone, Hunan University of Arts and Science, Changde 415000, China
2College of Liberal Arts and Science, National University of Defense Technology, Changsha 410073, China

Correspondence should be addressed to Liu Yang; ylazx@126.com

Received 15 September 2019; Accepted 3 December 2019; Published 22 December 2019

Academic Editor: Yukihiko Nakata

Copyright © 2019 Liu Yang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this work, a diffusive Leslie-Gower predator-prey model with additive Allee effect on prey under a homogeneous Neumann boundary condition is reconsidered. We establish new sufficient conditions for the global stability of the unique positive equilibrium point of the system by using the comparison method rather than the Lyapunov function method. It is shown that our result supplements and complements one of the main results of Yang and Zhong, 2015. Furthermore, numerical simulations are performed to consolidate the analytic finding.

1. Introduction

Taking into account the inhomogeneous distribution of the predators and their preys in different spatial locations, the authors [1] established the following diffusive Leslie-Gower predator-prey model with additive Allee effect:

\[
\begin{align*}
\frac{\partial u}{\partial t} & = d_1 \Delta u + u (r_1 - a_1 u) - \frac{m u}{u + b} - c_1 u v, \quad t > 0, \ x \in \Omega, \\
\frac{\partial v}{\partial t} & = d_2 \Delta v + v \left[ r_2 - \frac{c_2 v}{u + k} \right], \quad t > 0, \ x \in \Omega, \\
\frac{\partial u}{\partial n} & = \frac{\partial v}{\partial n} = 0, \quad t > 0, \ x \in \partial \Omega, \\
u(x, 0) & = u_0(x), \ v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(1)

where \( u(x, t) \) and \( v(x, t) \) denote the densities of prey and predator at time \( t \) and position \( x \), respectively. \( m \) and \( b \) are constants that indicate the severity of the Allee effect that has been modeled. In this model, the Allee effect is induced by predation. In such a case, the predator rate consumption is conveniently modeled by a monotonic function \( p(u) = au \), corresponding to a Holling I-type functional response. \( \Delta \) is the Laplacian operator, \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary \( \partial \Omega \), and \( n \) is the outward unit normal vector of the boundary \( \partial \Omega \). \( d_1 \) and \( d_2 \) are the diffusion coefficients of prey and predator, and initial data \( u_0(x), v_0(x) \) are non-negative continuous functions due to its biological sense.

For the global stability of the diffusive system (1), the authors established the sufficient conditions with the Lyapunov function method in [1], which was used in most studies [1–4]. In this paper, we will obtain a new global stability conclusion by the comparison method, which was used in [5, 6].

2. Globally Asymptotical Stability

Obviously, if \( r_1 - (m/b) > (kc_1 r_2/c_2) \), then system (1) has a unique positive equilibrium point \((u^*, v^*)\) (coexistence of prey and predator). Firstly, we recall the following results from [1].

**Theorem 1.** If \( r_1 - (m/b) > (c_1 r_2 (r_1 + a_1 k)/a_1 c_2) \), then system (1) is permanent.
Theorem 2. The positive constant steady state \((u^*, v^*)\) of (1) is globally asymptotically stable if

\[
(H_1): r_1 - \frac{m}{b} > \frac{c_1 r_2 (r_1 + a_1 k)}{a_1 c_2},
\]

\[
(H_2): a_1 - \frac{m}{b(b + u^* + b)} - \frac{c_2 v^*}{2k(u^* + k)} - \frac{c_1}{2} > 0,
\]

\[
(H_3): \frac{c_2}{c_1 + k} - \frac{c_1}{2} - \frac{c_2 v^*}{2k(u^* + k)} > 0.
\]

The inequalities (6) show that \((\xi_1, \xi_2)\) and \((\xi_1, \xi_2)\) are a pair of coupled upper and lower solutions of system (1) as in the definition [7, 8], as the nonlinearities in (1) are mixed quasimonotone. It is clear that there exists \(K > 0\) such that

\[
\left| u_1 (r_1 - a_1 u_1) - \frac{m u_1}{u_1 + b} - c_1 u_1 v_1 - u_2 (r_1 - a_1 u_2)
\right.
\]

\[
\left. + \frac{m u_2}{u_2 + b} + c_2 u_2 v_2 \right| \leq K \left( |u_1 - u_2| + |v_1 - v_2| \right).
\]

We define two iteration sequences \((\xi_1^{(n)}, \xi_2^{(n)})\) and \((\xi_1^{(n)}, \xi_2^{(n)})\) as follows: for \(n \geq 1\)

\[
\xi_1^{(n)} = \xi_1^{(n+1)} + \frac{1}{K \xi_1^{(n+1)}} \left( r_1 - a_1 \xi_1^{(n+1)} - \frac{m}{\xi_1^{(n+1)} + b} - c_1 \xi_2^{(n+1)} \right),
\]

\[
\xi_2^{(n)} = \xi_2^{(n+1)} + \frac{1}{K \xi_2^{(n+1)}} \left( r_2 - \frac{c_2 \xi_2^{(n+1)}}{\xi_1^{(n+1)} + k} \right),
\]

\[
\xi_1^{(n)} = \xi_1^{(n+1)} + \frac{1}{K \xi_1^{(n+1)}} \left( r_1 - \xi_1^{(n+1)} - \frac{m}{\xi_1^{(n+1)} + b} - c_1 \xi_2^{(n+1)} \right),
\]

\[
\xi_2^{(n)} = \xi_2^{(n+1)} + \frac{1}{K \xi_2^{(n+1)}} \left( r_2 - \frac{c_2 \xi_2^{(n+1)}}{\xi_1^{(n+1)} + k} \right).
\]

Proof. From the proof of Theorem 1 (i.e., Theorem 2 in [1]), if condition (II) holds, then system (1) is permanent and has a unique positive equilibrium position and there exist positive constants \(\xi_1, \xi_1, \xi_2, \xi_2\) so that

\[
\xi_1 \leq u(x, t) \leq \xi_1,
\]

\[
\xi_2 \leq v(x, t) \leq \xi_2,
\]

for \(t\) sufficiently large, and \(\xi_1, \xi_2, \xi_1, \xi_2\) satisfy

\[
\frac{r_1 - a_1 \xi_1}{\xi_1 + b} - c_1 \xi_2 \leq 0,
\]

\[
r_2 - \frac{c_2 \xi_2}{\xi_1 + k} \leq 0,
\]

\[
\frac{r_1 - a_1 \xi_1}{\xi_1 + b} - c_1 \xi_2 \geq 0,
\]

\[
r_2 - \frac{c_2 \xi_2}{\xi_1 + k} \geq 0.
\]
Figure 1: Global stability of the positive equilibrium point \((3.4581, 1.4916)\) of system (1) with the initial value \((u(0, x), v(0, x)) = (4, 3)\). (a) The numerical solution surface of \(u(x, t)\), (b) the numerical solution curve of \(u(t, 2)\) and \(u(t)\), (c) the numerical solution surface of \(v(t, x)\), (d) the numerical solution curve of \(v(t, 2)\) and \(v(t)\), (e) the numerical solution surface of \(u(t)\) and \(v(t)\), and (f) the numerical solution curve of \(u(t, 2)\) and \(v(t, 2)\).
\[ \begin{align*}
\lim_{t \to \infty} z_1^{(n)} &= \bar{c}_1, \\
\lim_{t \to \infty} z_2^{(n)} &= \bar{c}_2,
\end{align*} \]

and then
\[ \left( \bar{c}_1, \bar{c}_2 \right) \leq \left( \bar{c}_1, \bar{c}_2 \right) \leq \left( \bar{c}_1, \bar{c}_2 \right) \leq \left( \bar{c}_1, \bar{c}_2 \right). \]

Therefore, we can obtain that
\[ \begin{align*}
(r_1 - a_1 \bar{c}_1) - \frac{m}{\bar{c}_1 + b} - c_1 \bar{c}_2 &= 0, \\
r_2 - c_2 \bar{c}_2 &= 0, \\
(r_1 - a_1 \bar{c}_1) - \frac{m}{\bar{c}_1 + b} - c_1 \bar{c}_2 &= 0, \\
r_2 - c_2 \bar{c}_2 &= 0.
\end{align*} \]

Simplifying (12), we get
\[ \begin{align*}
(r_1 - a_1 \bar{c}_1) - \frac{m}{\bar{c}_1 + b} &= \frac{c_1 r_2}{c_2} (\bar{c}_1 + k), \\
(r_1 - a_1 \bar{c}_1) - \frac{m}{\bar{c}_1 + b} &= \frac{c_1 r_2}{c_2} (\bar{c}_1 + k).
\end{align*} \]

Subtracting the first equation of (13) from the second equation of (13), we have
\[ \left( \bar{c}_1 - \bar{c}_2 \right) \left[ \frac{c_1 r_2}{c_2} \left( \frac{m}{\bar{c}_1 + b} - a_1 \right) \right] = 0. \]

If we assume that \( \bar{c}_1 \neq \bar{c}_2 \), then
\[ \left( \bar{c}_1 + b \right) \left( \bar{c}_1 + b \right) = \frac{mc_2}{a_1 c_2 - r_2 c_1} \Leftrightarrow p > 0. \]

Substituting (15) into (13), we have
\[ \begin{align*}
(r_1 - a_1 \bar{c}_1) - \frac{m}{\bar{c}_1 + b} &= \frac{c_1 r_2}{c_2} \left( \frac{p}{\bar{c}_1 + b} - b + k \right), \\
(r_1 - a_1 \bar{c}_1) - \frac{m}{\bar{c}_1 + b} &= \frac{c_1 r_2}{c_2} \left( \frac{p}{\bar{c}_1 + b} - b + k \right).
\end{align*} \]

Hence, we get the following equation:
\[ \begin{align*}
(r_1 - a_1 x) - \frac{m}{x + b} &= \frac{c_1 r_2}{c_2} \left[ \frac{p}{x + b} - b + k \right],
\end{align*} \]

which has two positive roots \( \bar{c}_1, \bar{c}_1 \). Equation (17) can be rewritten as follows:
\[ \begin{align*}
a_1 x^2 - \left[ r_1 - a_1 b - \frac{c_1 r_2}{c_2} (k - b) \right] x \\
- \left[ r_1 b - m - \frac{c_1 r_2}{c_2} (p - b^2 + bk) \right] = 0.
\end{align*} \]

Since \( r_1 - (m/b) > (c_1 r_2/c_2) ((p/b) - b + k) \), we can easily get that (18) cannot have two positive roots. Hence, \( \bar{c}_1 = \bar{c}_1 \), and consequently, \( \bar{c}_2 = \bar{c}_2 \). And then by the results in [7, 8], the solution \( (u(x,t), v(x,t)) \) of system (1) satisfies
\[ \begin{align*}
\lim_{t \to \infty} u(x,t) = u^*, \quad \lim_{t \to \infty} v(x,t) = v^* \quad \text{uniformly for} \quad x \in \Omega.
\end{align*} \]

Then, the constant equilibrium \( (u^*, v^*) \) is globally asymptotic stable for system (1). Thus, the whole proof is completed.

**Remark 1.** Obviously, the parameter region in Theorem 3 is not contained in the set given by Theorem 2. That is, if the conditions of Theorem 2 hold, then the ones of Theorem 3 may not hold. Then, our global stable conclusion complements the one in [1]. Moreover, it is inconvenient and unnecessary to utilize the positive equilibrium point \( (u^*, v^*) \) to conclude the global stability. In addition, we depend only on the parameter value in Theorem 3 to come to conclusion. Therefore, it is more reasonable.

### 3. Numerical Simulations

In this section, we give the numerical simulation to consolidate our theoretical finding.

**Example 1.** In system (1), suppose \( r_1 = 8, a_1 = 1, m = 0.5, b = 4, c_1 = 3, r_2 = 4, c_2 = 20, k = 4, d_1 = 0.05, \quad d_2 = 0.05 \) and initial value \( (u(0,x), v(0,x)) = (4, 3) \) for all \( x \in \Omega = [0, 4] \). Obviously, \( r_1 - (m/b) - (c_1 r_2/c_2) = 5.475 > 0 \). Then, system (1) exists a unique positive equilibrium point \( (u^*, v^*) = (3.4581, 1.4916) \). Straightforward calculation shows that \( (c_2 r_2) - (c_1/a_1) = 2 > 0 \),
\[ r_1 - \frac{m}{b} - \max_{u_1 + b} \left( \frac{c_1 r_2}{c_2} \left( \frac{r_1}{a_1} \right), k-b + \frac{mc_2}{b(a_2 c_2 - c_1 r_2)} \right) \]
\[ = 2.7 > 0. \]

Then, all conditions of Theorem 3 hold. Hence, by Theorem 3, we know that the positive constant equilibrium state \( (u^*, v^*) \) of system (1) is globally asymptotic stable. Figure 1 shows the dynamics behavior of system (1).

**Remark 2.** As \( a_1 - (c_1/2) = 1 - (3/2) < 0 \), then the condition \( H_2 \) of Theorem 2 is not satisfied, so we cannot judge the global stability of system (1) with the above parameters by using Theorem 2 (i.e., Theorem 2 in [1]).

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

The author declares that there are no conflicts of interest.
Acknowledgments

This work was supported by the Natural Science Foundation of Hunan Province of China (Grant no. 2019JJ50399), the Scientific Research Fund of Hunan Provincial Education Department (Grant no. 19C1248), and the PhD Start-Up Fund from Hunan University of Arts and Science (Grant no. 17BSQD04).

References


