Research Article

Existence of a Nontrivial Steady-State Solution to a Parabolic-Parabolic Chemotaxis System with Singular Sensitivity

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Received 23 August 2018; Accepted 25 October 2018; Published 1 January 2019

Academic Editor: Rodica Luca

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This paper establishes the existence of a nontrivial steady-state solution to a parabolic-parabolic coupled system with singular (or logarithmic) sensitivity and nonlinear source arising from chemotaxis. The proofs mainly rely on the maximum principle, the implicit function theorem, and the Hopf bifurcation theorem.

1. Introduction

Chemotaxis is the biased movement of cells toward the concentration gradient of a chemical. It plays a critical role in a wide range of biological phenomena. For example, cells migrate toward resources of food and stay away from harmful substances. The first mathematical model of chemotaxis was introduced by Patlak in [1] and Keller and Segel in [2]. There are numerous works dedicated to the analysis of chemotaxis models. For example, Othmer and Stevens in [3] modeled myxobacteria as individual random walkers and proposed a microscopic model based on a velocity jump process. By taking the parabolic limit of the microscopic model, they obtain the macroscopic chemotaxis model, which is the well-known Keller-Segel system

\[ u_t = \nabla \cdot (\mu \nabla u) - \nabla \cdot (\chi u \nabla \Theta(v)) + f(u, v), \]
\[ \tau \Phi = d \Delta \Phi + g(u, v), \]

where \( \Omega \subset \mathbb{R}^n \) is a bounded connected domain with a smooth boundary \( \partial \Omega \). The function \( u = u(x, t) \) denotes the cell density and \( v = v(x, t) \) represents the chemical concentration, for example, oxygen. The constant \( \chi \) is called the chemotactic coefficient, and the sign of \( \chi \) corresponds to chemoattraction if \( \chi > 0 \) and chemorepulsion if \( \chi < 0 \). Parameters \( \mu \) and \( d \) are the diffusion coefficients of the cells and the chemical, respectively. The function \( f(u) \) represents the kinetic function describing production and degradation of cells, and \( \Theta(v) \) is commonly referred to as the chemotactic potential function. Function \( g(u, v) \) describes the production and degradation of the chemical.

The existence of global solutions, blow-up, and traveling wave solutions to the chemotactic system (1) were extensively studied during the past four decades (see, e.g., [4–12] and references therein). The authors studied the roles of growth, death and random in promoting population persistence through band population in [4], and the work was related to the significance of cell motility and chemotaxis in microbial ecology.

X.F. Wang addressed the trivial and nontrivial steady states with small \( \chi \) and \( \mu \) of the quasi-linear system (1) in [13]. In the absence of population dynamics (i.e., \( f(u) \equiv 0 \)), there have been extensive studies. The main feature of solutions to the Keller-Segel model is the possibility of blow-up in finite time in [9, 14, 15]. Moreover, recent results in [6, 10, 11, 16, 17] proved the global existence of solutions under...
some conditions. Moreover, the global existence, asymptotic
behavior, and steady states of classical solutions were studied
in [18] for the one-dimensional case.

The purpose of this paper is to study the existence of the
nontrivial steady-state solution to a parabolic-parabolic coupled
system arising from chemotaxis with singular sensitivity. We
consider the following system

\[ u_t = \varepsilon u_{xx} - \chi (u (\ln v))_x + (\mu H(v) - \sigma) u, \quad x \in \Omega, \; t > 0, \]

\[ v_t = v_{xx} - uH(v), \quad x \in \Omega, \; t > 0, \]

with the initial-boundary value conditions

\[ u(x, 0) = u_0 (x), \quad v(x, 0) = v_0 (x), \quad x \in \Omega, \]

\[ u'|_{x=0} = 0, \]

\[ (\varepsilon u' - \chi (u (\ln v'))|_{x=1} = 0, \]

\[ v'|_{x=0} = 0, \]

\[ (v' + v)|_{x=1} = 1, \]

\[ x \in \partial \Omega, \; t > 0, \]

where \( \Omega = [0, 1] \) and \( "\partial" = \partial/\partial x \). The function \( u = u(x,t) \)
denotes the density of the cells and \( v = v(x,t) \) denotes the
concentration of the chemical. The parameters \( \chi, \varepsilon, \mu, \) and \( \sigma \)
are all positive constants, with \( \varepsilon \) being the diffusion coefficient
of cells and \( \chi \) the chemotactic sensitivity coefficient as above.
The function \( H(v) \) is assumed to be smooth.

The steady-state problem corresponding to system (2)-(3) is

\[ \varepsilon u'' = \chi (u (\ln v'))' - (\mu H(v) - \sigma) u, \]

\[ x \in \Omega, \; t > 0, \]

\[ v'' = uH(v), \quad x \in \Omega, \; t > 0, \]

\[ u(x, 0) = u_0 (x), \]

\[ v(x, 0) = v_0 (x), \]

\[ x \in \Omega, \]

where \( \Omega = [0, 1] \). The linear chemotactic potential \( \Theta(v) = v \)
was considered in [19], and X.F. Wang’s model in [13] is
similar to ours.

In this paper, we establish the existence of nontrivial
steady-state solutions to the parabolic-parabolic coupled
chemotactic system (2)-(3) with singular sensitivity. Here we
assume that the function \( H(v) \) satisfies

\[ H(0) = 0, \]

\[ H'(v) > 0, \quad \text{as } v > 0. \]

Obviously, \( u \equiv 0 \) and \( v \equiv 1 \) are the trivial solutions to system
(4).

Our main results in this paper are presented in the
following theorem.

**Theorem 1.** Consider system (4) with condition (5). Then the
following alternative holds:

1. If \( \sigma \geq \mu H(1) \), then system (4) has a pair of nonnegative
   steady-state solutions \( u \equiv 0 \) and \( v \equiv 1 \).
2. If \( \sigma < \mu H(1) \), then there exists at least a pair of
   nontrivial steady-state solutions \( (u(x), v(x)) \).

The rest of this paper is organized as follows. In Section 2,
we give some preliminary lemmas. In Section 3, we complete
the proof of Theorem 1.

**2. Lemmas**

To prove Theorem 1, we need to establish boundary estimates
of solutions to system (4). Since the state variables represent
densities, we only consider nonnegative solutions, that is,
\( u(x) \geq 0 \) and \( v(x) \geq 0 \) on \([0,1]\). Firstly, we state one result
concerning the estimate of a solution \( (u,v) \).

**Lemma 2.** Assume the solution \( (u,v) \) is a nonnegative steady
state. Then we obtain the following:

\[ i) \ v' \geq 0, \quad \text{and} \quad 0 \leq v(x) \leq 1. \]
\[ ii) \ u' \geq 0. \]

**Proof.**

\[ i) \ \text{By the second equation of (4), we have } v'' \geq 0 \text{ and,}
\]
\[ \text{integrating once, we obtain } v' \geq 0. \text{ Hence, } v'(x) \geq 0 \]
\[ \text{and } v(x) \leq v(1). \]
\[ ii) \ \text{Integrating the first equation of (4) yields} \]
\[ u'(x) = \frac{1}{\varepsilon} \int_0^x u''(x) \, dx \]
\[ = \frac{1}{\varepsilon} \int_0^x [\chi (u (\ln v'))' - (\mu H(v) - \sigma) u] \, dx \]
\[ = \frac{\chi}{\varepsilon} u (\ln v') - \frac{1}{\varepsilon} \int_0^x (\mu H(v) - \sigma) u \, dx \geq 0. \]

**Lemma 3.** Let \( v(x) \) solve system (2) with the initial-boundary
value condition of (3). Then we have the lower-bound estimate

\[ \frac{1}{v(1)} \leq M, \quad \text{where } M = \left( G^{-1} \left( \int_0^1 u(s) \, ds \right) \right)^{-1}, \]

and \( G \) is defined in (11) below.
Proof. From the second equation in (4), we obtain
\[
\frac{v''(x)}{H'(v(x))} = u(x).
\]
Integrating the above equation once, we have
\[
\int_0^x \frac{v''(s)}{H'(v(s))} ds = \int_0^x u(s) ds.
\]
Integrating by part and applying the condition \(v'(0) = 0\), we get
\[
\int_0^x v''(s) ds = \left[ \frac{v'(s)}{v} \right]_0^x = v'(x).
\]
From the definition of the function \(V\) in (3), we have
\[
V(0) = 0.
\]
Proof. By the steady-state system (4), we know \(v'' \geq 0\), and for every \(x \in (0, 1)\), we have
\[
v'(x) = \int_0^x v''(x) dx \leq \int_0^1 v''(x) dx = v'(1).
\]
Applying the condition \(v''(1) + v(1) = 1\), we obtain the following equation
\[
v'(x) = \frac{1 - v(1)}{y}.
\]
Similarly, we also get
\[
\int_0^1 H(v) u dv = \int_0^1 v''(x) dx = v'(1) - v'(0) \leq \frac{1}{y}.
\]
On the other hand, integrating once the first equation of (4), we get
\[
\int_1^x eu'' ds = \int_1^x \left[ \frac{\chi u (\ln v)' - (\mu H(v) - \sigma) u}{v} \right] ds,
\]
which is equal to
\[
eu'(x) - eu'(1) dx = \chi u (\ln v)' - \chi u (\ln v)'|_{x=1}
\]
\[
- \int_1^x (\mu H(v) - \sigma) u dv.
\]
Then, applying the boundary condition \((eu' - \chi u (\ln v)'|_{x=1} = 0\) and rewriting (18), we have the following estimate
\[
u'(x) = \frac{1}{\varepsilon} \left[ \chi u (\ln v)' - \int_1^x (\mu H(v) - \sigma) u dx \right]
\]
\[
= \frac{1}{\varepsilon} \left[ \chi u V' - \int_1^x (\mu H(v) - \sigma) u dx \right]
\]
\[
\leq \frac{1}{\varepsilon} \left[ \chi u V' + \int_0^1 (\mu H(v) u) dv \right].
\]

Next, we review the condition \((yv' + v)|_{x=1} = 1\), which implies that \(v'(1) \leq 1\). Combined with the condition of \(1/v \leq M\) in Lemma 3, inequality (19) can be rewritten as
\[
u'(x) \leq \frac{\chi M}{\varepsilon y} u(x) + \frac{1}{\varepsilon} \int_0^1 (\mu H(v) u) dv
\]
\[
\leq \frac{\chi M}{\varepsilon y} u(x) + \mu.
\]
Integrating (20) once, we have
\[
\int_0^x u'(s) ds \leq \int_0^x \left( u(s) + \frac{\mu}{\chi M} \right) ds
\]
\[
\leq C_1 \int_0^1 (u(s) + C_2) ds.
\]
Here \(C_1 = \chi M/\varepsilon y\) and \(C_2 = \mu/\chi M\). Hence, we obtain
\[
u(x) - u(0) \leq C_1 \int_0^x (u(s) + C_2) ds
\]
which implies that
\[
u(x) \leq u(0) + C_1 \int_0^x (u(s) + C_2) ds.
\]
Define the function \(S(x) = u(x) + C_2\). Then we have \(S(0) = u(0) + C_2\). Combining the above inequalities, we obtain the following bound
\[
S(x) \leq S(0) + C_1 \int_0^x S(s) ds.
\]
By Gronwall’s inequality, we get
\[
S(x) \leq S(0) \exp \{C_1 x\}.
\]
Substituting the definition of \(S(x)\) into inequality (25), we obtain
\[
u(x) + C_2 \leq (u(0) + C_2) \exp \{C_1 x\}.
\]
Combining this result with \(u'(x) \geq 0\) in Lemma 2 (ii), we obtain estimate (13).

Next, we suppose that \(F(u) = \max[0, u]\) for any \(u(x) \in C^0[0, 1]\). Given \(u \in C^0[0, 1]\), we consider the equation
\[
v'(x) = F(u) H(v),
\]
\[
\left( yv' + v \right)|_{x=1} = 1.
\]
Then, we prove the following lemma.
Lemma 5. For any small $\sigma > 0$, we pick a sufficiently small $\varepsilon > 0$ such that $\mu H(\varepsilon) < \sigma$. Next, we choose

$$x_0 = \frac{2(\mu H(1) - \sigma)}{2(\mu H(1) - \sigma) - C(\mu H(\varepsilon) - \sigma)}; \quad (0 < x_0 < 1)$$

and set

$$N = \max \left\{ \frac{2D}{C}, \frac{4}{CH(1)(1-x_0)^2} \right\}.$$ (29)

Finally, we denote by $(u(x), v(x))$ a nonnegative solution to the system (4) when $\sigma \in [\sigma_0, +\infty)$. Then we have $|u(x)| \leq N$.

Proof. If not, there exists some $\sigma_0 \in [\sigma_0, +\infty)$ such that $u(x) > N$. According to Lemma 2 and the definition of $N$, we have

$$u(1) \geq |u(x)| > N > \frac{2D}{C}. \quad (30)$$

From (13), we can easily get

$$u(0) \geq Cu(1) - D > Cu(1) - \frac{Cu(1)}{2} = \frac{Cu(1)}{2}. \quad (31)$$

However, $v(x_0) < \delta$. In fact, if not we have

$$v(1) - v(x_0) = v'(x_0)(1 - x_0) + \frac{1}{2}v''(x_0)(1 - x_0)^2$$

$$+ \cdots (1 - x_0)^n$$

$$\geq \frac{1}{2}H(v(\delta))u(\delta)(1 - x_0)^2$$

$$\geq \frac{1}{2}H(v(x_0))u(0)(1 - x_0)^2$$

$$\geq \frac{1}{2}H(H(1)(1 - x_0)^2$$

$$\geq \frac{1}{4}H(H(1)(1 - x_0)^2 > 1$$

which contradicts the fact that $v(x) \leq 1$. Next, we consider

$$0 = \int_0^1 (\mu H(v) - \sigma) u dx = \int_0^{x_0} (\mu H(v) - \sigma) u dx$$

$$+ \int_0^1 (\mu H(v) - \sigma) u dx < (\mu H(\varepsilon) - \sigma)u(0)x_0$$

$$+ (\mu H(1) - \sigma)u(1)(1 - x_0) < (\mu H(\varepsilon) - \sigma)u(0)$$

$$\cdot x_0 + (\mu H(1) - \sigma) \frac{2u(0)}{C}(1 - x_0)$$

$$= \frac{u(0)}{C} [2(\mu H(1) - \sigma)(1 - x_0)$$

$$+ Cx_0(\mu H(\varepsilon) - \sigma)] = \frac{u(0)}{C} [2(\mu H(1) - \sigma)$$

$$- x_0(2(\mu H(1) - \sigma) - C) \mu H(\varepsilon) - \sigma)] = 0$$

which is a contradiction. In other words, we have $|u(x)| \leq N$. \(\square\)

Lemma 6.

(i) For each $u(x) \in C^0[0, 1]$, system (27) has a unique solution $v = A_u$.

(ii) $\mathcal{A} : u \rightarrow A_u$ is continuous.

(iii) If $u_1(x) \geq u_2(x)$, then $A_{u_1} \leq A_{u_2}$.

Proof. For any given $u(x) \in C^0[0, 1]$, it is obvious that $v(x) = 1$ and $v(x) = 0$ are a pair of sup-subsolutions. According to the standard comparison theorem, we easily obtain the estimate of the solution to system (27). If $v_1(x)$ and $v_2(x)$ are solutions to system (27), then we have

$$v''_1 = F(u)H(v_1),$$

$$v''_2 = F(u)H(v_2).$$

The difference between the first and second equation of (34) is

$$- (v_1 - v_2)^{''} + F(u)H(v_1 - v_2) = 0,$$

$$\left(v_1 - v_2\right)\bigg|_{x=0} = 0,$$

$$\left(v_1 - v_2\right)\bigg|_{x=1} = 0.$$ (35)

with the notation $H_\sigma = dH/d\sigma$.

By the maximum principle, we obtain $v_1 \equiv v_2$. Moreover, $A_u$ is uniquely defined and $0 \leq A_u \leq 1$.

Next we give the proof of (ii) and (iii) in Lemma 6, respectively.

(i) Assume that there exists a sequence $(u_n(x))_{n=1}^\infty$, which converges to $u(x)$. Then we only point out that there exists a subsequence $A_{u_{n_k}}$ of $A_u$, such that $A_{u_{n_k}} \rightarrow A_u$ (on $C^0[0, 1]$). By regularity theory, we know that $u(x)$ is the solution to system (34) and $u(x) \neq A_u$, which contradicts the uniqueness of $A_u$.

(ii) We can directly prove (iii) from the maximum principle. \(\square\)

3. Proof of Theorem 1

We dedicate this section to the proof of Theorem 1. Assume there exists $u(x) > 0$ such that $(u, A_u)$ satisfies the following system

$$eu'' = \chi (u \ln (A_u))_x - (\mu H(A_u) - \sigma)u,$$

$x \in (0, 1)$,
We can reformulate (39) as the following operator equation:

\[ (\varepsilon u' - \chi u (\ln (\lambda u))') |_{x=1} = 0. \]  

(36)

It is easy to see that \( \lambda u \) is the nonnegative solution to system (36) by the definition of \( \lambda \) and \( H(\lambda u) \).

Define \( R \psi(\overline{u}) \) to be the unique solution to the system

\[ -\varepsilon u'' + (q - \mu H(1)) u = \varphi(x), \quad x \in (0,1), \]

\[ u'|_{x=0} = 0, \]

\[ u'|_{x=1} = \psi(\overline{u}), \]

where \( q \) is a constant satisfying \( q > \mu H(1) \). For any given \( \varphi(x) \in C^0[0,1] \), we define \( K \varphi \) to be the unique solution to the following system

\[ -\varepsilon u'' + (q - \mu H(1)) u = \varphi(x), \quad x \in (0,1), \]

\[ u'|_{x=0} = 0, \]

\[ u'|_{x=1} = 0. \]

Then we rewrite (36) and obtain the following operator equation

\[ u = K \varphi + R \psi(u) = K \left[ -\varepsilon u'' + (q - \mu H(1)) u \right] \]

\[ + R \psi(u) = K \left[ -\chi (u (\ln (\lambda u))_x)_x + (\mu H(\lambda u) - \mu H(1)) u \right] \]

\[ + R \psi(u). \]

(38)

We can reformulate (39) as the following operator equation

\[ u = (q - \sigma) K u \]

\[ + K \left[ -\chi (u (\ln (\lambda u))_x)_x + \mu (H(\lambda u) - H(1)) u \right] \]

\[ + R \psi(u). \]

(40)

Next, we introduce some notation:

\[ E = C^1[0,1], \]

\[ F = \{ u(x) \in E; \ u(x) > 0, \ x \in [0,1]\}. \]

(41)

It is easy to see that the operator \( K : E \to E \) is compact and linear.

Define the operator \( L \) as the map

\[ L u = -\chi (u (\ln (\lambda u))_x)_x + \mu (H(\lambda u) - H(1)) u. \]

(42)

Similar to the proof of Lemma 5(i), we can show that the operator \( L \) is continuous and bounded from \( E \) to \( E \). We also have

\[ \| L u \|_E = o(\| u \|_E) \] as \( \| u \|_E \to 0. \)

(43)

Hence, \( K \circ L \) is compact operator from \( E \) to \( E \) with the norm \( \| K \circ L u \|_E = o(\| u \|_E) \). Moreover, the operator \( R \) is also compact from \( E \) to \( E \) and \( \| (R \ast \psi(u)) \|_E = o(\| u \|_E) \).

From the above lemmas, we can show the proof of Theorem 1.

**Proof of Theorem 1.** (1) Assume that \((u, \psi) \neq (1,0)\) is the nonnegative solution to system (2). Then we have

\[ \int_0^1 (\mu H(\psi) - \sigma) u \, dx = \int_0^1 [\chi (u (\ln \varphi))_x - \varepsilon \varphi] \, dx \]

\[ = 0. \]

(44)

However, from the proof of Lemma 3, we have

\[ \int_0^1 (\mu H(\psi) - \sigma) u \, dx < \int_0^1 (\mu H(1) - \sigma) u \, dx \leq 0 \]

(45)

which is a contradiction. Thus, we obtain \((u, \psi) \equiv (1,0)\).

(2) Assume that there is a constant \( \sigma \) such that \( \sigma > \mu H(1) \).

Next, we choose a variable \( t = \psi - \sigma \) as the parameter of bifurcation. Furthermore, \( \sigma - \mu H(1) \) is a single-value characteristic root. Thus we know that system produces a bifurcation in a small neighbourhood of the point \((\sigma - \mu H(1), 0)\) according to the classical bifurcation theorem. Then there is a bifurcating solution \((\tilde{u}, \tilde{\varphi})\), represented as

\[ i = i(\zeta), \]

\[ \tilde{\varphi}(\zeta) = \zeta (u_0 + v(\zeta)), \]

(46)

where \( |\zeta| \leq \varepsilon \) and \( \varepsilon \) is sufficiently small. Furthermore, \( i(0) = \sigma - \mu H(1) \) and \( \psi(0) = 0 \). Therefore, when \( \zeta \) is sufficiently small, the bifurcation solution is in the set of

\[ S^+ = \{ i(\zeta) \mid \zeta u_0 + \zeta \psi(\zeta), \ 0 < \zeta < \varepsilon \}. \]

(47)

where \( u_0 \) is the characteristic eigenvector of characteristic eigenvalue \( (\sigma - \mu H(1)) \) and \( u_0 \) is a positive constant. There is a nontrivial connected set \( \overline{S} \subseteq R \times E \) satisfying the following alternative conclusion:

(i) \( \overline{S} \) connects the two points \((\sigma - \mu H(1), 0)\) and \((\mu, 0)\), where \( \mu \) is another characteristic eigenvalue of \( K \).

(ii) \( \overline{S} \) is unbounded and \( \overline{S} \subseteq R \times E \).

We firstly prove \( \overline{S} \subseteq R \times E \). Otherwise, there exists a point \((\overline{u}, \overline{\varphi}) \in \overline{S} \cap (R \times \partial E)\) such that \((\overline{u}, \overline{\varphi}) \neq (\sigma - \mu H(1), 0)\) and

\[ \lim_{n \to \infty} (t_n, u_n) = (\overline{u}, \overline{\varphi}), \quad (t_n, u_n) \subseteq (\overline{S} \cap (R \times \partial E)). \]

(48)

Choose a sufficient large constant \( L \). Then by (36), we can obtain

\[ \varepsilon u'' - \chi (\ln (\lambda u))_x u' - Lu = [\chi (\ln (\lambda u))_x - (\mu H(\ln (\lambda u)) - \sigma) - L] u \leq 0. \]

(49)

However, \( \overline{u} \in \partial E \) which means that \( \overline{u} \) is equal to zero at some point. We can directly have \( \overline{u} \equiv 0 \) by the maximum principle. That is, \((0, \overline{\varphi})\) is a bifurcation point of (40).
Denote $\omega_n = u_n/\|u_n\|_E$. Because $K$ is a compact operator, there exists a convergent subsequence of $\omega_n$. For the convenience, we still write it as $\omega_n$ which satisfies $K\omega_n \rightarrow z$. Next, we divide by $\|u_n\|_E$ both sides of (40) and let $n$ go to infinity. Then we obtain

$$\bar{z} = iK\bar{z},$$

(50)

where $\bar{z} = K^{-1}Z$. By the definition of $z$ we know that $\bar{z} > 0$. However, $K$ has a unique characteristic eigenvalue with positive characteristic function. Thus, we have $i = \sigma - \mu H(1)$ which contradicts the assumption. So conclusion (i) is impossible. In other words, $\overline{S}$ is unbounded and $\overline{S} \subseteq R \times E$.

According to (i) in Theorem 1, we know that if $(u, t) \in \overline{S}$, then $i = \sigma > \mu H(1)$, which is equivalent to the condition $\sigma < \mu H(1)$. However, from Lemma 5, we have that $u$ is bounded for any $\sigma > 0$. By Lemma 4, we have that $\|u(x)\|_E$ is bounded. Furthermore, because the set $\overline{S}$ is unbounded and connected, we can obtain that $(\sigma - \mu H(1)) \in \{(u, t), i \in \overline{S}, \text{which means that when } \sigma < \mu H(1), \text{system (4)} \text{has nontrivial steady-state solution } (u, v)$. Finally, we show that system (2)-(3) has a positive solution $(u(x), v(x))$ when $x \in [0, 1]$. In fact, we know that $u(x) > 0$ because $u(x) \in E$. Next we prove that $v(x) > 0$. If $v(x)$ is zero at some point, then the initial value problem $v'' = uH(v), v(0) = 0, v'(0) = 0$ has a solution $v \equiv 0$, which contradicts the boundary condition $(yv' + v)|_{x=1} = 1$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (No. 41701054), the China Postdoctoral Science Foundation (No. 2018M631890), the Scientific and Technological Research Project of Jilin Province’s Education Department (No. 2016285), Teaching Reform of Higher Education Research Project of Jilin Province (No. SJYB17-01). Finally, the author is grateful for the support by the China Scholarship Council (No. 201707535005).

References


