Hindawi
Discrete Dynamics in Nature and Society
Volume 2019, Article ID 8052497, 7 pages
https://doi.org/10.1155/2019/8052497

Research Article

Infinitely Many Positive Solutions for a Coupled Discrete Boundary Value Problem

Liuming Li[1] and Zhan Zhou[1,2]

[1] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong 510006, China
[2] Center for Applied Mathematics, Guangzhou University, Guangzhou, Guangdong 510006, China

Correspondence should be addressed to Zhan Zhou; zzhou0321@hotmail.com

Received 12 December 2018; Accepted 27 January 2019; Published 14 February 2019

Academic Editor: Douglas R. Anderson

In this paper, we obtain some results for the existence of infinitely many positive solutions for a coupled discrete boundary value problem. The approach is based on variational methods.

1. Introduction

Let Z and R denote the sets of integers and real numbers, respectively. For $a, b \in Z$, define $[a, b] = \{a, a+1, \ldots, b\}$, when $a \leq b$.

In this paper, we consider the following coupled discrete boundary value problem:

$$\begin{aligned} &-\Delta\left(p(k)\Delta u_i(k-1)\right) + q(k)u_i(k) \\ &\quad = \lambda F_{u_i}\left(k, u_1(k), \ldots, u_m(k)\right), \\ &u_i(0) = u_i(N+1) = 0, \end{aligned} \tag{1}$$

for all $k \in [1, N]$, $N \in Z^+$, where $1 \leq i \leq m$, and λ is a positive parameter, $\Delta u_i(k) = u_i(k+1) - u_i(k)$ is the forward difference operator, $p(k) > 0$ for all $k \in [1, N+1]$, and $q(k) \geq 0$ for all $k \in [1, N]$, $F(k, \cdot, \ldots, \cdot)$ is a C^1–function in R^m satisfying $F(k, 0, \ldots, 0)$=0 for every $k \in [1, N]$, and F_{u_i} denotes the partial derivative of $F(k, u_1, u_2, \ldots, u_m)$ with respect to u_i for $i = 1, \ldots, m$.

As we know, results for existence of solutions for difference equations have been widely studied because of their applications to various fields of applied sciences, like mechanical engineering, control systems, computer science, economics, artificial or biological neural networks, and many others. Many scholars have studied such problems and main tools are fixed point methods, Brouwer degree theory, and upper and lower solution techniques; see [1–4] and references therein. In recent years, variational methods have been employed to study difference equation and various results have been obtained. See, for instance, [5–22].

More recently, especially, in [23–31], by starting from the seminal papers [32, 33], many results for the existence and multiplicity of solutions for discrete boundary value problems have been obtained also by adopting variational methods.

However, these papers only deal with a single equation. For instance, in [23], by studying the Dirichlet discrete boundary value problem

$$\begin{aligned} -\Delta^2 u(k-1) &= \lambda f_k(u(k)), \quad k \in [1, N], \\ u(0) &= u(N+1) = 0, \end{aligned} \tag{2}$$

the author obtained the existence of two positive solutions of the problem through appropriate variational methods. In this paper, we consider system (1) with m difference equations by using Ricceri's variational principle proposed in [33]. In Theorem 4 and Remark 5, we prove the existence of an interval (λ_1, λ_2) such that, for each $\lambda \in (\lambda_1, \lambda_2)$, problem (1) admits a sequence of positive solutions which is unbounded in X. To the best of our knowledge, this is the first time to deal with coupled discrete boundary value problems. This method has already been used for the continuous counterparts [34–36]. The monographs [37–39] are related books of the critical point theory and difference equations.

The rest of the paper is organized as follows: Section 1 consists of some definitions and mathematical symbols. In Section 2, we emphasize that a strong maximum principle (Lemma 1) is presented so that if $F_{u_i}(k, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m) \geq 0$, for all $k \in [1, N]$ and $i = 1, \ldots, m$, our results guarantee the existence of infinitely many positive solutions (Remark 5 in Section 3). Section 3 contains a more precise version (Lemma 3) of Ricceri's variational principle, the statements and proofs of the main results (Theorem 4 and Remark 5), two corollaries (Corollaries 6 and 7), and an example (Example 10).

Throughout this paper, we let $S = \{u : [0, N+1] \longrightarrow R : u(0) = u(N+1) = 0\}$ and X be the Cartesian product of m Banach spaces $S, \ldots, S$; i.e.,

$$X \coloneqq \underbrace{S \times \cdots \times S}_{m} \tag{3}$$

endowed with the norm

$$\|u\| = \|(u_1, \ldots, u_m)\| \coloneqq \left(\sum_{i=1}^{m} \|u_i\|_*^2\right)^{1/2}, \tag{4}$$

where

$$\|v\|_* \coloneqq \left(\sum_{k=1}^{N+1} p(k) |\Delta v(k-1)|^2 + \sum_{k=1}^{N} q(k) |v(k)|^2\right)^{1/2}, \quad \text{for } v \in S, \tag{5}$$

which is a norm in S.

Put

$$A \coloneqq \liminf_{y \longrightarrow +\infty} \frac{\sum_{k=1}^{N} \max_{(t_1,\ldots,t_m) \in K(y)} F(k, t_1, \ldots, t_m)}{y^2}, \tag{6}$$

where

$$K(y) \coloneqq \left\{(t_1, \ldots, t_m) \in R^m : \sum_{i=1}^{m} |t_i| \leq y\right\}, \tag{7}$$

for every $y \in R^+$, and

$$B \coloneqq \limsup_{\sum_{i=1}^{m} t_i^2 \longrightarrow +\infty \, (t_1,\ldots,t_m) \in R_+^m} \frac{\sum_{k=1}^{N} F(k, t_1, \ldots, t_m)}{\sum_{i=1}^{m} t_i^2}, \tag{8}$$

where $R_+^m = \{(t_1, \ldots, t_m) \in R^m : t_i \geq 0 \text{ for all } 1 \leq i \leq m\}$.

2. Preliminaries

First, we establish a strong maximum principle.

Lemma 1. *Fix $u \in S$ such that either $u(k) > 0$ or*

$$\begin{gathered} \text{if } u(k) \leq 0 \\ \text{then } -\Delta(p(k)\Delta u(k-1)) + q(k) u(k) \geq 0. \end{gathered} \tag{9}$$

Then either $u > 0$ in $[1, N]$ or $u \equiv 0$.

Proof. Let $j \in [1, N]$ such that

$$u(j) = \min\{u(k) : k \in [1, N]\}. \tag{10}$$

If $u(j) > 0$, then it is clear that $u > 0$ in $[1, N]$.

If $u(j) \leq 0$, then by (9), we have

$$-\Delta(p(j)\Delta u(j-1)) \geq -\Delta q(j) u(j) \geq 0, \tag{11}$$

that is

$$p(j+1)\Delta u(j) \leq p(j)\Delta u(j-1). \tag{12}$$

On the other hand, by the definition of $u(j)$, we see that

$$\begin{gathered} \Delta u(j-1) \leq 0, \\ \Delta u(j) \geq 0. \end{gathered} \tag{13}$$

Thus, by (12), we obtain that $u(j+1) = u(j) = u(j-1)$.

By similar arguments applied to $u(j-1)$ and $u(j+1)$ and continuing in this way, we have $u(k) = u(N+1) = u(0) = 0, \forall k \in [1, N]$. □

Remark 2. Let $F : [1, N] \times R^m \longrightarrow R$ be such that $F_{t_i}(k, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m) \geq 0$ for all $k \in [1, N]$ and $i = 1, \ldots, m$.

Put

$$F_{t_i}^*(k, t_1, \ldots, t_m) = \begin{cases} F_{t_i}(k, t_1, \ldots, t_m), & \text{if } t_i > 0, \\ F_{t_i}(k, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m), & \text{if } t_i \leq 0. \end{cases} \tag{14}$$

Clearly, $F_{t_i}^*(k, \cdot)$ is continuous in R^m for each $k \in [1, N]$. Owing to Lemma 1, all solutions of problem

$$\begin{gathered} -\Delta(p(k)\Delta u_i(k-1)) + q(k) u_i(k) \\ = \lambda F_{u_i}^*(k, u_1(k), \ldots, u_m(k)), \\ u_i(0) = u_i(N+1) = 0, \end{gathered} \tag{15}$$

are either zero or positive and hence are also solutions for problem (1). Hence we emphasize that when (15) admits nontrivial solutions, then problem (1) admits positive solutions, independently of the sign of F_{u_i}.

3. Main results

Let X be a reflexive real Banach space and let $I_\lambda : X \longrightarrow R$ be a function satisfying the following structure hypothesis:

(Λ) $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are two functions of class C^1 on X with Φ coercive; i.e., $\lim_{\|u\|\longrightarrow+\infty}\Phi(u) = +\infty$, and λ is a real positive parameter.

Provided that $\inf_X \Phi < r$, put

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \tag{16}$$

and

$$\gamma := \liminf_{r\longrightarrow+\infty} \varphi(r). \tag{17}$$

Clearly, $\gamma \geq 0$. When $\gamma = 0$, in the sequel, we agree to read $1/\gamma$ as $+\infty$.

For the readers' convenience, we recall a more precise version of Theorem 2.1 of [32] (see also Theorem 2.5 of [33]) which is the main tool used to investigate problem (1).

Lemma 3. *Assume that condition (Λ) holds. If $\gamma < +\infty$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(a_1) I_λ possesses a global minimum, or

(a_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\longrightarrow+\infty} \Phi(u_n) = +\infty \tag{18}$$

Put

$$\begin{aligned} P &:= p(1) + p(N+1), \\ Q &:= \sum_{k=1}^{N} q(k), \\ p* &:= \min\{p(k) : k \in [1, N+1]\}. \end{aligned} \tag{19}$$

Our main result is the following theorem.

Theorem 4. *Assume that*

(i) F is nonnegative in $[1, N] \times R_+^m$

(ii) $A < (p*/m(P+Q)(N+1))B$, where A and B are given by (6) and (8), respectively

Then, for each $\lambda \in ((P+Q)/2B, p*/2m(N+1)A)$, system (1) admits an unbounded sequence of solutions.

Proof. For each $u = (u_1(k), \ldots, u_m(k)) \in X$, put

$$\begin{aligned} \Phi(u) &:= \frac{1}{2}\|u\|^2, \\ \Psi(u) &:= \sum_{k=1}^{N} F(k, u_1(k), \ldots, u_m(k)), \end{aligned} \tag{20}$$

and

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u). \tag{21}$$

Standard arguments show that $I_\lambda \in C^1(X, R)$ and that critical points of I_λ are exactly the solutions of problem (1). In fact, $\Phi, \Psi \in C^1(X, R)$; that is, Φ and Ψ are continuously Fréchet differentiable in X. Using the summation by parts formula and the fact that $u_i(0) = u_i(N+1) = 0$ for any $u \in X$, we get

$$\begin{aligned} \Phi'(u)(v) &= \lim_{t\longrightarrow 0} \frac{\Phi(u+tv) - \Phi(u)}{t} = \sum_{k=1}^{N+1}\sum_{i=1}^{m} p(k) \\ &\quad \cdot \Delta u_i(k-1)\Delta v_i(k-1) + \sum_{k=1}^{N}\sum_{i=1}^{m} q(k) u_i(k) v_i(k) \\ &= \sum_{i=1}^{m} p(N+1) u_i(N) v_i(N) + \sum_{i=1}^{m} \left[p(k)\Delta u_i(k-1)\right. \\ &\quad \left.\cdot v_i(k-1)\right]_1^{N+1} - \sum_{k=1}^{N}\sum_{i=1}^{m} \Delta\left(p(k)\Delta u_i(k-1)\right) v_i(k) \\ &\quad + \sum_{k=1}^{N}\sum_{i=1}^{m} q(k) u_i(k) v_i(k) \\ &= -\sum_{k=1}^{N}\sum_{i=1}^{m} \left[\Delta\left(p(k)\Delta u_i(k-1)\right) v_i(k)\right. \\ &\quad \left. - q(k) u_i(k) v_i(k)\right] \end{aligned} \tag{22}$$

and

$$\begin{aligned} \Psi'(u)(v) &= \lim_{t\longrightarrow 0} \frac{\Psi(u+tv) - \Psi(u)}{t} \\ &= \sum_{k=1}^{N}\sum_{i=1}^{m} F_{u_i}(k, u_1(k), \ldots, u_m(k)) v_i(k), \end{aligned} \tag{23}$$

for any $u, v \in X$.

Moreover, we obtain

$$\begin{aligned} I'_\lambda(u)(v) &= \sum_{k=1}^{N}\sum_{i=1}^{m} \left[-\Delta\left(p(k)\Delta u_i(k-1)\right)\right. \\ &\quad \left. + q(k) u_i(k) - \lambda F_{u_i}(k, u_1(k), \ldots, u_m(k))\right] v_i(k), \end{aligned} \tag{24}$$

for all $u, v \in X$.

Now we verify that $\gamma < +\infty$. Let $\{b_n\}$ be a real sequence such that $\lim_{n\longrightarrow+\infty} b_n = +\infty$, and

$$\lim_{n\longrightarrow+\infty} \frac{\sum_{k=1}^{N} \max_{(t_1,\ldots,t_m)\in K(b_n)} F(k, t_1, \ldots, t_m)}{b_n^2} = A. \tag{25}$$

It follows from [24] that

$$\sup_{k\in[1,N]} |u_i(k)| \leq \frac{(N+1)^{1/2}}{2p*^{1/2}} \|u_i\|_*, \tag{26}$$

for $1 \le i \le m$. And put $r_n := 2p * b_n^2/m(N+1)$ for all $n \in Z^+$. Hence a computation ensures that $\sum_{i=1}^m |u_i(k)| \le b_n$ whenever $u \in \Phi^{-1}(-\infty, r_n)$.

Taking into account the fact that $\sum_{k=1}^N F(k,0,\dots,0) = 0$, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&= \inf_{\sum_{i=1}^m(\|u_i\|_*^2/2)<r_n} \frac{\sup_{\sum_{i=1}^m(\|v_i\|_*^2/2)<r_n}\sum_{k=1}^N F(k, v_1(k),\dots,v_m(k)) - \sum_{k=1}^N F(k,u_1(k),\dots,u_m(k))}{r_n - \sum_{i=1}^m\left(\|u_i\|_*^2/2\right)} \\
&\le \frac{\sup_{\sum_{i=1}^m(\|v_i\|_*^2/2)<r_n}\sum_{k=1}^N F(k,v_1(k),\dots,v_m(k))}{r_n} \le \frac{m(N+1)}{2p*}\frac{\sum_{k=1}^N \max_{(t_1,\dots,t_m)\in K(b_n)} F(k,t_1,\dots,t_m)}{b_n^2}.
\end{aligned} \tag{27}$$

Therefore, since from assumption (ii) one has $A < +\infty$, we obtain

$$\gamma \le \liminf_{n\longrightarrow+\infty}\varphi(r_n) \le \frac{m(N+1)}{2p*}A < +\infty. \tag{28}$$

Now fix $\lambda \in ((P+Q)/2B, p*/2m(N+1)A)$. We claim that I_λ is unbounded from below. Let $\{\xi_{i,n}\}$ be m positive real sequences such that $\lim_{n\longrightarrow+\infty}(\sum_{i=1}^m \xi_{i,n}^2) = +\infty$, and

$$\lim_{n\longrightarrow+\infty}\frac{\sum_{k=1}^N F(k,\xi_{1,n},\dots,\xi_{m,n})}{\sum_{i=1}^m \xi_{i,n}^2} = B, \tag{29}$$

for all $n \in Z^+$.

For each $n \in Z^+$, let $\omega_{i,n}(k) := \xi_{i,n}$ for all $k \in [1,N]$, $\omega_{i,n}(0) = \omega_{i,n}(N+1) = 0$.

Clearly, $\omega_n = (\omega_{1,n},\dots,\omega_{m,n}) \in X$, and

$$\begin{aligned}
\|\omega_{i,n}\|_*^2 &= \sum_{k=1}^{N+1} p(k)\left|\Delta\omega_{i,n}(k-1)\right|^2 \\
&\quad + \sum_{k=1}^N q(k)\left|\omega_{i,n}(k)\right|^2 \\
&= p(1)\left|\omega_{i,n}(1)-\omega_{i,n}(0)\right|^2 + \cdots \\
&\quad + p(N+1)\left|\omega_{i,n}(N+1)-\omega_{i,n}(N)\right|^2 \\
&\quad + \sum_{k=1}^N q(k)\xi_{i,n}^2 \\
&= \left[p(1)+p(N+1)+\sum_{k=1}^N q(k)\right]\xi_{i,n}^2 \\
&= (P+Q)\xi_{i,n}^2.
\end{aligned} \tag{30}$$

Therefore, we have

$$\begin{aligned}
\Phi(\omega_n) - \lambda\Psi(\omega_n) &= \sum_{i=1}^m \frac{\|\omega_{i,n}\|_*^2}{2} \\
&\quad - \lambda\sum_{k=1}^N F(k,\xi_{1,n},\dots,\xi_{m,n}) \\
&= \frac{P+Q}{2}\sum_{i=1}^m \xi_{i,n}^2 \\
&\quad - \lambda\sum_{k=1}^N F(k,\xi_{1,n},\dots,\xi_{m,n}),
\end{aligned} \tag{31}$$

for all $n \in Z^+$.

If $B < +\infty$, let $\varepsilon \in ((P+Q)/2\lambda B, 1)$. By (29) there exists N_ε such that

$$\sum_{k=1}^N F(k,\xi_{1,n},\dots,\xi_{m,n}) > \varepsilon B\sum_{i=1}^m \xi_{i,n}^2, \tag{32}$$

for all $n > N_\varepsilon$. Moreover,

$$\begin{aligned}
\Phi(\omega_n) - \lambda\Psi(\omega_n) &< \frac{P+Q}{2}\sum_{i=1}^m \xi_{i,n}^2 - \lambda\varepsilon B\sum_{i=1}^m \xi_{i,n}^2 \\
&= \left(\frac{P+Q}{2} - \lambda\varepsilon B\right)\sum_{i=1}^m \xi_{i,n}^2,
\end{aligned} \tag{33}$$

for all $n > N_\varepsilon$. Taking into account the choice of ε, we have

$$\lim_{n\longrightarrow+\infty}\left[\Phi(\omega_n) - \lambda\Psi(\omega_n)\right] = -\infty. \tag{34}$$

If $B = +\infty$, let us consider $M > (P+Q)/2\lambda$. By (29) there exists N_M such that

$$\sum_{k=1}^N F(k,\xi_{1,n},\dots,\xi_{m,n}) > M\sum_{i=1}^m \xi_{i,n}^2, \tag{35}$$

for all $n > N_M$. Moreover,

$$\begin{aligned}\Phi(\omega_n) - \lambda\Psi(\omega_n) &< \frac{P+Q}{2}\sum_{i=1}^{m}\xi_{i,n}^2 - \lambda M\sum_{i=1}^{m}\xi_{i,n}^2 \\ &= \left(\frac{P+Q}{2} - \lambda M\right)\sum_{i=1}^{m}\xi_{i,n}^2,\end{aligned} \quad (36)$$

for all $n > N_M$. Taking into account the choice of M, in this case we also have

$$\lim_{n\longrightarrow+\infty}\left[\Phi(\omega_n) - \lambda\Psi(\omega_n)\right] = -\infty. \quad (37)$$

Due to Lemma 3, for each $\lambda \in ((P+Q)/2B, p*/2m(N+1)A)$, the functional I_λ admits an unbounded sequence of critical points, and the conclusion is proven. □

Remark 5. When $F_{t_i}(k, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m) \geq 0$ for all $k \in [1, N]$ and $i = 1, \ldots, m$, owing to Remark 2, the solutions in the conclusion of Theorem 4 are positive.

It is interesting to list some special cases of the above results.

Corollary 6. *Assume that*

(i′) F is nonnegative in $[1, N] \times R_+^m$

(ii′) $A < B/m(2+Q)(N+1)$

Then, for each $\lambda \in ((2+Q)/2B, 1/2m(N+1)A)$, the system

$$\begin{aligned}&-\Delta^2 u_i(k-1) + q(k)u_i(k) \\ &\quad = \lambda F_{u_i}(k, u_1(k), \ldots, u_m(k)), \quad k \in [1, N], \\ &u_i(0) = u_i(N+1) = 0,\end{aligned} \quad (38)$$

for $1 \leq i \leq m$, admits an unbounded sequence of solutions.

Corollary 7. *Let $F : R^m \longrightarrow R$ be a C^1-function and assume that*

(i″) F is nonnegative in R_+^m

(ii″) $C < (p*/m(P+Q)(N+1))D$,

where

$$C := \liminf_{y\longrightarrow+\infty}\frac{\max_{(t_1,\ldots,t_m)\in K(y)}F(t_1,\ldots,t_m)}{y^2} \quad (39)$$

and

$$D := \limsup_{\sum_{i=1}^m t_i^2\longrightarrow+\infty\,(t_1,\ldots,t_m)\in R_+^m}\frac{F(t_1,\ldots,t_m)}{\sum_{i=1}^m t_i^2}. \quad (40)$$

Then, for each $\lambda \in ((P+Q)/2D, p*/2m(N+1)C)$, the system

$$\begin{aligned}&-\Delta(p(k)\Delta u_i(k-1)) + q(k)u_i(k) \\ &\quad = \lambda F_{u_i}(u_1(k), \ldots, u_m(k)), \quad k \in [1, N], \\ &u_i(0) = u_i(N+1) = 0,\end{aligned} \quad (41)$$

for $1 \leq i \leq m$, admits an unbounded sequence of solutions.

Remark 8. When $F_{t_i}(k, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m) \geq 0$ for all $k \in [1, N]$ and $i = 1, \ldots, m$, owing to Remark 2, the solutions in the conclusion of Corollary 6 are positive.

Remark 9. When $F_{t_i}(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m) \geq 0$ for $i = 1, \ldots, m$, owing to Remark 2, the solutions in the conclusion of Corollary 7 are positive.

Now we give an example to illustrate our results.

Example 10. Let $m = 2$, $p(k) = q(k) = 1$ and consider the increasing sequence of positive real numbers given by

$$a_n := 2^{n(n+1)/2}, \quad (42)$$

for every $n \in Z^+$.

Define the function $F : R^2 \longrightarrow R$ as follows: If $(t_1, t_2) \in B((a_n, a_n), 1)$ for some positive integer n, then

$$F(t_1, t_2) = (a_n)^2\left[1 - (t_1 - a_n)^2 - (t_2 - a_n)^2\right]^2; \quad (43)$$

otherwise,

$$F(t_1, t_2) = 0, \quad (44)$$

where $B((a_n, a_n), 1)$ denotes the open unit ball of center (a_n, a_n).

By the definition of F, we see that it is nonnegative in R_+^2 and $F(0, 0) = 0$. Further it is a simple matter to verify that $F \in C^1(R^2)$. We will denote by $F_{t_1}(t_1, t_2)$ and $F_{t_2}(t_1, t_2)$, respectively, the partial derivative of $F(t_1, t_2)$ with respect to t_1 and t_2. Now, for every $n \in Z^+$, the restriction $F(t_1, t_2)|_{B((a_n,a_n),1)}$ attains its maximum in (a_n, a_n) and one has $F(a_n, a_n) = (a_n)^2$. Obviously,

$$\limsup_{|t_1|^2+|t_2|^2\longrightarrow+\infty\,(t_1,t_2)\in R_+^2}\frac{F(t_1, t_2)}{|t_1|^2 + |t_2|^2} = \frac{1}{2}, \quad (45)$$

owing to the fact that

$$\lim_{n\longrightarrow+\infty}\frac{F(a_n, a_n)}{2(a_n)^2} = \frac{1}{2}. \quad (46)$$

On the other hand, by setting $y_n = a_{n+1} - 1$ for every $n \in Z^+$, one has

$$\max_{(t_1,t_2)\in K(y_n)}F(t_1, t_2) = (a_n)^2, \quad \forall n \in Z^+. \quad (47)$$

Then

$$\lim_{n\longrightarrow+\infty}\frac{\max_{(t_1,t_2)\in K(y_n)}F(t_1, t_2)}{y_n^2} = 0, \quad (48)$$

and hence

$$\liminf_{y\longrightarrow+\infty}\frac{\max_{(t_1,t_2)\in K(y)}F(t_1, t_2)}{y^2} = 0. \quad (49)$$

Finally

$$0 = C < \frac{1}{2(N+2)(N+1)}D = \frac{1}{4(N+2)(N+1)}. \quad (50)$$

The previous observations and computations ensure that all the hypotheses of Corollary 7 are satisfied. Then, for each $\lambda \in (N+2, +\infty)$, the problem

$$\begin{aligned}
&-\Delta^2 u_1(k-1) + u_1(k) = \lambda F_{u_1}(u_1(k), u_2(k)), \\
&\qquad\qquad\qquad\qquad\qquad\qquad k \in [1, N], \\
&-\Delta^2 u_2(k-1) + u_2(k) = \lambda F_{u_2}(u_1(k), u_2(k)), \\
&\qquad\qquad\qquad\qquad\qquad\qquad k \in [1, N], \\
&u_1(0) = u_1(N+1) = u_2(0) = u_2(N+1) = 0,
\end{aligned} \tag{51}$$

admits an unbounded sequence of solutions.

Taking partial derivative to $F(t_1, t_2)$ with respect to t_1 gives

$$\begin{aligned}
&F_{t_1}(t_1, t_2) \\
&\quad = -4(t_1 - a_n)(a_n)^2 \left[1 - (t_1 - a_n)^2 - (t_2 - a_n)^2\right],
\end{aligned} \tag{52}$$

if $(t_1, t_2) \in B((a_n, a_n), 1)$ for some positive integer n; otherwise, $F_{t_1}(t_1, t_2) = 0$.

It is easy to see that

$$F_{t_1}(0, t_2) = 0. \tag{53}$$

In a similar way, we obtain

$$F_{t_2}(t_1, 0) = 0. \tag{54}$$

Consequently, according to Remark 9, problem (51) admits an unbounded sequence of positive solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11571084) and the Program for Changjiang Scholars and Innovative Research Team in University (Grant No. IRT-16R16).

References

[1] Y. Tian and W. Ge, "Multiple positive solutions of boundary value problems for second-order discrete equations on the half-line," *Journal of Difference Equations and Applications*, vol. 12, no. 2, pp. 191–208, 2006.

[2] C. Bereanu and J. Mawhin, "Existence and multiplicity results for nonlinear second order difference equations with Dirichlet boundary conditions," *Mathematica Bohemica*, vol. 131, no. 2, pp. 145–160, 2006.

[3] J. Chu and D. Jiang, "Eigenvalues and discrete boundary value problems for the one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 2, pp. 452–465, 2005.

[4] D. Jiang, J. Chu, D. O'Regan, and R. P. Agarwal, "Positive solutions for continuous and discrete boundary value problems to the one-dimension p-Laplacian," *Mathematical Inequalities and Applications*, vol. 7, no. 4, pp. 523–534, 2004.

[5] Z. M. Guo and J. S. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China Series A: Mathematics*, vol. 46, no. 4, pp. 506–515, 2003.

[6] Z. Zhou and J. Yu, "On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems," *Journal of Differential Equations*, vol. 249, no. 5, pp. 1199–1212, 2010.

[7] Z. Zhou, J. Yu, and Y. Chen, "Homoclinic solutions in periodic difference equations with saturable nonlinearity," *Science China Mathematics*, vol. 54, no. 1, pp. 83–93, 2011.

[8] Z. Zhou and J. S. Yu, "Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity," *Acta Mathematica Sinica, English Series*, vol. 29, no. 9, pp. 1809–1822, 2013.

[9] H. Shi, "Periodic and subharmonic solutions for second-order nonlinear difference equations," *Applied Mathematics and Computation*, vol. 48, no. 1-2, pp. 157–171, 2015.

[10] Z. Zhou and D. Ma, "Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials," *Science China Mathematics*, vol. 58, no. 4, pp. 781–790, 2015.

[11] G. Lin and Z. Zhou, "Homoclinic solutions of discrete ϕ-Laplacian equations with mixed nonlinearities," *Communications on Pure and Applied Analysis*, vol. 17, no. 5, pp. 1723–1747, 2018.

[12] X. H. Tang, "Non-Nehari manifold method for periodic discrete superlinear Schrödinger equation," *Acta Mathematica Sinica, English Series*, vol. 32, no. 4, pp. 463–473, 2016.

[13] Q. Q. Zhang, "Homoclinic orbits for a class of discrete periodic Hamiltonian systems," *Proceedings of the American Mathematical Society*, vol. 143, no. 7, pp. 3155–3163, 2015.

[14] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular and nonsingular discrete problems via variational methods," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 58, no. 1-2, pp. 69–73, 2004.

[15] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular discrete p-Laplacian problems via variational methods," *Advances in Difference Equations*, vol. 2005, no. 2, pp. 93–99, 2005.

[16] M. Mihăilescu, V. Rădulescu, and S. Tersian, "Eigenvalue problems for anisotropic discrete boundary value problems," *Journal of Difference Equations and Applications*, vol. 15, no. 6, pp. 557–567, 2009.

[17] M. Avci and A. Pankov, "Nontrivial solutions of discrete nonlinear equations with variable exponent," *Journal of Mathematical Analysis and Applications*, vol. 431, no. 1, pp. 22–33, 2015.

[18] M. Avci, "Existence results for anisotropic discrete boundary value problems," *Electronic Journal of Differential Equations*, vol. 148, pp. 1–11, 2016.

[19] M. Galewski, "A note on the existence of solutions for difference equations via variational methods," *Journal of Difference Equations and Applications*, vol. 17, no. 4, pp. 643–646, 2011.

[20] Z. Balanov, C. Garca-Azpeitia, and W. Krawcewicz, "On variational and topological methods in nonlinear difference equations," *Communications on Pure and Applied Analysis*, vol. 17, no. 6, pp. 2813–2844, 2018.

[21] L. Erbe, B. G. Jia, and Q. Q. Zhang, "Homoclinic solutions of discrete nonlinear systems via variational method," *Journal of Applied Analysis and Computation*, vol. 9, no. 1, pp. 271–294, 2019.

[22] Q. Zhang, "Homoclinic orbits for discrete Hamiltonian systems with local super-quadratic conditions," *Communications on Pure and Applied Analysis*, vol. 18, no. 1, pp. 425–434, 2019.

[23] G. Bonanno, P. Jebelean, and C. Serban, "Superlinear discrete problems," *Applied Mathematics Letters*, vol. 52, pp. 162–168, 2016.

[24] L. Jiang and Z. Zhou, "Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 345916, 10 pages, 2008.

[25] G. Bonanno and P. Candito, "Infinitely many solutions for a class of discrete non-linear boundary value problems," *Applicable Analysis: An International Journal*, vol. 88, no. 4, pp. 605–616, 2009.

[26] M. K. Moghadam and S. Heidarkhani, "Existence of a non-trivial solution for nonlinear difference equations," *Differential Equations and Applications*, vol. 6, no. 4, pp. 517–525, 2014.

[27] G. M. Bisci and D. Repovs, "Existence of solutions for p-Laplacian discrete equations," *Applied Mathematics and Computation*, vol. 242, pp. 454–461, 2014.

[28] Z. Zhou and M. Su, "Boundary value problems for 2n-order ϕ_c-Laplacian difference equations containing both advance and retardation," *Applied Mathematics Letters*, vol. 41, no. 1, pp. 7–11, 2015.

[29] G. D'Agua, J. Mawhin, and A. Sciammetta, "Positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 447, no. 1, pp. 383–397, 2017.

[30] M. K. Moghadam and M. Avci, "Existence results to a nonlinear $p(k)$-Laplacian difference equation," *Journal of Difference Equations and Applications*, vol. 23, no. 10, pp. 1652–1669, 2017.

[31] Z. Zhou and J. Ling, "Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with ϕ_c-Laplacian," *Applied Mathematics Letters*, vol. 91, no. 1, pp. 28–34, 2019.

[32] G. Bonanno and G. M. Bisci, "Infinitely many solutions for a boundary value problem with discontinuous nonlinearities," *Boundary Value Problems*, vol. 2009, Article ID 670675, pp. 1–20, 2009.

[33] B. Ricceri, "A general variational principle and some of its applications," *Journal of Computational and Applied Mathematics*, vol. 113, no. 1-2, pp. 401–410, 2000.

[34] G. Bonanno, G. M. Bisci, and D. O'Regan, "Infinitely many weak solutions for a class of quasilinear elliptic systems," *Mathematical and Computer Modelling*, vol. 52, no. 1-2, pp. 152–160, 2010.

[35] G. A. Afrouzi and A. Hadjian, "Infinitely many weak solutions for a class of two-point boundary value systems," *Journal of Nonlinear Analysis and Application*, vol. 2012, pp. 1–8, 2012.

[36] S. Shokooh, G. A. Afrouzi, and H. Zahmatkesh, "Infinitely many weak solutions for fourth-order equations depending on two parameters," *Boletim da Sociedade Paranaense de Matematica*, vol. 36, no. 4, pp. 131–147, 2018.

[37] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, New York, NY, USA, 1989.

[38] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, CRC Press, Boca Raton, Fla, USA, 2000.

[39] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, San Diego, Calif, USA, 2nd edition, 2001.