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Research Article

Some Types of Convergence for Negatively Dependent Random Variables under Sublinear Expectations

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In this paper, we research complete convergence and almost sure convergence under the sublinear expectations. As applications, we extend some complete and almost sure convergence theorems for weighted sums of negatively dependent random variables from the traditional probability space to the sublinear expectation space.

1. Introduction

As is well known that limit theorems are important research topics in probability and statistics, classical limit theorems only hold in the case of model certainty. However, in financial and other field research, statistical model uncertainty, risk random fluctuation, and other problems are emerging, and the classical probabilistic space cannot meet the need of solving such problems. Academician Peng [1–3] of Shandong University creatively proposed the concept and system of the sublinear expectation. Since it was introduced, it has been widely concerned and studied by a large number of scholars, and many excellent results have been achieved. For instance, Chen [4] studied Kolmogorov's strong law of larger numbers. Zhang and Chen [5] established a weighted central limit theorem for independent random variables under sublinear expectation. Zhang [6–8] got the expectation inequalities, Rosenthals inequalities, and strong law of larger numbers. Yu and Wu [9] obtained complete convergence for weighted sums of END random variables under sublinear expectations, and so on.

Complete convergence and almost sure convergence are important problems in limit theorems. The initial definition of complete convergence was presented by Hsu and Robbins [10] in 1947. From then on, lots of results on complete convergence for different sequences have been found under classical probability space. For instance, Wu and Jiang [11] obtained complete convergence for negatively associated sequences of random variables. Wang et al. [12, 13] studied complete convergence for martingale difference sequence and complete convergence for a type of random variables satisfying Rosenthal-type inequality. Some results on almost sure convergence can be found in Sung et al. [14, 15], Wu [16], Chen and Gan [17], Xu and Yu [18], and so on. In this article, we establish the complete convergence and almost sure convergence for weighted sums of ND random variables under sublinear expectations. The results obtained by Sung [15] have been generalized to the sublinear expectation space.

2. Preliminaries

We also use the framework and notations of Peng [1–3]. Here, we do not discuss notations, concepts, and properties of the sublinear expectation space in detail. By the properties of sublinear expectations and capacity, we can get Markov inequality:

$$\forall X \in \mathscr{H},\ \mathbb{V}(|X| \geq x) \leq \frac{\widehat{\mathbb{E}}\left(|X|^{p}\right)}{x^{p}}, \quad \forall x > 0,\ p > 0. \tag{1}$$

Definition 1 ([6]). (i) (Negative dependence) [6] In a sublinear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, a random vector $\mathrm{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathscr{H}$ is said to be negatively dependent (ND) to another random vector $\mathrm{X} = (X_1, \ldots, X_m)$, $X_i \in \mathscr{H}$ under $\widehat{\mathbb{E}}$ if for each pair of test functions $\varphi_1 \in C_{l,Lip}(\mathbb{R}_m)$ and

$\varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$ we have $\widehat{\mathbb{E}}(\varphi_1(X)\varphi_2(Y)) \le \widehat{\mathbb{E}}(\varphi_1(X))\widehat{\mathbb{E}}(\varphi_2(Y))$, whenever φ_1, φ_2 are coordinate-wise nondecreasing or φ_1, φ_2 are coordinate-wise nonincreasing with $\varphi_1(X) \ge 0$, $\widehat{\mathbb{E}}\varphi_2(Y) \ge 0$, $\widehat{\mathbb{E}}|\varphi_1(X)\varphi_2(Y)| < \infty$, $\widehat{\mathbb{E}}|\varphi_1(X)| < \infty$, $\widehat{\mathbb{E}}|\varphi_2(Y)| < \infty$.

(ii) (ND random variables) [6] Let $\{X_n;\ n \ge 1\}$ be a sequence of random variables in the sublinear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$. $X_1, X_2, \ldots$ are said to be negatively dependent if X_{i+1} is negatively dependent to $(X_1, \ldots, X_i)$ for each $i \ge 1$.

Obviously, if $\{X_n;\ n \ge 1\}$ is a sequence of ND random variables and functions $f_1(x), f_2(x), \ldots \in C_{l,Lip}(\mathbb{R})$ are all nondecreasing (resp., all nonincreasing), then $\{f_n(X_n);\ n \ge 1\}$ is also a sequence of ND random variables.

The symbol c stands for a positive constant which may have different values in different places. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \le cb_n$ for sufficiently large n.

Lemma 2. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of row-wise ND random variables with $\widehat{\mathbb{E}}X_{ni} = 0$. Suppose that*

(a) $\max_{1\le i\le n}|X_{ni}| = O((\log n)^{-1})$,

(b) $\sum_{i=1}^{n} \widehat{\mathbb{E}}X_{ni}^2 = o((\log n)^{-1})$.

Then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) < \infty. \tag{2}$$

Proof. Taking $b_n = (\log n)^{-1}$ in Lemma 2.2 of [9], we can get conditions (a) and (b). The condition (iii) of Lemma 2.2 of [9] is clearly satisfied for $\alpha > 1$ and $b_n = (\log n)^{-1}$. So, we can get Lemma 2. □

Lemma 3. *Assume $X \in \mathscr{H}$, $p > 1$, $c > 0$. Then*

$$C_{\mathbb{V}}(|X|^p) < \infty \Longleftrightarrow \sum_{i=1}^{\infty} \mathbb{V}\left(|X| > ci^{1/p}\right) < \infty. \tag{3}$$

$$C_{\mathbb{V}}(|X|^p) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} n\mathbb{V}\left(|X| > cn^{2/p}\right) < \infty. \tag{4}$$

Proof. $C_{\mathbb{V}}(|X|^p) < \infty$ is equivalent to $C_{\mathbb{V}}(|X|^p/c^p) < \infty$. Note that

$$\begin{gathered} C_{\mathbb{V}}\left(\frac{|X|^p}{c^p}\right) = \int_0^{\infty} \mathbb{V}\left(|X| > cx^{1/p}\right) \mathrm{d}x < \infty \Longleftrightarrow \\ \sum_{i=1}^{\infty} \mathbb{V}\left(|X| > ci^{1/p}\right) < \infty. \end{gathered} \tag{5}$$

Therefore, (3) holds. Then, let $x = y^2$

$$\begin{aligned} C_{\mathbb{V}}\left(\frac{|X|^p}{c^p}\right) &= \int_0^{\infty} \mathbb{V}\left(|X| > cx^{1/p}\right) \mathrm{d}x \\ &= \int_0^{\infty} 2y\mathbb{V}\left(|X| > cy^{2/p}\right) \mathrm{d}y < \infty \Longleftrightarrow \\ &\sum_{n=1}^{\infty} n\mathbb{V}\left(|X| > cn^{2/p}\right) < \infty. \end{aligned} \tag{6}$$

Thus, (4) holds. □

Lemma 4 (Borel-Cantelli Lemma, Zhang [6], Lemma 3.9). *Let $\{A_n;\ n \ge 1\}$ be a sequence of events in $\mathscr{F}$. Suppose that V is a countably subadditive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then $V(A_n; i.o.) = 0$, where $A_n; i.o. = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m$.*

In the sublinear expectation space, because of the uncertainty of expectation and capacity, the study of complete convergence and almost sure convergence is much more complex and difficult. As we all know, in the probability space, there is an equality: $EI(|X| \le a) = P(|X| \le a)$. However, the expression $\widehat{\mathbb{E}}I(|X| \le a)$ does not exist in the sublinear expectation space. This needs to modify the indicator function by functions in $C_{l,Lip}$. To this end, for $0 < \mu < 1$, we define the even function $g(x) \in C_{l,Lip}(\mathbb{R})$, $0 \le g(x) \le 1$ for all x, and $g(x) = 1$ if $|x| \le \mu$, $g(x) = 0$ if $|x| > 1$. Then

$$\begin{gathered} I(|x| \le \mu) \le g(x) \le I(|x| \le 1), \\ I(|x| > 1) \le 1 - g(x) \le I(|x| > \mu). \end{gathered} \tag{7}$$

3. Main Results

Theorem 5. *Let $p \ge 1$, and $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of row-wise negatively dependent random variables under sublinear expectations. There exist a r.v. X and a constant c satisfying*

$$\begin{gathered} \widehat{\mathbb{E}}(h(X_{ni})) \le c\widehat{\mathbb{E}}(h(X)), \\ \text{for all } n \ge 1,\ 1 \le i \le n,\ 0 \le h \in C_{l,Lip}(\mathbb{R}), \end{gathered} \tag{8}$$

$$\widehat{\mathbb{E}}(|X|^p) \le C_{\mathbb{V}}(|X|^p) < \infty. \tag{9}$$

Let $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of positive constants such that

$$\sum_{i=1}^{n} {a_{ni}}^p = O\left(\frac{1}{n^{\alpha}}\right), \quad \text{for some } \alpha > 0. \tag{10}$$

$$\sum_{i=1}^{n} a_{ni}^2 \widehat{\mathbb{E}}X^2 g(a_{ni}X\log n) = o\left((\log n)^{-1}\right). \tag{11}$$

Then, for any positive whole number N satisfying $\alpha N > 1$, and $\forall\varepsilon > 0$,

$$\begin{gathered} \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\mathbb{E}}X_{ni}\right) I\left(\left|a_{ni}X_{ni}\right| \le \frac{\varepsilon}{N}\right) > \varepsilon\right) \\ < \infty, \end{gathered} \tag{12}$$

and

$$\begin{gathered} \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\varepsilon}X_{ni}\right) I\left(\left|a_{ni}X_{ni}\right| \le \frac{\varepsilon}{N}\right) < -\varepsilon\right) \\ < \infty. \end{gathered} \tag{13}$$

In particular, if $\widehat{\mathbb{E}}X_{ni} = \widehat{\varepsilon}X_{ni}$, then

$$\begin{gathered} \sum_{n=1}^{\infty} \mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\mathbb{E}}X_{ni}\right) I\left(\left|a_{ni}X_{ni}\right| \le \frac{\varepsilon}{N}\right)\right| > \varepsilon\right) \\ < \infty. \end{gathered} \tag{14}$$

Remark 6. Condition (8) is similar to stochastic domination condition in Theorem 2.1 of [15]. Condition (9) corresponds to the moment condition of probability space. Condition (11) is similar to condition (ii) of Theorem 2.1 of [15]. Our results extend Theorem 2.1 of [15] from the traditional probability space to sublinear expectation space.

Theorem 7. *Suppose that $p \geq 1$, and $\mathbb{V}$ is countably subadditive. Let $\{X_n, n \geq 1\}$ be a sequence of negatively dependent random variables. There exist a r.v. X and a constant c satisfying*

$$\widehat{\mathbb{E}}\left(h\left(X_n\right)\right) \leq c\widehat{\mathbb{E}}\left(h\left(X\right)\right), \quad \text{for all } n \geq 1,\ 0 \leq h \in C_{l,Lip}\left(\mathbb{R}\right), \tag{15}$$

$$\widehat{\mathbb{E}}\left(|X|^p\right) \leq C_{\mathbb{V}}\left(|X|^p\right) < \infty. \tag{16}$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive constants such that

$$\max_{1\leq i\leq n} a_{ni} = O\left(\frac{1}{n^{1/p}}\right), \tag{17}$$

$$\sum_{i=1}^{n} a_{ni}^{p} = O\left(\frac{1}{n^{\alpha}}\right), \quad \text{for some } \alpha > 0, \tag{18}$$

$$\sum_{i=1}^{n} a_{ni}^{2}\widehat{\mathbb{E}}X^{2}g\left(a_{ni}X\log n\right) = o\left(\left(\log n\right)^{-1}\right). \tag{19}$$

Then

$$\limsup_{n\longrightarrow\infty}\sum_{i=1}^{n} a_{ni}\left(X_i - \widehat{\mathbb{E}}X_i\right) \leq 0 \quad a.s.\mathbb{V}, \tag{20}$$

and

$$\liminf_{n\longrightarrow\infty}\sum_{i=1}^{n} a_{ni}\left(X_i - \widehat{\varepsilon}X_i\right) \geq 0 \quad a.s.\mathbb{V}. \tag{21}$$

In particular, if $\widehat{\mathbb{E}}X_i = \widehat{\varepsilon}X_i$, then

$$\lim_{n\longrightarrow\infty}\sum_{i=1}^{n} a_{ni}\left(X_i - \widehat{\mathbb{E}}X_i\right) = 0 \quad a.s.\mathbb{V}. \tag{22}$$

Theorem 8. *Suppose that $1 \leq p \leq 2$, and $\mathbb{V}$ is countably subadditive. Let $\{X_n, n \geq 1\}$ be a sequence of negatively dependent random variables. There exist a r.v. X and a constant c satisfying (15), (16). Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive constants such that (17), (18) hold. Then we also can get (20), (21), and (22).*

Theorem 9. *Suppose that $p \geq 1$, and $\mathbb{V}$ is countably subadditive. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of row-wise negatively dependent random variables. There exist a r.v. X and a constant c satisfying (8) and (9). Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive constants such that*

$$\max_{1\leq i\leq n} a_{ni} = O\left(\frac{1}{n^{2/p}}\right), \tag{23}$$

$$\sum_{i=1}^{n} a_{ni}^{2}\widehat{\mathbb{E}}X^{2}g\left(a_{ni}X\log n\right) = o\left(\left(\log n\right)^{-1}\right). \tag{24}$$

Then

$$\limsup_{n\longrightarrow\infty}\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\mathbb{E}}X_{ni}\right) \leq 0 \quad a.s.\mathbb{V}, \tag{25}$$

and

$$\liminf_{n\longrightarrow\infty}\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\varepsilon}X_{ni}\right) \geq 0 \quad a.s.\mathbb{V}. \tag{26}$$

In particular, if $\widehat{\mathbb{E}}X_{ni} = \widehat{\varepsilon}X_{ni}$, then

$$\lim_{n\longrightarrow\infty}\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\mathbb{E}}X_{ni}\right) = 0 \quad a.s.\mathbb{V}. \tag{27}$$

Remark 10. Theorems 7–9 extend Corollary 2.1, Corollary 2.2, and Corollary 2.5 of Sung [15] from the probability space to sublinear expectation space.

Proof of Theorem 5. To prove our main results, we just need to prove (12). Because of considering $\{-X_{ni}, 1 \leq i \leq n, n \geq 1\}$ instead of $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ in (12), we can get (13). Without loss of generality, we can assume that $\widehat{\mathbb{E}}X_{ni} = 0$.

For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$\begin{aligned}
X_{ni}(1) &= X_{ni}I\left(\left|a_{ni}X_{ni}\right| \leq \left(\log n\right)^{-1}\right) + a_{ni}^{-1}\left(\log n\right)^{-1} \\
&\quad \cdot I\left(a_{ni}X_{ni} > \left(\log n\right)^{-1}\right) - a_{ni}^{-1}\left(\log n\right)^{-1} \\
&\quad \cdot I\left(a_{ni}X_{ni} < -\left(\log n\right)^{-1}\right), \\
X_{ni}(2) &= \left(X_{ni} - a_{ni}^{-1}\left(\log n\right)^{-1}\right) \\
&\quad \cdot I\left(\left(\log n\right)^{-1} < a_{ni}X_{ni} \leq \frac{\varepsilon}{N}\right), \\
X_{ni}(3) &= \left(X_{ni} + a_{ni}^{-1}\left(\log n\right)^{-1}\right) \\
&\quad \cdot I\left(\frac{-\varepsilon}{N} \leq a_{ni}X_{ni} < -\left(\log n\right)^{-1}\right), \\
X_{ni}(4) &= -a_{ni}^{-1}\left(\log n\right)^{-1} I\left(a_{ni}X_{ni} > \frac{\varepsilon}{N}\right) \\
&\quad + a_{ni}^{-1}\left(\log n\right)^{-1} I\left(a_{ni}X_{ni} < -\frac{\varepsilon}{N}\right).
\end{aligned} \tag{28}$$

Then, we have

$$\begin{aligned}
&a_{ni}X_{ni}I\left(\left|a_{ni}X_{ni}\right| \leq \frac{\varepsilon}{N}\right) \\
&\quad = a_{ni}X_{ni}(1) + a_{ni}X_{ni}(2) + a_{ni}X_{ni}(3) + a_{ni}X_{ni}(4).
\end{aligned} \tag{29}$$

On account of the arbitrariness of ε, it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}I\left(\left|a_{ni}X_{ni}\right|\le\frac{\varepsilon}{N}\right)>4\varepsilon\right)\\
&\quad\le\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}(1)>\varepsilon\right)\\
&\qquad+\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}(2)>\varepsilon\right)\\
&\qquad+\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}(3)>\varepsilon\right)\\
&\qquad+\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}(4)>\varepsilon\right)\coloneqq I_1+I_2+I_3+I_4.
\end{aligned}\tag{30}$$

Because of $a_{ni}>0$, by the definition of ND, $\{a_{ni}X_{ni},\ 1\le i\le n\}$ is also a sequence of ND random variables. Obviously, $\{a_{ni}X_{ni}(1)\}$ is nondecreasing by $a_{ni}X_{ni}$. Therefore, $\{a_{ni}X_{ni}(1)\}$ is also a sequence of ND random variables. Firstly, we prove $I_1<\infty$.

For the array $\{a_{ni}(X_{ni}(1)-\widehat{\mathbb{E}}X_{ni}(1)),\ 1\le i\le n,\ n\ge1\}$, we can make use of Lemma 2, noting that $\max_{1\le i\le n}|a_{ni}(X_{ni}(1)-\widehat{\mathbb{E}}X_{ni}(1))|\le 2(\log n)^{-1}$, and condition (a) of Lemma 2 holds.

By (7) and the C_r inequality, for any $r>0$,

$$\begin{aligned}
&\left|X_{ni}(1)\right|^{r}\\
&\quad\ll\left|X_{ni}\right|^{r}I\left(\left|a_{ni}X_{ni}\right|\le(\log n)^{-1}\right)\\
&\qquad+a_{ni}^{-r}(\log n)^{-r}I\left(\left|a_{ni}X_{ni}\right|>(\log n)^{-1}\right)\\
&\quad\le\left|X_{ni}\right|^{r}g\left(\mu a_{ni}X_{ni}\log n\right)\\
&\qquad+a_{ni}^{-r}(\log n)^{-r}\left(1-g\left(a_{ni}X_{ni}\log n\right)\right).\\
&\widehat{\mathbb{E}}\left(\left|X_{ni}(1)\right|^{r}\right)\\
&\quad\ll\widehat{\mathbb{E}}\left(|X|^{r}g\left(\mu a_{ni}X\log n\right)\right)\\
&\qquad+a_{ni}^{-r}(\log n)^{-r}\widehat{\mathbb{E}}\left(1-g\left(a_{ni}X\log n\right)\right)\\
&\quad\le\widehat{\mathbb{E}}\left(|X|^{r}g\left(\mu a_{ni}X\log n\right)\right)\\
&\qquad+a_{ni}^{-r}(\log n)^{-r}\mathbb{V}\left(\left|a_{ni}X\right|>\mu(\log n)^{-1}\right).
\end{aligned}\tag{31}$$

Then, by (1), (8), (9), (10), (11), (31), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\widehat{\mathbb{E}}\left(a_{ni}\left(X_{ni}(1)-\widehat{\mathbb{E}}X_{ni}(1)\right)\right)^{2}\\
&\quad\le\sum_{i=1}^{n}a_{ni}^{2}\widehat{\mathbb{E}}\left(2X_{ni}^{2}(1)+2\left(\widehat{\mathbb{E}}X_{ni}(1)\right)^{2}\right)\\
&\quad\le c\sum_{i=1}^{n}a_{ni}^{2}\widehat{\mathbb{E}}\left(X_{ni}(1)\right)^{2}\\
&\quad\le\sum_{i=1}^{n}a_{ni}^{2}\widehat{\mathbb{E}}X^{2}g\left(\mu a_{ni}X\log n\right)\\
&\qquad+(\log n)^{-2}\sum_{i=1}^{n}\mathbb{V}\left(\left|a_{ni}X\right|>\mu(\log n)^{-1}\right)\\
&\quad\ll o\left((\log n)^{-1}\right)+(\log n)^{p-2}\sum_{i=1}^{n}a_{ni}^{p}\widehat{\mathbb{E}}\left(|X|^{p}\right)\\
&\quad\le o\left((\log n)^{-1}\right)+(\log n)^{p-2}O\left(\frac{1}{n^{\alpha}}\right)C_{\mathbb{V}}\left(|X|^{p}\right)\\
&\quad=o\left((\log n)^{-1}\right).
\end{aligned}\tag{32}$$

Now, conditions (a) and (b) of Lemma 2 hold. So, by Lemma 2, we can get that

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}\left(X_{ni}(1)-\widehat{\mathbb{E}}X_{ni}(1)\right)>\frac{\varepsilon}{2}\right)<\infty.\tag{33}$$

So to prove $I_1<\infty$, it suffices to prove that

$$\left|\sum_{i=1}^{n}a_{ni}\widehat{\mathbb{E}}X_{ni}(1)\right|\longrightarrow0,\quad n\longrightarrow\infty.\tag{34}$$

By $\widehat{\mathbb{E}}X_{ni}=0$, (1), (7), (8), (9), (10), (31), we can get

$$\begin{aligned}
&\left|\sum_{i=1}^{n}a_{ni}\widehat{\mathbb{E}}X_{ni}(1)\right|=\left|\sum_{i=1}^{n}a_{ni}\left(\widehat{\mathbb{E}}X_{ni}(1)-\widehat{\mathbb{E}}X_{ni}\right)\right|\le\sum_{i=1}^{n}a_{ni}\\
&\quad\cdot\widehat{\mathbb{E}}\left|X_{ni}-X_{ni}(1)\right|\ll\sum_{i=1}^{n}a_{ni}\widehat{\mathbb{E}}\left(|X|\right.\\
&\quad\left.\cdot\left(1-g\left(a_{ni}X\log n\right)\right)\right)\le\sum_{i=1}^{n}a_{ni}\widehat{\mathbb{E}}\left(|X|^{p}\right.\\
&\quad\left.\cdot a_{ni}^{p-1}(\log n)^{p-1}\mu^{1-p}\left(1-g\left(a_{ni}X\log n\right)\right)\right)\\
&\quad\ll(\log n)^{p-1}\sum_{i=1}^{n}a_{ni}^{p}\widehat{\mathbb{E}}|X|^{p}\le c(\log n)^{p-1}O\left(\frac{1}{n^{\alpha}}\right)\longrightarrow\\
&\quad0.
\end{aligned}\tag{35}$$

So, we get (34).

Now, we prove $I_2<\infty$. By the condition of $0\le a_{ni}X_{ni}(2)\le\varepsilon/N$, $|\sum_{i=1}^{n}a_{ni}X_{ni}(2)|=\sum_{i=1}^{n}a_{ni}X_{ni}(2)>\varepsilon$ means that there is at least one N that makes $X_{ni}(2)\ne0$. By the definition of ND, (1), (7), (8), (9), (10), we can obtain that

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}(2)>\varepsilon\right)\le\sum_{1\le k_1<\cdots<k_N\le n}\mathbb{V}\left(X_{n,k_1}(2)\ne0,\right.\\
&\quad\left.\ldots,X_{n,k_N}(2)\ne0\right)\le\sum_{1\le k_1<\cdots<k_N\le n}\mathbb{V}\left(\left|a_{n,k_1}X_{n,k_1}\right|\right.\\
&\quad\left.\ge(\log n)^{-1},\ldots,\left|a_{n,k_N}X_{n,k_N}\right|\ge(\log n)^{-1}\right)
\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{1\le k_1<\cdots<k_N\le n} \widehat{\mathbb{E}}\left(\left(1-g\left(a_{n,k_1}X\log n\right)\right)\right.\\
&\left.\cdots\left(1-g\left(a_{n,k_N}X\log n\right)\right)\right) \le \sum_{1\le k_1<\cdots<k_N\le n} \widehat{\mathbb{E}}\left(1\right.\\
&\left.-g\left(a_{n,k_1}X\log n\right)\right)\cdots\widehat{\mathbb{E}}\left(1-g\left(a_{n,k_N}X\log n\right)\right)\\
&\le \left(\sum_{i=1}^{n}\widehat{\mathbb{E}}\left(1-g\left(a_{ni}X\log n\right)\right)\right)^N \le \left(\sum_{i=1}^{n}\mathbb{V}\right.\\
&\left.\cdot\left(\left|a_{ni}X\right|>\mu\left(\log n\right)^{-1}\right)\right)^N \ll \left(\left(\log n\right)^p\sum_{i=1}^{n}a_{ni}^p\right.\\
&\left.\cdot\widehat{\mathbb{E}}\left(|X|^p\right)\right)^N \ll \left(\left(\log n\right)^p\left(\frac{1}{n^\alpha}\right)\widehat{\mathbb{E}}\left(|X|^p\right)\right)^N\\
&\le c\left(\left(\log n\right)^p\left(\frac{1}{n^\alpha}\right)\right)^N.
\end{aligned} \tag{36}$$

When $\alpha N > 1$, we can get $I_2 < \infty$.

In the same way, we know that $-\varepsilon/N \le a_{ni}X_{ni}(3) \le 0$, so, $|\sum_{i=1}^{n} a_{ni}X_{ni}(3)| = -\sum_{i=1}^{n} a_{ni}X_{ni}(3) > \varepsilon$ also means that there is at least one N that makes $X_{ni}(3) \ne 0$. Using the same method of proving $I_2 < \infty$, we can have $I_3 < \infty$. At last, we prove $I_4 < \infty$. Using the similar method to I_2. By the value of $X_{ni}(4)$, we can get $a_{ni}X_{ni}(4) \le (\log n)^{-1}I(|a_{ni}X_{ni}| > \varepsilon/N) \le \varepsilon/N$. Hence, by (1), (7), (8), (9), (10), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}(4)>\varepsilon\right)\\
&\quad\le\sum_{n=1}^{\infty}\mathbb{V}\left(\textit{there are at least N such that } \left|a_{ni}X_{ni}\right|\right.\\
&\quad\left.>\frac{\varepsilon}{N}\right)\ll\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\widehat{\mathbb{E}}\left(1-g\left(\frac{a_{ni}X}{\varepsilon/N}\right)\right)\right)^N\\
&\quad\le\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\mathbb{V}\left(\left|a_{ni}X\right|>\frac{\varepsilon\mu}{N}\right)\right)^N\\
&\quad\le\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}a_{ni}^p\widehat{\mathbb{E}}|X|^p\left(\frac{\varepsilon\mu}{N}\right)^{-p}\right)^N\\
&\quad\ll\sum_{n=1}^{\infty}\left(\widehat{\mathbb{E}}|X|^pO\left(\frac{1}{n^\alpha}\right)\right)^N<\infty.
\end{aligned} \tag{37}$$

Because of $\alpha N > 1$, we can get $I_4 < \infty$. Thus, this completes the proof of Theorem 5. □

Proof of Theorem 7. Firstly, we prove (20). Without loss of generality, we assume that $\widehat{\mathbb{E}}X_i = 0$.

By Theorem 5, $\forall\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_iI\left(\left|a_{ni}X_i\right|\le\frac{\varepsilon}{N}\right)>\varepsilon\right)<\infty. \tag{38}$$

Thus, we can get

$$\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_iI\left(\left|a_{ni}X_i\right|\le\frac{\varepsilon}{N}\right)>\varepsilon;\ i.\ o.\right)=0. \tag{39}$$

Equation (39) follows from Lemma 4, and $\mathbb{V}$ is countably subadditive. $\forall\varepsilon > 0$, on account of the arbitrariness of ε, it implies

$$\limsup_{n\longrightarrow\infty}\left(\sum_{i=1}^{n}a_{ni}X_iI\left(\left|a_{ni}X_i\right|\le\frac{\varepsilon}{N}\right)\right)\le\varepsilon\quad a.s.\mathbb{V}. \tag{40}$$

Thus, to prove (20), we just need to check that

$$\sum_{i=1}^{n}a_{ni}X_iI\left(\left|a_{ni}X_i\right|>\frac{\varepsilon}{N}\right)\longrightarrow 0\quad a.s.\mathbb{V}. \tag{41}$$

By (7) and (3) of Lemma 3, we can get

$$\begin{aligned}
\sum_{i=1}^{\infty}\mathbb{V}\left(\left|X_i\right|\ge ci^{1/p}\right)&\le\sum_{i=1}^{\infty}\widehat{\mathbb{E}}\left(1-g\left(\frac{X_i}{ci^{1/p}}\right)\right)\\
&\ll\sum_{i=1}^{\infty}\widehat{\mathbb{E}}\left(1-g\left(\frac{X}{ci^{1/p}}\right)\right)\\
&\le\sum_{i=1}^{\infty}\mathbb{V}\left(|X|>\mu ci^{1/p}\right)<\infty.
\end{aligned} \tag{42}$$

Thus, $\mathbb{V}(|X_i| \ge ci^{1/p};\ i.\ o.) = 0$ follows from Lemma 4 and $\mathbb{V}$ being countably subadditive. By (17), it is easily checked that

$$\begin{aligned}
&\left|\sum_{i=1}^{n}a_{ni}X_iI\left(\left|a_{ni}X_i\right|>\frac{\varepsilon}{N}\right)\right|\\
&\quad\ll\frac{1}{n^{1/p}}\sum_{i=1}^{n}\left|X_i\right|I\left(\frac{1}{n^{1/p}}\left|X_i\right|>\frac{\varepsilon}{N}\right)\\
&\quad\ll\frac{1}{n^{1/p}}\sum_{i=1}^{n}\left|X_i\right|I\left(\left|X_i\right|>\frac{i^{1/p}\varepsilon}{N}\right)\longrightarrow 0\quad a.s.\mathbb{V}.
\end{aligned} \tag{43}$$

Hence, (41) holds; because of the arbitrariness of ε, we can get (20). Considering $\{-X_n;\ n \ge 1\}$ instead of $\{X_n;\ n \ge 1\}$ in (20), we can obtain

$$\limsup_{n\longrightarrow\infty}\sum_{i=1}^{n}a_{ni}\left(-X_i-\widehat{\mathbb{E}}\left(-X_i\right)\right)\le 0. \tag{44}$$

We know that $\widehat{\varepsilon}X := -\widehat{\mathbb{E}}(-X)$, $\forall X \in \mathscr{H}$. Replacing $-\widehat{\mathbb{E}}(-X_i)$ with $\widehat{\varepsilon}X_i$ in (44), we can get (21). When $\widehat{\mathbb{E}}X_i = \widehat{\varepsilon}X_i$, combine (20) with (21), and we will have (22). Thus, we obtain the result of Theorem 7. □

Proof of Theorem 8. For $1 \le p \le 2$, by (16), (18), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n}a_{ni}^2\widehat{\mathbb{E}}X^2g\left(a_{ni}X\log n\right)\\
&\quad\le\left(\log n\right)^{p-2}\sum_{i=1}^{n}a_{ni}^p\widehat{\mathbb{E}}\left(|X|^p\right)g\left(a_{ni}X\log n\right)
\end{aligned}$$

$$\leq (\log n)^{p-2} \sum_{i=1}^{n} a_{ni}^{p} \widehat{\mathbb{E}} \left(|X|^{p}\right)$$
$$\leq (\log n)^{p-2} O\left(\frac{1}{n^{\alpha}}\right) C_{\mathbb{V}} \left(|X|^{p}\right) = o\left((\log n)^{-1}\right). \tag{45}$$

We get condition (19) of Theorem 7. So the result of Theorem 8 can be proved. □

Proof of Theorem 9. Without loss of generality, we still assume that $\widehat{\mathbb{E}} X_{ni} = 0$. By (23), obviously

$$\sum_{i=1}^{n} a_{ni}^{p} \leq n \max_{1\leq i\leq n} a_{ni}^{p} \leq O\left(\frac{1}{n}\right). \tag{46}$$

Then, by Theorem 5, to prove (25), we just need to prove that

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} I\left(\left|a_{ni} X_{ni}\right| > \frac{\varepsilon}{N}\right) > \varepsilon\right) < \infty. \tag{47}$$

By (4) of Lemma 3, (7), (8), (23), it is easy to check that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} I\left(\left|a_{ni} X_{ni}\right| > \frac{\varepsilon}{N}\right) > \varepsilon\right) \\
&\quad \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbb{V}\left(\left|a_{ni} X_{ni}\right| > \frac{\varepsilon}{N}\right) \\
&\quad \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \widehat{\mathbb{E}}\left(1 - g\left(\frac{a_{ni} X_{ni}}{\varepsilon/N}\right)\right) \\
&\quad \ll \sum_{n=1}^{\infty}\sum_{i=1}^{n} \widehat{\mathbb{E}}\left(1 - g\left(\frac{a_{ni} X}{\varepsilon/N}\right)\right) \\
&\quad \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbb{V}\left(\left|a_{ni} X\right| > \frac{\mu\varepsilon}{N}\right) \\
&\quad \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbb{V}\left(c \cdot \frac{1}{n^{2/p}} |X| > \frac{\mu\varepsilon}{N}\right) \\
&\quad \leq \sum_{n=1}^{\infty} n \mathbb{V}\left(|X| > \frac{\mu\varepsilon n^{2/p}}{cN}\right) < \infty.
\end{aligned} \tag{48}$$

Therefore, (47) holds, then, we obtain (25). Considering $\{-X_{ni};\ n \geq 1\}$ instead of $\{X_{ni};\ n \geq 1\}$ in (25), we have

$$\limsup_{n\longrightarrow\infty} \sum_{i=1}^{n} a_{ni} \left(-X_{ni} - \widehat{\mathbb{E}}\left(-X_{i}\right)\right) \leq 0. \tag{49}$$

Replacing $-\widehat{\mathbb{E}}(-X_{ni})$ with $\widehat{\varepsilon} X_{ni}$ in (49), we can get (26). When $\widehat{\mathbb{E}} X_{ni} = \widehat{\varepsilon} X_{ni}$, combine (25) with (26), and we will have (27). Therefore, we obtain the result of Theorem 9. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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