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Research Article

Applying Graph Theory to Some Problems of Economic Dynamics

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This paper studies dynamic models of production and exchange on graph with consideration of transportation costs. Using graph and set of matrices, we introduce superlinear multivalued mappings which describe the exchange ratio in considered system. Effective trajectories of these models are studied. It is shown that trajectories can be constructed using the simplest equilibrium type mechanisms. Characteristics of effective trajectories in Neumann type models are given. Conditions for the existence of equilibrium state of the considered model are found.

1. Introduction

A lot of works appeared lately dealing with the applications of graph theory to some models of economic dynamics [1–3] and related extremal problems [2, 4–9]. In this paper, an attempt is made to apply the elements of graph theory to the models of economic dynamics with consideration of transportation costs. We consider production mappings which define the Neumann-Gale model [10].

Consider a digraph with no multiple arcs $\gamma=(J,G)$, where $J = \{1, 2 \dots m\}$ is a set of vertices and $G \subset J \times J$ is a set of arcs. Multivalued mapping with the graphic G is denoted by the same symbol G. Thus, $G(i)$ consists of those vertices j for which there exists an arc from i to j. For convenience, we will assume that each vertex is provided with a loop, i.e., $i \in G(i)$ for every i. The simulated economic system operates with n products. If there is a product vector $y^j = (y^{j1}, y^{j2}, \dots, y^{jn}) \in R^n_+$ at the vertex $j \in G$, where $R^n_+ = \{x \in R^n : x \geq 0\}$ then any part u^{jk} of this vector ($0 \leq u^{jk} \leq y^j$) can be transferred (transported) to the vertex $k \in G(j)$. Each arc $(j, k) \in G$ is associated with some nonnegative matrix C^{jk}, by means of which "transport costs" are taken into account in some generalized sense. This means that if the vector u^{jk} at the vertex j is selected for transmission to the vertex k, then the vector $C^{jk}u^{jk}$ moves to the latter vertex. The additional vector u^{jk} -$C^{jk}u^{jk}$, in case of its nonnegativity, can be considered as a transportation fee, which is withdrawn from the system. We can assume that C^{jk} is a diagonal matrix with nonnegative diagonal elements $\nu^{jkl} \leq 1$, where 1-ν^{jkl} coincides with the fraction of the unit of the l-th product, which should be paid for the transportation of this unit along the arc (j, k). But this kind of matrix C^{jk} will not be used in the sequel. We assume below that C^{jj} coincides with the identity matrix E for all $j \in J$ (no need to pay for the transportation from a vertex to itself).

Consider the complete graph $\widetilde{\gamma}$ with the same set of vertices J, and associate every pair $(j, k) \in J \times J$ with the matrix $\widetilde{C}^{jk}$ by letting $\widetilde{C}^{jk} = C^{jk}$ if $(j, k) \in G$ and $\widetilde{C}^{jk} = 0$ otherwise. The problems below may be considered on a complete graph γ; i.e., instead of the system $(J,G,(C^{jk})_{(j,k)\in G})$ one may consider $(J,J \times J,(C^{jk})_{(j,k)\in J})$. In other words, the arc $(j, k) \in J \times J$ is obviously forbidden in these problems if $\widetilde{C}^{jk} = 0$. But it would be convenient for us to express the set of arcs G in explicit form. Using the graph $\gamma=(J,G)$ and the set of the matrices $(C^{jk})_{(j,k)\in G}$, we can introduce the mapping B, which describes the exchange relation in the simulated system. This mapping is defined on the cone $(R^n_+)^m$. Let $Y = (y^1, y^2, \dots, y^m)$, where y^i is a vector of products in the vertex i. Then

$$B(Y) = \left\{ Z = \left(z^1, \dots, z^m\right) : z^k = \sum_{j \in G^{-1}(k)} c^{jk} u^{jk} \right\}$$

$$K = 1, 2, \dots, m; \; u^{ij} \geq 0 \text{ for all } (i, j) \in G$$

$$\sum_{j\in G(i)} u^{ij} = y^i, \quad i = \{1, 2, \dots, m\}. \tag{1}$$

It is easy to verify that the mapping B is superlinear, i.e., it has the following three properties:

(1) $B(Y^1 + Y^2) \supset B(Y^1) + B(Y^2)$

(2) $B(\lambda Y) = \lambda B(Y); \lambda \geq 0$

(3) The graph of the mapping B, i.e., the set *gr B* $= \{(Y, Z) : Z \in B(Y)\}$, is closed; besides, the following conditions are satisfied:

(4) $B(0) = \{0\}$

(5) $B((R_+^n)^m) = (R_+^n)^m$

Equality (9) follows from the fact that $Y \in B(Y)$ for all Y. Production capabilities of the vertex $j \in J$ are described by the superlinear mapping a^j: $R_+^n \longrightarrow 2^{R_+^n}$.

It is assumed that $a^j(0) = \{0\}$ for all j. Production capabilities of the entire system are given by the mapping A defined on the cone $(R_+^n)^m$.

If $X = (X^1, \dots, X^m) \in (R_+^n)^m$, then

$$A(X) = a^1(x^1) \cdot a^2(x^2) \cdot \dots \cdot a^m(x^m), \tag{2}$$

In other words,

$$A(X) = \left\{ Y = (y^1, y^2, \dots, y^m) : y^i \in a^i(X^i), i = 1, 2, \dots, m \right\} \tag{3}$$

Main Part. The operations performed by the entire system over a period of time consist of production and exchange. These operations can be carried out in different order. If the production comes first followed by the exchange, then the work of the system is described by the composition $a = B \circ A$ of the mappings A and B:

$$a(X) = \bigcup_{Y\in A(X)} B(Y), \quad X \in (R_+^n)^m. \tag{4}$$

Conversely, if the exchange happens first and then comes production, then we should consider the composition $b = A \circ B$ of the mappings B and A:

$$b(Y) = \bigcup_{Z\in B(Y)} A(Z), \quad Y \in (R_+^n)^m. \tag{5}$$

It is obvious that the mappings *a* and *b* are superlinear and, besides, $a(0) = b(0) = \{0\}$

However, since the equality $a((R_+^n)^m) = R_+^n$ may not be satisfied, the relation $a((R_+^n)^m) = (R_+^n)^m$ may not hold. If it is true, then the mapping *a* defines the Neumann-Gale model [11]. The same is also true for the mapping *b*.

In what follows, we will need description of the mappings conjugate to *a* and *b*. Recall that, for the superlinear mapping c: $R_+^n \longrightarrow 2^{R_+^k}$, its conjugate c^* is defined by the equality

$$c^*(g) = \left\{ f \in R_+^k : [f, x] \geq [g, y] \ \ \forall x, y \in c(x) \right\}, \quad g \in R_+^k \tag{6}$$

The symbol [x, y] denotes the scalar product of the vectors x and y. Sometimes, instead of the conjugate c^*, it is convenient to use its inverse mapping c', called the dual mapping [11]:

$$c'(f) = \left\{ g \in R_+^k : [g, y] \leq [f, x] \ \ \forall x, y \in c(x) \right\}, \quad f \in R_+^k \tag{7}$$

The following statements are valid.

Proposition 1 (see [10]). *Let* $G = (g^1, g^2, \dots, g^m) \in (R_+^n)^m$.

Then

$$A^*(G) = \left\{ F = (f^1, \dots, f^m) \in (R_+^n)^m : f_i \in (a^i)^* (g^i) \right\}. \tag{8}$$

Let us introduce the following notations. Let $H - (h^1, \dots, h^m) \in (R_+^n)^m$ *and*

$$q^i(H) = \sup_{i\in G(j)} \left((c^{ji})^* h^i \right); \tag{9}$$

The supremum of the vectors $(c^{ji})^* h$ *is calculated here coordinatewise (* $*$ *is a sign of matrix transposition). Further, let*

$$Q(H) = \left(q^1(H), \dots, q^m(H) \right). \tag{10}$$

Proposition 2. *Let* $H = (h^1, \dots, h^m) \in (R_+^n)^m$, *then*

$$B^*(H) = Q(H) + (R_+^n)^m;$$

In other words,

$$B^*(H) = \left\{ G = (g^1, \dots, g^m) : g^k \geq q^k(H), k = 1, 2, \dots, m \right\}. \tag{11}$$

Proof. Let $S_H(Y) = \max_{z\in B(Y)} [H, Z]$. It is well known that the set B(H) coincides with the super differential $\partial S_H = \{G : [G, Y] \geq S_H(Y), Y \in (R_+^n)^m\}$ of the superlinear functional S_H. Let us calculate this functional. First we find the quantity [H,Z], where $Z \in B(Y)$; i.e., Z is representable in the form $Z = (z^1, z^2, \dots, z^m)$, where $z^i = \sum_{j\in G^{-1}(i)} c^{ji} u^{ji}$, and the elements $u^{ji} \geq 0$ are such that $\sum_{j\in G^{-1}(i)} u^{ji} = y^i$, j=1,2,…,m.

We have

$$\begin{aligned} [H, Z] &= \sum_{i=1}^m [h^i, z^i] = \sum_i \left[h^i, \sum_{j\in G^{-1}(i)} c^{ji} u^{ji} \right] \\ &= \sum_{(j,i)\in G} [h^i, c^{ji} u^{ji}] = \sum_j \sum_{i\in G(j)} [h^i, c^{ji} u^{ji}] \\ &= \sum_j \sum_{i\in G(j)} \left[(c^{ji})^* h^i, u^{ji} \right]. \end{aligned} \tag{12}$$

It follows that $S_H(X) = \max[H, Z] = \max_{\sum_{i\in G(j)} u^{ji} = y^j;\ u^{ji} \geq 0} \sum_j \sum_{i\in G(j)} [(c^{ji})^* h^i, u^{ji}]$.

The maximum here is calculated over independent sets; that is, the elements on which the maximum is attained for some j depend only on y^j. Therefore,

$$S_H(Y) = \sum_j \max_{\sum u^{ji} = y^j;\ u^{ji} \geq 0} \sum_{i\in G(j)} \left[(c^{ji})^* h^i, u^{ji} \right]. \tag{13}$$

It is known from the theory of semiordered spaces [5] that the maximum under the first sign of sum can be written in the form $[q^i(H), y^j]$, $q^i(H)$ where is the element defined by (9). Thus,

$$S(H) = \sum_j \left[q^j(H), y^j\right] = [Q(H), Y], \tag{14}$$

i.e., the functional S_H is linear and is determined by the element Q(H). It immediately follows that $\partial S_H = Q(H) + (R^n_+)^m$. This completes the proof. □

Remark 1. Since $j \in G(j)$ and C^{jj} is an identity operator, we have $q^i(H) \geq h^j$ for all j, and, consequently, $Q(H) \geq H$.

Proposition 3. *Let* $\overline{z} \in B(\overline{Y})$, *where*

$$\overline{Z} = \left(\overline{z}^1, \overline{z}^2, \dots, \overline{z}^m\right), \qquad \overline{Y} = \left(\overline{y}^1, \overline{y}^2, \dots, \overline{y}^m\right) \tag{15}$$

and the elements $\overline{u}^{ji}(i, j \in G)$ *are such that*

$$\overline{y}^j = \sum_{i \in G(j)} \overline{u}^{ji}, \qquad \overline{z} = \sum_{j \in G^{-1}(i)} c^{ji} \overline{u}^{ji} \tag{16}$$

Further, let $G \in B^*(H)$, *where* $G = (g^1, \dots, g^m)$, $H = (h^1, \dots, h^m)$. *Then the equality* $[H, \overline{Z}] = [G, \overline{Y}]$ *is valid if and only if when*

$$\left[q^i - \left(c^{ji}\right)^* h^j, \overline{u}^{ji}\right] = 0 \tag{17}$$

for all $(j, i) \in G$.

Proof. It follows from (12) that

$$\left[H, \overline{Z}\right] = \sum_{(j,i) \in G} \left[\left(c^{ji}\right)^* h^j, \overline{u}^{ji}\right]. \tag{18}$$

At the same time,

$$\left[G, \overline{Y}\right] = \sum_j \left[q^j, \overline{y}^j\right] = \sum_{(j,i) \in G} \left[\left(c^{ji}\right)^* h^j, q^j, \overline{u}^{ji}\right] \tag{19}$$

By Proposition 2, each term in the last sum is nonpositive. Therefore, the sum is zero if and only if each term is zero.

The proposition is proved. □

Proposition 4. *Suppose that under the conditions of Proposition 3 all the matrices* $c^{ji}(j, i) \in G$ *coincide with the identity matrix and, in addition, the vector Y is strictly positive. Then G=Q(H)=H.*

Proof. From Proposition 2 and Remark 1 it follows that $G \geq Q(H) \geq H$. On the other hand, since c^{ji} is an identity operator, it follows from (17) that

$$\left[g^j - h^j, y^j\right] = \left[g^j - h^j, \sum_{i=1}^m u^{ji}\right] = \sum \left[g^j - h^j, y^j\right] = 0 \tag{20}$$

Strict positivity of the vector y^j implies the validity of the equality $g^j = h^j$.

The proposition is proved. □

The results obtained make it possible to describe the characteristics of the effective trajectories in Neumann type models defined by the production mappings of the form (4) and (5). Therefore, these models can be called models of production and exchange on graph.

Consider the model Z_a defined by the mapping a of the form (4). Let T be a positive integer. As is known [1], T-step trajectory of the model Z_a is defined as a finite sequence such that $X_{t+1} \in a_t(X_t)(t=0,1,\dots,T-1)$. Let $F=(f^1,\dots, f^m)$ be a price vector. The trajectory $X_0, X_1, \dots, X_T$ is called optimal in the sense of F if $[F,X]=\max[F, \tilde{X}_T]$ where the maximum is taken over all the trajectories $(\tilde{X}_0, \dots, \tilde{X}_t, \dots, \tilde{X}_T)$ starting at the point X_0. Under natural conditions, the optimal trajectory in the sense of F admits a characteristic [1]; that is, there exists a sequence

$$\begin{gathered} F_0, F_1, \dots, F_T, \quad \text{such that } F_T = F \\ \left[F, X_0\right] = \dots = \left[F_T, X_T\right], \left[F, \tilde{X}_0\right] \geq \dots \geq \left[F_T, \tilde{X}_T\right] \end{gathered} \tag{21}$$

for every T-step trajectory $(\tilde{X}_0, \dots, \tilde{X}_T)$.

Consider the trajectory $X_0, X_1, \dots, X_T$. Denote by y^i_t, u^{ji}_t $(t=1,\dots,T; (j,i) \in G)$ the elements R^n_+, with the property

$$\begin{aligned} y^i_t &\in a_t\left(x^i_{t-1}\right), \\ y^i_t &= \sum_{j \in G^{-1}(i)} u^{ji}_t, \\ x^i_t &= \sum_{j \in G^{-1}(i)} c^{ji} u^{ji}_t. \end{aligned} \tag{22}$$

Theorem 1. *The sequence* $F_0, \dots, F_T$, *where* $F_t = (f^i_t, \dots, f^m_t)$, *is a characteristic of the trajectory* $X_0, \dots X_T$ *if and only if*

$$(1) \quad f^i_{t-1} \in \left(a^i_t\right)^* \left(q^i\left(F_t\right)\right) \quad t = 1, 2, \dots, T; \; i = \overline{1, m}; \tag{23}$$

here q^i *is an operator defined by (9).*

$$(2) \quad \left[q^i\left(F_t\right) - \left(c^{ji}\right)^* f^j_i, u^{ji}_t\right] = 0 \quad (\forall (j, i) \in G) \tag{24}$$

Proof. Let (F_t) be a characteristic for the trajectory (X_t). Then for every $t=0,1,\dots,T$ we have

$$\begin{gathered} (1) \quad F_{t+1} \in a^*\left(F_t\right); \\ (2) \quad \left[F_{t-1}, X_{t-1}\right] = \left[F_t, X_t\right] \end{gathered} \tag{25}$$

Since $a = B \circ A$, by the theorem on the conjugate to the composition [1], we have $a^* = B^* \circ A^*$. Therefore, there exists a price vector $G_t = (g^1_t, \dots, g^m_t)$ such that

$$G_t \geq Q\left(F_t\right) \quad \text{i.e. } q^i_t \geq q^i\left(F_t\right) \text{ and } f^i_{t-1} \in \left(a^i\right)^* \left(q^i\right) \tag{26}$$

The inequality $g^i_t \geq q^i_t(F_t)$ implies the inclusion $(a^i)^*(g^i) \subset (a^i)^*(f^i_t)$. The validity of this simple statement was proved, e.g., in [1]. The validity of relation (23) follows immediately from this statement.

The inclusions $y_t^i \in a(x_{t-1}^i)$, $f_{t-1}^i \in (a^i)^*(q^i(F_t))$ imply the inequality $[Q(F_t), Y_t] \le [F_{t-1}, X_{t-1}]$, which, in turn, combined with (25) implies the inequality

$$[Q(F_t), Y_t] \le [F_t, X_t]. \tag{27}$$

At the same time, the relation $Q(F_t) \in B^*(F_t)$ shows that $[F_t, X_t] \le [Q(F_t), Y_t]$.

Thus, $[Q(F_t), Y_t] = [F_t, X_t]$. Then, by virtue of Proposition 3, we get the validity of (24). So, if $F_0, F_1, \ldots, F_T$ are the characteristics of the trajectory $X_0, \ldots, X_T$ then relations (23) and (24) are true. Converse statement can be easily verified. □

Remark 2. The statement of Theorem 1 remains valid for infinite trajectories, as well as in the case where the production mappings a_t of the model Z_a depend on time.

Characteristic prices of the effective trajectory can be interpreted as the equilibrium prices in some model of distribution economy.

Now recall the definition of fixed income distribution model. This model contains m participants (consumers), with i-th participant defined by his utility function u^i and his income λ^i. It is assumed that the resource vector of the entire economy X is known. This model is denoted as (u, λ, x), where $u = (u^1, u^2, \ldots, u^m)$, $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^m)$.

Equilibrium state of the model (u, λ, x) is defined as the set $(x^1, x^2, \ldots, x^m, p)$, where P is a price vector, $x^1, x^2, \ldots, x^m$ is a resource vector with $x^i \ge 0$,

$$\sum x^i = x \tag{28}$$

and x^i is a solution of the problem

$$\begin{gathered} u^i(x) \longrightarrow \max_{[p,x] \le \lambda^i} \\ \text{subject to } x \ge 0 \end{gathered} \tag{29}$$

In the sequel, we will assume that all utility functions u^i are first degree positively homogeneous functions. Suppose $\lambda^i > 0$. In this case, the vector x^i is a solution of problem (29) if and only if it satisfies the equality $[p, x^i] = \lambda^i$ and in addition is a solution of the problem

$$\frac{u^i(x)}{[p, x]} = \max \tag{30}$$

subject to $x \ge 0$.

The quantity $[p,x]$ represents the cost of resources at the prices P. If $u^i(x)$ may be interpreted as a cost of production (at some price) and $u^i(x) - [p,x]$ may be interpreted as an income, then the problem (30) is reduced to the maximization of the growth rate of profit.

Consider problem (30) under the assumption that $\lambda^i = 0$ and this problem has a solution.

Then the equality $[p,x]=0$ implies $u^i(x) = 0$ (otherwise the solution does not exist). Assume that $0/0=0$. Then every vector $x \ge 0$ with $[p,x]=0$ is a solution of both problem (29) and problem (30). Let us define the nonfixed income distribution model. As above, this model has m participants. However, the i-th participant is characterized only by the utility function u^i. This function is assumed to be positively homogeneous of the first degree. Besides, we are given a total resources vector X. Assuming $u = (u^1, u^2, \ldots, u^m)$, we denote this model by (u, x).

The equilibrium state of the model (u, x) is a set $(x^1, x^2, \ldots, x^m, p)$, where x^i is a solution of problem (30) and $\sum x^i = x$.

When considering problem (30), we assume $0/0=0$; $c/0=+\infty$, for $c > 0$. Let the set $(x^1, x^2, \ldots, x^m, p)$ be an equilibrium state of the model (u, x) and $\lambda^i = [p, x^i]$ $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^m)$.

Then this set is an equilibrium state of the model (u, λ, x). On the other hand, the equilibrium state of (u, λ, x) is the equilibrium state of (u, x) for any λ.

Back to the above considered mappings A, B. Let $b = A \circ B$. Using the theorem on the mapping conjugate to the composition [1], we obtain $b^* = B^* \circ A^*$. In other words, if $F = (f^1, f^2, \ldots, f^m)$ $G = (g^1, g^2, \ldots, g^m)$ are price vectors and $F \in b^*(G)$, then there exists a price vector $H = (h^1, h^2, \ldots, h^m)$, such that $H \in A^*(G)$, $F \in b^*(H)$. By Proposition 1, $h^i \in (a^i)^*(g^i)$, i.e., $[g^i, y] \le [h^i, z]$ for all z and $y \in a^i(z)$.

The last relation can be rewritten as

$$V^i(z) \le [h^i, z] \quad (z \ge 0), \tag{31}$$

where

$$V^i(z) = \max_{y \in a^i(z)} [g^i, y]. \tag{32}$$

Proposition 5. *Let the vectors* $\overline{X} = (\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^m)$, $\overline{Y} = (\overline{y}^1, \overline{y}^2, \ldots, \overline{y}^m)$, *and* $\overline{Z} = (\overline{z}^1, \overline{z}^2, \ldots, \overline{z}^m)$ *and the price vectors* $F = (f^1, f^2, \ldots, f^m)$, $H = (h^1, h^2, \ldots, h^m)$, *and* $G = (g^1, g^2, \ldots, g^m)$ *satisfy* $\overline{Z} \in B(\overline{X})$, $\overline{Y} \in A(\overline{Z})$, $H \in A^*(G)$, $F \in B^*(H)$, *and, in addition,* $[F, \overline{X}] = [G, \overline{Y}]$. *Then the vector Z is a solution of the problem*

$$\begin{gathered} \frac{V^i(z)}{[h^i, z]} \longrightarrow \max, \\ \text{subject to } Z \ge 0. \end{gathered} \tag{33}$$

Here V^i *is a function defined by equality (32), and, as above, it is assumed that 0/0=0.*

Proof. Since

$$\begin{gathered} F \in B^*(H), \\ H \in A^*(G) \end{gathered} \tag{34}$$

we have

$$[F, \overline{X}] \le [H, \overline{Z}] \le [G, \overline{Y}]. \tag{35}$$

Then it follows from the condition of the proposition that $[F, \overline{X}] = [H, \overline{Z}] = [G, \overline{Y}]$. Using relations (32) and (33), we get

$$\begin{gathered}\left[G, \overline{Y}\right] \leq \sum \left[g^i, \overline{y}^{\mathrm{i}}\right] \leq \sum V^i, \\ \left(\overline{Z}^{\mathrm{i}}\right) \leq \sum \left[h^i, \overline{Z}\right] = \left[\mathrm{H}, \overline{Z}\right].\end{gathered} \tag{36}$$

As $[G, \overline{Y}] = [\mathrm{H}, \overline{Z}]$, we have $\mathrm{V}^{\mathrm{i}}(\overline{Z})$ for all i. Then it follows from (31) that $\overline{Z}$ is a solution of problem (33). □

Remark 3. It immediately follows from the proof that the value of problem (33) coincides with either zero or unity.

Consider the case where under the conditions of Proposition 5 the graph $\gamma = (\mathrm{J}, \mathrm{G})$ is complete, all the matrices $\mathrm{C}^{\mathrm{ji}}(\mathrm{i}, \mathrm{j} \in \mathrm{J})$ coincide with the identity matrices, and the vector X is strictly positive. Then, as follows from Proposition 4, Q(H)=H.

Let $h = \sup(H)$. Since the graph is complete, we have $\mathrm{q}^{\mathrm{j}}(\mathrm{H}) = \mathrm{H}$. Thus, the equality $\mathrm{h} = \mathrm{q}^{\mathrm{j}}(\mathrm{H}) = \mathrm{h}^{\mathrm{j}}$ holds for every j, and therefore problem (33) may be rewritten in the form

$$\frac{\mathrm{V}^{\mathrm{j}}(\mathrm{z})}{[\mathrm{h}, \mathrm{z}]} \longrightarrow \max \tag{37}$$

$$\text{subject to } \mathrm{Z} \geq 0.$$

It follows that the set $(\overline{z}^1, \overline{z}^2, \ldots, \overline{z}^{\mathrm{m}}, \mathrm{h})$ is an equilibrium state of the model (V,x), where $\mathrm{V}=(\mathrm{V}^1,\mathrm{V}^2,\ldots,\mathrm{V}^{\mathrm{m}})$, $\mathrm{x}=\sum \overline{x}^i$. The last relation means that there exist the elements $\mathrm{u}^{\mathrm{ji}} \geq 0$ such that

$$\begin{gathered}\sum_{\mathrm{i},\mathrm{j}} u^{ji} = x, \\ \sum_{\mathrm{j}=1}^{\mathrm{m}} u^{ji} = \overline{x}^{\mathrm{i}}, \\ \mathrm{i} = 1, 2, \ldots, \mathrm{m}.\end{gathered} \tag{38}$$

We now consider the general situation, that is, the model of distribution economy on the graph (J, G) with the system of matrices C^{ji}, $(\mathrm{i},\mathrm{j})\in\mathrm{G}$. The participant i in this model is defined by the utility function $\mathrm{u}^{\mathrm{i}}(\mathrm{i}\in\mathrm{J})=\{1, 2, \ldots, \mathrm{m}\}$ and the resource vector x^{i}. Denote the considered model by (U, X), where $\mathrm{U}=(\mathrm{u}^1,\mathrm{u}^2,\ldots,\mathrm{u}^{\mathrm{m}})$, $\mathrm{X}=(\mathrm{x}^1,\mathrm{x}^2,\ldots,\mathrm{x}^{\mathrm{m}})$. Proposition 5 suggests the following definition.

Definition. By the equilibrium state for the model (U, X) we mean a set (Z, H) with $\mathrm{Z}=(\mathrm{z}^1,\mathrm{z}^2,\ldots,\mathrm{z}^{\mathrm{m}})$, $\mathrm{H}=(\mathrm{h}^1,\mathrm{h}^2,\ldots,\mathrm{h}^{\mathrm{m}})$ where z^{i} is a resource vector, h^{i} is a price vector, Z^{i} is a solution of the problem $\mathrm{U}^{\mathrm{i}}(\mathrm{Z})/[\mathrm{h}^{\mathrm{i}}, \mathrm{z}] \longrightarrow \max$ subject to $\mathrm{Z} \geq 0$, and there exist the vectors u^{ji}, $(\mathrm{j},\mathrm{i})\in\mathrm{G}$ such that

$$\begin{gathered}\sum_{\mathrm{i}\in\mathrm{G}(\mathrm{j})} \mathrm{u}^{\mathrm{ji}} = \mathrm{x}^{\mathrm{i}}, \\ \sum_{\mathrm{j}\in\mathrm{G}^{-1}(\mathrm{i})} \mathrm{c}^{\mathrm{ji}}\mathrm{u}^{\mathrm{ji}} = \mathrm{z}^{\mathrm{i}}.\end{gathered} \tag{39}$$

The vector $\mathrm{H}=(\mathrm{h}^1,\mathrm{h}^2,\ldots,\mathrm{h}^{\mathrm{m}})$ is related to the vectors u^{ji} by the relations of type

$$\left[\mathrm{q}^{\mathrm{i}}(\mathrm{H}) - \left(\mathrm{c}^{\mathrm{ji}}\right)^* \mathrm{h}, \mathrm{u}^{\mathrm{ji}}\right] = 0 \tag{40}$$

where $\mathrm{q}^{\mathrm{j}}(\mathrm{H})$ is a vector defined by (9).

Consider the Neumann-Gale model Z_{b} given by the production mapping $\mathrm{b} = \mathrm{A}\circ\mathrm{B}$. Also consider the Neumann type model m that has B as a production mapping at even moments of time and A as a production mapping at odd moments of time. Let $(\mathrm{X}_{\mathrm{t}})$ be the effective trajectory of the model Z_{b} admitting the characteristics $(\mathrm{F}_{\mathrm{t}})$. Then there exist the resource vectors Z_{t} and the price vectors H_{t} such that the sequences

$$\begin{gathered}\mathrm{X}_0, \mathrm{Z}_0, \mathrm{X}_1, \mathrm{Z}_1, \ldots, \mathrm{X}_{\mathrm{t}}, \mathrm{Z}_{\mathrm{t}}, \mathrm{X}_{\mathrm{t}+1}, \ldots \\ \mathrm{F}_0, \mathrm{H}_0, \mathrm{F}_1, \mathrm{H}_1, \ldots, \mathrm{F}_{\mathrm{t}}, \mathrm{H}_{\mathrm{t}}, \mathrm{F}_{\mathrm{t}+1}, \ldots\end{gathered} \tag{41}$$

are the trajectories of the model m and its dual m', respectively, with the second sequence being a characteristic of the first one. Besides, the sequence $\mathrm{Z}_0,\mathrm{Z}_1,\ldots,\mathrm{Z}_{\mathrm{t}}\ldots$ makes a trajectory of the model Z_{b}, while the sequence $\mathrm{H}_0,\mathrm{H}_1,\ldots,\mathrm{H}_{\mathrm{t}}\ldots$ forms the characteristics of this trajectory.

It follows directly from Proposition 4 that the pair $(\mathrm{H}_{\mathrm{t}}, \mathrm{Z}_{\mathrm{t}})$ represents the equilibrium in a nonfixed income model defined by the resource vector $\mathrm{X}_{\mathrm{t}} = (\mathrm{x}_{\mathrm{t}}^1,\mathrm{x}_{\mathrm{t}}^2,\ldots,\mathrm{x}_{\mathrm{t}}^{\mathrm{m}})$ and utility functions $\mathrm{V}_{\mathrm{t}}=(\mathrm{V}_{\mathrm{t}}^1,\mathrm{V}_{\mathrm{t}}^2,\ldots,\mathrm{V}_{\mathrm{t}}^{\mathrm{m}})$, where $\mathrm{V}_{\mathrm{t}}^{\mathrm{i}}$ is given by the formula (32) for $\mathrm{g}^{\mathrm{i}}=\mathrm{f}_{\mathrm{t}+1}^{\mathrm{i}}$. This equilibrium is characterized by the fact that the value of the problem $\max \mathrm{V}_{\mathrm{t}}^{\mathrm{i}}(\mathrm{z})/[\mathrm{h}_{\mathrm{t}}^{\mathrm{i}},\mathrm{z}]$ coincides with either zero or unity.

Suppose that we have a graph (J, G) equipped with a system of matrices $(\mathrm{c}^{\mathrm{ji}})$ $(\mathrm{j},\mathrm{i}\in\mathrm{G})$, each i being associated with the resource vector x^{i} and the utility function U^{i}. Using the theorem on characteristics, under some conditions it is possible to prove the existence of an equilibrium (Z, H) for the model (U, X) with an additional property that the value of the problem $\max \mathrm{U}^{\mathrm{i}}(\mathrm{z})/[\mathrm{h}^{\mathrm{i}}, \mathrm{z}]$ coincides for all i with either zero or unity. Let us give the outlines of the proof. Consider some strictly positive vectors

$$\begin{gathered}\mathrm{G} = \left(\mathrm{g}^1, \mathrm{g}^2, \ldots, \mathrm{g}^{\mathrm{m}}\right) \\ \text{and } \mathrm{W} = \left(\mathrm{w}^1, \mathrm{w}^2, \ldots, \mathrm{w}^{\mathrm{m}}\right)\end{gathered} \tag{42}$$

which satisfy the condition $[\mathrm{g}^{\mathrm{i}}, \mathrm{w}^{\mathrm{i}}] = 1$. Define the mapping a^{i} by letting $\mathrm{a}^{\mathrm{i}}(\mathrm{x}) = \{\mathrm{y} : 0 \leq \mathrm{y} \leq \mathrm{U}(\mathrm{x})\mathrm{w}^{\mathrm{i}}\}$. Let V^{i} be a function given by (32). Then, by definition, $\mathrm{U}(\mathrm{x}) = \mathrm{V}^{\mathrm{i}}(\mathrm{x})$ for all $\mathrm{x} \geq 0$.

Suppose that the vectors x^{i} are strictly positive and consider one-step trajectory of the model Z_{b} starting at the point X and maximizing the price vector G on the set $\mathrm{Z}_b(\mathrm{X})$, where, as above, $\mathrm{b} = \mathrm{A}\circ\mathrm{B}$, the mapping B is defined in (1) by the graph (J, G) and the matrices c^{ji}, and the mapping A is defined in (4) by the mapping a^{i}. Let this trajectory have the form (X, Y). Then there exists a vector Z such that $\mathrm{Z}\in\mathrm{B}(\mathrm{X})$, $\mathrm{Y}\in\mathrm{A}(\mathrm{Z})$. By the well-known theorems [1], there exists a price vector F such that the pair (F,G) is a characteristic of the trajectory (X, Y). There exists a vector H such that $\mathrm{F}\in\mathrm{B}^*(\mathrm{H})$, $\mathrm{H}\in\mathrm{A}^*(\mathrm{G})$

$$[\mathrm{F}, \mathrm{X}] = [\mathrm{H}, \mathrm{Z}] = [\mathrm{G}, \mathrm{Y}]. \tag{43}$$

It is clear that the pair $[\mathrm{Z}, \mathrm{H}]$ is the sought one.

2. Conclusions

The results obtained in this work allow applying well-known facts about graph theory to some models of production and exchange. A description of the characteristics of the effective trajectories of the Neumann type models is given. It is shown that the characteristic prices can be considered as equilibrium prices in some distribution models. Besides, the resource vector of considered problem is a solution of some extremal problem. The existence of equilibrium is proved under some conditions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that they have no conflicts of interest.

References

[1] C. Berge, *The Theory of Graphs and its Application*, IL, Moscow, Russia, 1962.

[2] J. Little, M. T. Murty, D. Sweeney, and Carrel, "Algorithms for solving the problems of the traveling salesman," *Economics and Mathematical Methods*, no. 1, 1965.

[3] O. Ore, *Graph Theory*, Nauka, Moscow, Russia, 1980.

[4] M. C. Alvares and D. Ehnts, "Graph theory and macroeconomic regimes in stock-flow consistent modeling," Working paper No. 854, Lewi Economic Institute, 2015.

[5] V. Gazda, D. Horvath, and M. Resovsky, *An Application of Graph Theory in the Process of Mutual Debt Compensation*, 1980.

[6] E. N. Kuzbozhev, "Application of graph theory in planning," *Economics and Mathematical Sciences*, 2007.

[7] J. Leskow, M. Puchet, and L. F. Punzo, *New Tools of Economic Dynamics Business and Economics*, 2006.

[8] E. Maynika, *Optimization Algorithms on Networks and Graphs*, Mir, Moscow, Russia, 1981.

[9] D. Mishra, *Basic Graph Theory with Applications to Economics*, 2010.

[10] V. L. Makarov and A. M. Rubinov, *Mathematical Theory of Economical Dynamics and Equilibrium*, 1998.

[11] L. V. Kantorovich, "Optimization Methods and Mathematical Models of Economics," *Progress in Mathematical Sciences*, 1970.


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