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Research Article

On Naturally Ordered Abundant Semigroups with an Adequate Monoid Transversal

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In this paper, we study a class of naturally ordered abundant semigroups with an adequate monoid transversal, namely, naturally ordered concordant semigroups with an adequate monoid transversal. After giving some properties of such semigroups, we obtain a structure theorem for naturally ordered concordant semigroups with an adequate monoid transversal.

1. Introduction

Suppose that S is a regular semigroup and S° is an inverse subsemigroup of S. Then S° is called an inverse transversal of S, if S° contains a unique inverse x° of each element x of S. The structure theorems for regular semigroups with an inverse transversal have been given by many authors (see [1–4]). An analogue of an inverse transversal which is termed an adequate transversal was introduced for abundant semigroups by El-Qallali (see [5]). An ordered semigroup $(S, \leq)$ is *naturally ordered* if, for all $e, f \in E(S)$, $e\omega f$ implies $e \leq f$, where ω is the natural partial order on $E(S)$. Recently, Blyth and Almeida Santos have investigated naturally ordered regular semigroups with an inverse monoid transversal and determined the structure of such regular semigroups. Since abundant semigroups generalize regular semigroups, it is natural that, in the first instance, we should approach their structure theory by looking for generalization of results from the theory of regular semigroups. The following papers contain some work along these lines: [5–20]. As we shall discuss below, this paper is one of a sequence in which we concentrate on the structure of a class of naturally ordered abundant semigroups with an adequate monoid transversal.

We proceed as follows: Section 2 presents some necessary notation and known results. In Section 3, we obtain some characterizations for naturally ordered concordant semigroups with an adequate transversal. In Section 4, we give several order properties of naturally ordered concordant semigroups with an adequate transversal. In Section 5, we give some characterizations of the regularity of Green $*$−relations. In Section 6, we establish a structure theorem for naturally ordered concordant semigroups with an adequate monoid transversal, which generalizes the main result of [21].

2. Preliminaries

In what follows, we shall use the notion and notation of [6, 21]. Other undefined terms can be found in [11, 22, 23]. Here we provide some known results repeatedly used without mention in the following. First we recall some of the basic facts about the relations $\mathscr{L}^*$ and $\mathscr{R}^*$.

Lemma 1 (see [11]). *Let a, b be elements of a semigroup S. Then the following statements are equivalent:*

(1) $a\mathscr{L}^* b (a\mathscr{R}^* b)$;

(2) for all $x, y \in S^1$, $ax = ay (xa = ya) \Longleftrightarrow bx = by (xb = yb)$.

As an easy but useful consequence, we have the following.

Corollary 2 (see [11]). *Let $a, e^2 = e \in S$. Then the following statements are equivalent:*

(1) $a\mathscr{L}^* e (a\mathscr{R}^* e)$;

(2) $ae = a (ea = a)$ and for all $x, y \in S^1$, $ax = ay (xa = ya) \Longrightarrow ex = ey (xe = ye)$.

It is well known that $\mathscr{L}^*$ is a right congruence while $\mathscr{R}^*$ is a left congruence. In general, $\mathscr{L} \subseteq \mathscr{L}^*$ and $\mathscr{R} \subseteq$

$\mathscr{R}^*$. But when a, b are regular elements, $a\mathscr{L}b$ [resp., $a\mathscr{R}b$] if and only if $a\mathscr{L}^*b$ [resp., $a\mathscr{R}^*b$]. For convenience, a^+ [resp. a^*] is denoted the typical idempotent related a by the relation $\mathscr{R}^*$ [resp. $\mathscr{L}^*$]. A semigroup S is called *abundant* if each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class contains at least an idempotent. An abundant semigroup is called *quasi-adequate* if the set of idempotents forms a subsemigroup. Moreover, a quasi-adequate semigroup is called *adequate* if its band of idempotents is a semilattice. In this case, each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class contains exactly one idempotent.

In this paper, $\leq_n$ denotes the natural partial order on an abundant semigroup S, defined by the rule that $a\leq_n b$ if and only if for some idempotents e and f, $a = eb = bf$. And $\omega(a)$ stands for the set $\{b \in S \mid b\leq_n a\}$. Following [9], S is *idempotent-connected* (for short, IC) provided, for every element a of S, and for some [for all] a^+, a^*, there exists a bijection $\theta : < a^+ > \longrightarrow < a^* >$ such that $xa = a(x\theta)$ for all x of $< a^+ >$, where $< a^+ >$ ($< a^* >$) is the subsemigroup of S generated by the set $\{y \in E(S) \mid y = ya^+ = a^+y\}$ (resp., $\{y \in E(S) \mid y = ya^* = a^*y\}$).

Lemma 3 (see [18]). *Let a, b be elements of a semigroup S. Then the following statements are equivalent:*

(1) S is IC;

(2) for each element a of S two conditions hold:

(i) for some[for all] a^* and for all $e \in \omega(a^*)$ there exists an idempotent $f \in \omega(a^+)$ such that $ae = fa$.

(ii) for some[for all] a^+ and for all $h \in \omega(a^+)$ there exists an idempotent $g \in \omega(a^*)$ such that $ha = ag$.

An abundant semigroup is called *concordant* if it is IC and satisfies the regularity condition.

If S is an abundant semigroup and U is an abundant subsemigroup of S; then we say that U is a $*$-*subsemigroup* of S if $\mathscr{L}^*(U) = \mathscr{L}^*(S) \cap (U\times U)$ and $\mathscr{R}^*(U) = \mathscr{R}^*(S)\cap(U\times U)$.

Lemma 4 (see [19]). *Let S be an abundant semigroup. If e is an idempotent of S, then eSe is a $*$-subsemigroup of S.*

Let S° be an adequate $*$-subsemigroup of S and E° be the idempotent semilattice of S°. As in [5], S° is called an *adequate transversal* for S if, for each element x in S, there exists a unique element $\overline{x} \in S^\circ$ and idempotents e, f in E such that $x = e\overline{x}f$, where $e\mathscr{L}\overline{x}^+$, $f\mathscr{R}\overline{x}^*$ for $\overline{x}^+, \overline{x}^*$ in E°. It is straightforward to show that such e and f are uniquely determined by x (see [5]). Hence we normally denote e by e_x, f by f_x. We define

$$\begin{aligned} I &= \{e_x \mid x \in S\}, \\ \Lambda &= \{f_x \mid x \in S\}, \\ L &= \{e_x\overline{x} \mid x \in S\}, \\ R &= \{\overline{x}f_x \mid x \in S\}. \end{aligned} \tag{1}$$

The adequate transversal S° is said to be a *quasi-ideal* if $S^\circ S S^\circ \subseteq S^\circ$ or, equivalently, $\Lambda I \subseteq S^\circ$ and *left simplistic* if $S^\circ I S^\circ \subseteq S^\circ$ or equivalently, $S^\circ I \subseteq S^\circ$. Let S° be a $*$-adequate subsemigroup of S and $a \in S^\circ$. Throughout this paper, we denote by a^+ [resp., a^*] the unique idempotent of L_a^*[resp. R_a^*] in S°. For our purpose, we list the following results.

Lemma 5 (see [6]). *Let S° be an adequate transversal of an abundant semigroup S and $x, y \in S$. Then*

(1) $e_x\mathscr{R}^*x$ and $f_x\mathscr{L}^*x$;

(2) $e_{\overline{x}}\mathscr{L}e_x$ and $f_{\overline{x}}\mathscr{R}f_x$;

(3) $\overline{e_x} = e_{\overline{x}} = \overline{x}^+ = f_{e_x}$, $\overline{f_x} = f_{\overline{x}} = \overline{x}^* = e_{f_x}$;

(4) if $x \in S^\circ$ then $e_x = x^+ \in E^\circ$, $\overline{x} = x$, $f_x = x^* \in E^\circ$;

(5) if $x \in E^\circ$ then $e_x = \overline{x} = f_x = x$;

(6) if $x \in I$, $y \in \Lambda$ then $e_x = x, \overline{x} = f_x = e_{\overline{x}}$ and $e_y = \overline{y} = f_{\overline{y}}$, $f_y = y$.

Lemma 6 (see [7]). *Let S° be an adequate transversal of an abundant semigroup S and $x, y \in S$. Then*

(1) $x\mathscr{R}^*y \Longleftrightarrow e_x = e_y$;

(2) $x\mathscr{L}^*y \Longleftrightarrow f_x = f_y$.

Lemma 7 (see [20]). *Let S be an abundant semigroup with a quasi-ideal adequate transversal S°. For any $x, y \in S$, then*

(1) $\overline{xy} = \overline{x}f_xe_y\overline{y}$;

(2) $e_{xy} = e_x(\overline{x}f_xe_y)^+$;

(3) $f_{xy} = (f_xe_y\overline{y})^*f_y$.

Let S be an abundant semigroup with an adequate transversal S°. If $x \in Reg(S)$, then, by Lemma 5(1) and [23, Theorem 2.3.4], there exists a unique $x^\circ \in V(x)$ with $xx^\circ = e_x$ and $x^\circ x = f_x$.

Theorem 8 (see [6]). *Let S be an abundant semigroup with an adequate transversal S°. If $x \in Reg(S)$ then $x^\circ \in S^\circ, \overline{x} = x^{\circ\circ}$, and $x^\circ = x^{\circ\circ\circ}$.*

Lemma 9 (see [6]). *Let S be an abundant semigroup with a quasi-ideal adequate transversal S°. If $Reg(S)$ is a subsemigroup of S then I is a left regular subband of S and Λ is a right regular subband of S.*

Lemma 10 (see [6]). *Let S be an abundant semigroup with an adequate transversal S°. Then $I \cap \Lambda = E^\circ, L \cap R = S^\circ, I = L \cap E(S), \Lambda = R \cap E(S)$.*

Theorem 11 (see [6]). *Let S be an abundant semigroup with an adequate transversal S°. If I is a band, then the following statements are equivalent:*

(1) S° is left simplistic;

(2) E° is a right ideal of I;

(3) I is a normal band.

Theorem 12 (see [6]). *Let S be an abundant semigroup with an adequate transversal S°. Then S° is both left and right simplistic if and only if S° is a quasi-ideal of S.*

3. Characterization

The object of this section is to give characterizations of naturally ordered concordant semigroups with an adequate transversal.

Lemma 13 (see [17]). *Let S be an ordered abundant semigroup. Then S is naturally ordered if and only if, for all $e \in E(S)$, eSe is $\mathscr{R}^*$-unipotent, $\mathscr{L}^*$-unipotent, and naturally ordered.*

As a consequence of Lemma 13, we have the following.

Corollary 14. *Let S be an ordered abundant semigroup which satisfies the regularity condition. Then S is naturally ordered if and only if S is locally adequate and, for all $e \in E(S)$, eSe is naturally ordered.*

Lemma 15 (see [16]). *Let S be an IC abundant semigroup. Then the following statements are equivalent:*

(1) S is locally adequate;

(2) $\leq_n$ is compatible with multiplication.

Combining with Corollary 14 and Lemma 15, we have the following.

Theorem 16. *A concordant semigroup can be naturally ordered if and only if it is locally adequate.*

Theorem 17. *Let S be a concordant semigroup with a quasi-ideal adequate transversal S°. Then S is locally adequate.*

Proof. Suppose that S° is a quasi-ideal adequate transversal of a concordant semigroup S. Let $T = Reg(S)$ and let $U = T \cap S^\circ$. It is clear that U is a regular subsemigroup of T. It is easy to see that $U = \{x^\circ : x \in T\} = \{x \in T : x = \overline{x} = x^{\circ\circ}\}$. Since S° is adequate, then we can easily deduce that U is inverse and that U is an inverse transversal of the regular semigroup T. It is reasonably clear that $E(T) = E(S)$, $E(U) = E(S^\circ)$. Since S is a concordant semigroup, $UTU \subseteq T$. Because S° is a quasi-ideal, $UTU \subseteq S^\circ S S^\circ \subseteq S^\circ$. Thus $TUT \subseteq T \cap S^\circ = U$. This implies that U is a quasi-ideal inverse transversal of T. By [21, Theorem 3], T is locally inverse. Let $e \in E(S) = E(T)$. Then eTe is an inverse subsemigroup of T and, by [18, Lemma 1.5], eSe is concordant. Assume that $x \in Reg(eSe)$. Then $x \in T$. Thus $x = exe \in eTe$. It follows that $Reg(eSe) \subseteq eTe \subseteq eSe$. Since S is concordant, $eTe \subseteq Reg(eSe)$ and so $Reg(eSe) = eTe$. Therefore, eSe is adequate. This shows that S is locally adequate. □

Furthermore, we have the following result, which generalizes [21, Theorem 3].

Theorem 18. *Let S be a concordant semigroup with an adequate transversal S°. Then the following statements are equivalent:*

(1) S can be naturally ordered;

(2) S is locally adequate;

(3) S° is a quasi-ideal of S.

Proof. We need to only prove that (2) implies (3). Suppose that S is locally adequate. Then, by Lemma 15, $\leq_n$ is compatible with multiplication. Hence, by Lemma 9 and [23, Exercise 4.7.18], I is a normal band whence by Theorem 11, S° is left simplistic. Similarly, S is right simplistic. Hence, by Theorem 12, S° is a quasi-ideal. □

4. Order Properties

Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. By Theorem 18 above the adequate transversal S° is a quasi-ideal. We now consider relationships between the imposed order $\leq$ and the natural order $\leq_n$ on S. The proof of the following lemma is essentially the same as that of [21, note].

Lemma 19. *Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. Then the orders $\leq$ and $\leq_n$ coincide on E°.*

In the following results, we shall see that the order $\leq$ also extends the natural order $\leq_n$ on the adequate transversal S°; and, as a consequence, the assignment $x \longmapsto \overline{x}$ is always isotone.

Lemma 20. *Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. If $x, y \in S$ are such that $x \leq_n y$ then $e_x \leq e_y$, $f_x \leq f_y$.*

Proof. Suppose that $x, y \in S$ are such that $x \leq_n y$; then there exist $e, f \in E(S)$ such that $x = ey = yf$. Observe that $x = yf$ gives $e_x = e_{yf} = e_y(\overline{y} f_y e_f)^+$ by Lemma 7(2), whence we have $e_y e_x = e_x$; and, by Lemma 9, $e_x e_y = e_x e_y e_x = e_x$. Hence we have $e_x \leq_n e_y$ and so $e_x \leq e_y$. Similarly $f_x \leq f_y$. □

The next results generalize [21, Theorem 5 and its Corollary].

Theorem 21. *Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. If $x, y \in S$ are such that $\overline{x} \leq_n \overline{y}$, then $\overline{x} \leq \overline{y}$.*

Proof. Suppose that $x, y \in S$ are such that $\overline{x} \leq_n \overline{y}$. Then, by Lemma 20, in E° we have $\overline{x}^+ = e_{\overline{x}} \leq_n e_{\overline{y}} = \overline{y}^+$. Consequently, $\overline{x}^+ = \overline{x}^+\overline{y}^+ = \overline{y}^+\overline{x}^+$ and $\overline{x}^+ \leq \overline{y}^+$. Since $\overline{x} \leq_n \overline{y}$, there exists $f' \in E$ such that $\overline{x} = f'\overline{y} = \overline{x}^+ f' \overline{y}^+ y$. Let $f = \overline{x}^+ f' \overline{y}^+$. Since $f'\overline{x} = \overline{x}$, $f'\overline{x}^+ = \overline{x}^+$ and so $\overline{x}^+ f' \in E(S)$. Because $\overline{x}^+ f' \overline{y}^+ \overline{x}^+ f' \overline{y}^+ = \overline{x}^+ f' \overline{x}^+ f' \overline{y}^+ = \overline{x}^+ f' \overline{y}^+ \in S^\circ$, $f = \overline{x}^+ f' \overline{y}^+ \in E(S^\circ)$. Since $E(S^\circ)$ is a semilattice, $f = \overline{x}^+ f = f\overline{x}^+$, which implies that $f \leq_n \overline{x}^+$. Hence, $f \leq \overline{x}^+ \leq \overline{y}^+$ by Lemma 19. It follows that $\overline{x} = f\overline{y} \leq \overline{y}^+\overline{y} = \overline{y}$. □

Corollary 22. *Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. If $x, y \in S$ are such that $x \leq y$, then $\overline{x} \leq \overline{y}$.*

Proof. If $x, y \in S$ are such that $x \leq y$, then, using the fact that S° is a quasi-ideal, we have

$$\overline{x} = e_{\overline{x}} x f_{\overline{x}} \leq e_{\overline{x}} y f_{\overline{x}} = e_{\overline{x}} e_y \overline{y} f_y f_{\overline{x}}. \tag{2}$$

By Theorems 11 and 12, $e_{\overline{x}} e_y, f_y f_{\overline{x}} \in E^\circ$. Writing $g = e_{\overline{x}} e_y$ and $b = \overline{y} f_y f_{\overline{x}}$, we have $gb = b^+ g b^+ b$. Since S is IC, there exists $h \in E(S)$ such that $gb = b^+ g b^+ b = bh$. Hence $e_{\overline{x}} e_y \overline{y} f_y f_{\overline{x}} \leq_n \overline{y} f_y f_{\overline{x}}$. Similarly $\overline{y} f_y f_{\overline{x}} \leq_n \overline{y}$. Since $e_{\overline{x}} e_y \overline{y} f_y f_{\overline{x}}, \overline{y} f_y f_{\overline{x}} \in S^\circ$ by Lemma 5(3) and S° is a quasi-ideal, by Lemma 5(4) and Theorem 21, $\overline{x} \leq \overline{y}$. □

By the proof of [21, Theorem 8], we have the following result:

Lemma 23. *Let $(S;\leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. If $e, f \in E(S)$ are such that $e\leq_n f$ then $e = fe^{\circ}e = ee^{\circ}f = fe^{\circ}f$.*

Theorem 24. *Let $(S;\leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. The following conditions are equivalent:*

(1) If $x, y \in Reg(S)$ are such that $x \leq y$, then $y^{\circ} \leq x^{\circ}$;

(2) If $e, f \in E(S)$ are such that $e \leq f$, then $f^{\circ} \leq e^{\circ}$;

(3) S is completely $\mathcal{J}^*$-simple;

(4) The adequate transversal S° is a cancellative monoid.

Proof. (1)⟹(2) It is trivial.

(2)⟹(3) Let $e, f \in E(S)$ be such that $e\leq_n f$. Then $e \leq f$. Hence by the hypothesis and Lemma 23, $f = ff^{\circ}f \leq fe^{\circ}f = e$. Thus $f = e$. Therefore, $\leq_n$ is equality on $E(S)$ and by [11, Corollary 5.2] S is completely $\mathcal{J}^*$-simple.

(3)⟹(4) Suppose that S is completely $\mathcal{J}^*$-simple. Then by [11, Corollary 5.2], we may assume S is a Rees matrix semigroup $\mu(T; I, \Lambda; P)$ without zero over a cancellative monoid T where each entry in P is a unit of T. Since S° is an adequate semigroup, S° contains an idempotent. Let $(i, p_{\lambda i}^{-1}, \lambda)$ and $(j, p_{\mu j}^{-1}, \mu)$ be idempotents of S°. Then they are inverses of $(i, p_{\mu i}, \mu)$. Since S° is an adequate transversal of S, by [7, Corollary 2.4], $(i, p_{\lambda i}^{-1}, \lambda) = (j, p_{\mu j}^{-1}, \mu)$, so that S° has only one idempotent. Thus S° is a cancellative monoid.

(4)⟹(1) If $x, y \in Reg(S)$ are such that $x \leq y$ then, by the Corollary 22, we have $x^{\circ\circ} = \overline{x} \leq \overline{y} = y^{\circ\circ}$. Because S° is a cancellative monoid, $Reg(S^{\circ})$ is a group. Thus $y^{\circ} \leq x^{\circ}$. □

Theorem 25. *Let $(S;\leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. The following conditions are equivalent:*

(1) The adequate transversal S° is a monoid;

(2) $(S^{\circ};\leq)$ has a biggest idempotent;

(3) $(\exists \xi \in E^{\circ})S^{\circ} = \xi S\xi$;

(4) $(\exists \xi \in E^{\circ})(\forall x \in S)\overline{x} = \xi x\xi$;

(5) $(\exists \xi \in E^{\circ})L = S\xi, R = \xi S$;

(6) L[resp. R] is an idempotent-generated principal left [resp. right] ideal.

Proof. (1)⟹(2) If (1) holds then for every $e \in E^{\circ}$ we have $e\leq_n 1_{S^{\circ}}$. Thus $e \leq 1_{S^{\circ}}$.

(2)⟹(1) Suppose that (2) holds and let $\xi = \max E^{\circ}$. If $\overline{x} \in S^{\circ}$, then by Lemma 5 (3) $\overline{x}^{+} = e_{\overline{x}} \leq \xi$. Since $\leq$ coincides with $\leq_n$ on E°, $\overline{x}^{+}\leq_n\xi$. Thus $\xi\overline{x} = \xi\overline{x}^{+}\overline{x} = \overline{x}^{+}\overline{x} = \overline{x}$. Similarly, $\overline{x}\xi = \overline{x}$. Consequently, $\xi = 1_{S^{\circ}}$.

(1)⟹(3) If (1) holds then we have

$$1_{S^{\circ}}S1_{S^{\circ}} \subseteq S^{\circ}SS^{\circ} \subseteq S^{\circ} = 1_{S^{\circ}}S^{\circ}1_{S^{\circ}} \subseteq 1_{S^{\circ}}S1_{S^{\circ}} \tag{3}$$

whence $S^{\circ} = \xi S\xi$, where $\xi = 1_{S^{\circ}}$.

(3)⟹(4) Let $T = Reg(S)$; let $U = T \cap S^{\circ}$. It is clear that U is a regular subsemigroup of T. It is easy to see that $U = \{x^{\circ} : x \in T\} = \{x \in T : x = \overline{x} = x^{\circ\circ}\}$. Since S° is adequate then we can easily deduce that U is inverse and that U is an inverse transversal of the regular semigroup T. It is reasonably clear that $E(T) = E(S)$, $E(U) = E(S^{\circ})$. Since $S^{\circ} = \xi S\xi$, $U = \xi T\xi$. By [6, Theorem 18(4)], for all $x \in T$, $x^{\circ\circ} = \xi x\xi$. Hence we have, for all $x \in S$,

$$\begin{aligned}\xi x\xi &= \xi e_x\overline{x}f_x\xi = \xi e_x\xi\overline{x}\xi f_x\xi = e_x^{\circ\circ}\overline{x}f_x^{\circ\circ} = \overline{e_x}\,\overline{x}\,\overline{f_x}\\ &= e_{\overline{x}}\overline{x}f_{\overline{x}} = \overline{x}.\end{aligned} \tag{4}$$

(4)⟹(1) It is trivial.

(4)⟹(5) If (4) holds, then for every $x \in S$ we have $e_x\overline{x} = e_x\xi x\xi \in S\xi$. Hence $L \subseteq S\xi$. Similarly $R \subseteq \xi S$. By Lemma 7, we have

$$e_{x\xi}\overline{x\xi} = e_x\left(\overline{x}f_x\xi\right)^{+}\overline{x}f_x\xi = e_x\overline{x}f_x\xi = x\xi. \tag{5}$$

Thus $S\xi \subseteq L$. Similarly $\xi S \subseteq R$ and we have (5).

(5)⟹(3) If (5) holds, then, by Lemma 10, $S^{\circ} = L \cap R = S\xi \cap \xi S = \xi S\xi$.

(5)⟹(6) This is clear.

(6)⟹(5) The proof is essentially the same as that of [21, Theorem 10]. □

5. The Regularity of $\mathcal{L}^*$ and $\mathcal{R}^*$

Lemma 20 and [21, Definition 3.2] motivate the next definition.

Definition 26. Let $(S;\leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. $\mathcal{L}^*$[resp. $\mathcal{R}^*$] is said to be regular if

$$\begin{gathered}(\forall x, y \in S)\\ x \leq y \Longrightarrow\\ f_x \leq f_y\\ \left[resp.\ e_x \leq e_y\right].\end{gathered} \tag{6}$$

Definition 27 (see [21]). An equivalence relation Θ on an ordered set A is said to satisfy the link property if

$$\begin{gathered}a\Theta x \leq b \Longrightarrow\\ (\exists y \in A)\\ a \leq y\Theta b;\end{gathered} \tag{7}$$

and the dual link property if

$$\begin{gathered}a \leq x\Theta b \Longrightarrow\\ (\exists y \in A)\\ a\Theta y \leq b.\end{gathered} \tag{8}$$

Theorem 28. *Let $(S;\leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. If $\mathcal{L}^*$[resp. $\mathcal{R}^*$] is regular then $\mathcal{L}^*$[resp. $\mathcal{R}^*$] satisfies the dual link property.*

Proof. If $a \leq x\mathcal{L}^*b$, then since $\mathcal{L}^*$ is regular, we have $f_a \leq f_x = f_b$ by Lemma 6 and so $bf_a \leq b$. Since $f_af_xf_a\leq_n f_a$, whence $f_a f_x f_a \leq f_a$. Since also $f_a \leq f_x$, it follows that $f_a = f_a f_a f_a \leq f_a f_x f_a$ and so $f_a f_x f_a = f_a$. If $y, z \in S$ such that $bf_a y = bf_a z$, then we have $f_x f_a y = f_x f_a z$. Hence $f_a f_x f_a y = f_a f_x f_a z$ and so $f_a y = f_a z$. Thus ay=az. If $c, d \in S$ such that $ac = ad$, then we have $f_a c = f_a d$ and so $bf_a c = bf_a d$. It follows that $a\mathscr{L}^* bf_a \leq b$. Similarly, when regular, $\mathscr{R}^*$ satisfies the dual link property. ◻

Definition 29. Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. $\mathscr{L}^*$ is said to be lower Λ-stable if

$$\begin{gathered}(\forall x \in S)\\ x \leq l \in \Lambda \Longrightarrow \\ f_x \leq l;\end{gathered} \tag{9}$$

and upper Λ-stable if

$$\begin{gathered}(\forall x \in S)\\ x \geq l \in \Lambda \Longrightarrow \\ f_x \geq l.\end{gathered} \tag{10}$$

Dually, we define $\mathscr{R}^*$ to be lower and upper I-stable.

Theorem 30. *Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. The following conditions are equivalent:*

(1) $\mathscr{L}^*$ is regular;

(2) $\mathscr{L}^*$ satisfies the dual link property and is lower Λ-stable.

Proof. (1)⟹(2) If $\mathscr{L}^*$ is regular then, by Theorem 28, it satisfies the dual link property. Moreover, if $x \leq l \in \Lambda$, then $f_x \leq f_l = l$ by Lemma 5(6) and so $\mathscr{L}^*$ is lower Λ-stable.

(2)⟹(1) Let $x, y \in S$ be such that $x \leq y$. Because $x \leq y\mathscr{L}^* f_y$ and $\mathscr{L}^*$ satisfies the dual link property, there exists $z \in S$ such that $x\mathscr{L}^* z \leq f_y$. Then $f_x = f_z$. Since $\mathscr{L}^*$ is lower Λ-stable, $f_z \leq f_y$. Thus $\mathscr{L}^*$ is regular. ◻

Theorem 31. *Let $(S; \leq)$ be a naturally ordered concordant semigroup with an adequate transversal S°. If S is upper directed then the following conditions are equivalent:*

(1) $\mathscr{L}^*$ is regular;

(2) $\mathscr{L}^*$ satisfies the link property and is upper Λ-stable.

Proof. (1)⟹(2) Suppose that $a\mathscr{L}^* x \leq b$ and let $k \in S$ be such that $a, b \leq k$. Then $f_a = f_x \leq f_b$ whence $a = af_a \leq kf_b$. Also, $f_b \leq f_k$ which, by [21, Theorem 1] and Lemma 9, gives $f_b = f_b f_k f_b = f_k f_b$ whence, by Lemmas 5 and 7, $f_{kf_b} = (f_k f_{\overline{b}})^* f_b = f_k f_{\overline{b}} f_b = f_k f_b = f_b$. Hence $a \leq kf_b \mathscr{L}^* b$. It is easy to see that $\mathscr{L}^*$ is upper Λ-stable.

(2)⟹(1) Let $x, y \in S$ be such that $x \leq y$. Since $f_x \mathscr{L}^* x \leq y$ and $\mathscr{L}^*$ satisfies the link property, there exists $z \in S$ such that $f_x \leq z\mathscr{L}^* y$. Then $f_z = f_y$. Because $\mathscr{L}^*$ is upper Λ-stable, $f_x \leq f_z$. Thus $\mathscr{L}^*$ is regular. ◻

6. Structure Theorem

The main objective in this section is to give a structure theorem for a naturally ordered concordant semigroup with an adequate monoid transversal.

Let S be an abundant semigroup with a set of idempotents E. An idempotent u of S is called *medial* if, for all $e \in E, e = eue$.

We have the following result, which generalizes [21, Theorem 17].

Theorem 32. *Let L and R be concordant semigroups with a common medial idempotent ξ and a common adequate submonoid $T = \xi L \xi = \xi R \xi$. Define a mapping $R \times L \longrightarrow Reg(T)$ by $(a, x) \longmapsto a \circ x$ with the following properties:*

(1) $(\forall a, b \in R)(\forall y, z \in L) a(a \circ y) b \circ z = [(a \circ y) b\xi]^* (b \circ z)$;

(2) $(\forall a, b \in R)(\forall y, z \in L) a \circ y(b \circ z) z = (a \circ y)[\xi y(b \circ z)]^+$;

(3) $(\forall a \in R)(\forall x \in L)(\forall t \in T) a \circ t = (a \circ \xi)(\xi \circ t), t \circ x = (t \circ \xi)(\xi \circ x)$;

(4) $(\forall a \in R)(\forall x \in L) a(a \circ \xi) = \xi a \xi, (\xi \circ x) x = \xi x \xi$;

(5) $(\forall a \in R, f \in E(L^*_a))(\forall x \in L, e \in E(R^*_x)) f \circ x = a \circ x = a \circ e$

On the set $L| \times |_\xi R = \{(x, a) \in L \times R \mid \xi x = a\xi\}$ define a multiplication by

$$(x, a)(y, b) = (x(a \circ y) y, a(a \circ y) b). \tag{11}$$

Then $L| \times |_\xi R$ is a locally adequate concordant semigroup that has an adequate monoid transversal.

Moreover, every such semigroup is obtained in this way. More precisely, if S is a locally adequate concordant semigroup with an adequate monoid transversal S°, that is a monoid with identity, ξ then $S^\circ = \xi S \xi$, ξ is a medial idempotent of both $S\xi$ and ξS, the mapping $\xi S \times S\xi \longrightarrow Reg(S^\circ)$ given by $(a, x) \longmapsto f_a e_x$ satisfies properties (1) to (5) above, and there is a semigroup isomorphism $S \cong S\xi| \times |_\xi \xi S$.

Proof. It is easy to see that the multiplication on $L| \times |_\xi R$ is well-defined.

We proceed in the following stages.

(i) $L| \times |_\xi R$ is a semigroup.

For associativity of the multiplication, let $(x, a), (y, b), (z, c) \in L| \times |_\xi R$. It is easy to see that the first component of $[(x, a)(y, b)](z, c)$ is $x(a \circ y) y[a(a \circ y) b \circ z] z$ which by (1) is $x(a \circ y) y[a(a \circ y) b\xi]^* (b \circ z) z$. Using the fact that $b\xi = \xi y$ and ξ is an identity of T, we see that this reduces to $x(a \circ y) y(b \circ z) z$. On the other hand, the first component of $(x, a)[(y, b)(z, c)]$ is $x[a \circ y(b \circ z) z] y(b \circ z) z$ which by (2) is $x(a \circ y)[\xi y(b \circ z)]^+ y(b \circ z) z$. It also reduces to $x(a \circ y) y(b \circ z) z$. Thus the first components of the products are the same and, similarly, so are the second components. Thus, with the above multiplication, $L| \times |_\xi R$ is a semigroup.

(ii)$L| \times |_\xi R$ is an abundant semigroup.

Suppose that $(x, a) \in L| \times |_\xi R$. Let $e \in E(L)$, $f \in E(R)$ be such that $e\mathscr{R}^* x$, $f\mathscr{R}^* a$. It is easy to see $e = e\xi$ and $f = \xi f$. We have for some $g \in E(L^*_x)$, $\xi x = \xi x g \xi g = \xi x \xi g = (\xi \circ x) x g = (\xi \circ x) x = \xi x \xi$. Similarly, $a\xi = \xi a \xi$. Since $\xi e \mathscr{R}^* \xi x$ and $\xi e \xi e = \xi e$, it follows that $\xi e = (\xi x \xi)^+$. If $b, c \in R$ are such that $ba\xi = ca\xi$ then we have for some $h \in E(L^*_a)$, $bah\xi h = cah\xi h$. Hence $ba = ca$ and so $a\mathscr{R}^* a\xi$. Thus $f\xi \mathscr{R}^* f \mathscr{R}^* a \mathscr{R}^* a\xi$ and so $f\xi = \xi f\xi = (\xi a\xi)^+ = (\xi x\xi)^+ = \xi e$ whence $(e, f\xi) \in L|\times|_\xi R$. We have

$$\begin{aligned}(e, f\xi)(e, f\xi) &= (e(f\xi \circ e)e, f\xi(f\xi \circ e)f\xi)\\ &= (e(f\xi \circ \xi)(\xi \circ e)e, f\xi(f\xi \circ \xi)(\xi \circ e)f\xi) \quad by\,(3)\\ &= (e\xi e(f\xi \circ \xi)\xi e\xi, \xi f\xi(\xi \circ e)\xi e) \quad by\,(4)\\ &= (ef\xi(f\xi \circ \xi)\xi e\xi, \xi f\xi\xi e\xi) \quad by\,(4)\\ &= (e\xi f\xi e\xi, \xi f\xi f\xi) \quad by\,(4)\\ &= (e\xi e\xi e, f\xi)\\ &= (e, f\xi)\end{aligned} \tag{12}$$

whence $(e, f\xi) \in E(L| \times |_\xi R)$. We also have

$$\begin{aligned}(e, f\xi)(x, a) &= (e(f\xi \circ x)x, f\xi(f\xi \circ x)a)\\ &= (e(f\xi \circ \xi)(\xi \circ x)x, f\xi(f\xi \circ \xi)(\xi \circ x)ah\xi h) \quad by\,(3)\\ &= (e\xi e(f\xi \circ \xi)\xi x\xi, \xi f\xi(\xi \circ x)\xi xh) \quad by\,(4)\\ &= (ef\xi(f\xi \circ \xi)\xi x\xi, \xi f\xi\xi x\xi h) \quad by\,(4)\\ &= (e\xi f\xi x\xi, \xi f\xi a\xi h) \quad by\,(4)\\ &= (e\xi e\xi x, f\xi a) = (x, a).\end{aligned} \tag{13}$$

If $(y, b), (z, c) \in L|\times|_\xi R$ are such that $(y, b)(x, a) = (z, c)(x, a)$, then $y(b \circ x)x = z(c \circ x)x$ and $b(b \circ x)a = c(c \circ x)a$. Hence by (5) and $e\mathscr{R}^* x$, $f\xi\mathscr{R}^* a$, $y(b \circ e)e = z(c \circ e)e$ and $b(b \circ e)f\xi = c(c \circ e)f\xi$. Thus we have

$$\begin{aligned}(y, b)(e, f\xi) &= (y(b \circ e)e, b(b \circ e)f\xi)\\ &= (z(c \circ e)e, c(c \circ e)f\xi) = (z, c)(e, f\xi)\end{aligned} \tag{14}$$

whence $(x, a)\mathscr{R}^*(e, f\xi) = (e, \xi e)$.

Dually, $(h\xi, h) = (\xi g, h) \in E(L|\times|_\xi R)$ and $(x, a)\mathscr{L}^*(h\xi, h)$, where $g \in E(L^*_x), h \in E(L^*_a)$. This shows that $L| \times |_\xi R$ is abundant.

(iii) $L| \times |_\xi R$ is IC.

Let $(e, \xi e) \in E(R^*_{(x,a)})$ and $(e', f') \in \omega((e, \xi e))$ where $e \in E(R^*_x), \xi e \in E(R^*_a)$. Then $(e', f')(e', f') = (e', f')$ and $(e', f')(e, \xi e) = (e', f') = (e, \xi e)(e', f')$. It follows that $e'(f' \circ e)e = e'$, $e(\xi e \circ e')e' = e'$, and $f'(f' \circ e)\xi e = f'$, whence $e'e = e'$, $ee' = e'$, and $f' = f'\xi = \xi e'$. Hence, we have, for some $\alpha \in E(R^*_{e'})$,

$$\begin{aligned}&e'(f' \circ e')e' = e' \Longrightarrow\\ &e'(\xi e' \circ e')e' = e' \Longrightarrow\\ &\alpha\xi\alpha e'(\xi e' \circ \xi)(\xi \circ e')e' = e' \quad by\,(3) \Longrightarrow\\ &\alpha\xi e'\xi\xi e'\xi = e' \quad by\,(4) \Longrightarrow\\ &\alpha\xi e'e' = e' \Longrightarrow\\ &e'e' = e'\end{aligned} \tag{15}$$

and so $e' \in \omega(e)$. Suppose that $k \in \omega(e)$, it is easy to see that $(k, \xi k) \in \omega(e, \xi e)$. Thus

$$\begin{aligned}&(e', f') \in \omega((e, \xi e)) \Longleftrightarrow\\ &e' \in \omega(e),\\ &f' = \xi e'.\end{aligned} \tag{16}$$

Dually, let $(h\xi, h) \in E(L^*_{(x,a)})$ where $h \in E(L^*_a), h\xi \in E(L^*_x)$. Then we have

$$\begin{aligned}&(g', h') \in \omega((h\xi, h)) \Longleftrightarrow\\ &h' \in \omega(h),\\ &g' = h'\xi.\end{aligned} \tag{17}$$

Let $(k, \xi k) \in \omega((e, \xi e))$. We have

$$\begin{aligned}(k, \xi k)(x, a) &= (k(\xi k \circ x)x, \xi k(\xi k \circ x)a)\\ &= (k(\xi k \circ \xi)(\xi \circ x)x, \xi k(\xi k \circ \xi)(\xi \circ x)ah\xi h) \quad by\,(3)\\ &= (k\xi k(\xi k \circ \xi)\xi x\xi, \xi k\xi(\xi \circ x)\xi xh) \quad by\,(4)\\ &= (k\xi k\xi x\xi, \xi k\xi a\xi h) \quad by\,(4)\\ &= (k\xi k\xi x, \xi ka) = (kx, \xi ka).\end{aligned} \tag{18}$$

Since R is IC and $\xi k \in \omega(\xi e)$, by Lemma 3, there exists $l \in \omega(h)$ such that $\xi ka = al$. Hence we have

$$\begin{aligned}(x, a)(l\xi, l) &= (x(a \circ l\xi)l\xi, a(a \circ l\xi)l)\\ &= (x(a \circ \xi)(\xi \circ l\xi)l\xi, a(a \circ \xi)(\xi \circ l\xi)l\xi l) \quad by\,(3)\\ &= (e\xi ex(a \circ \xi)\xi l\xi, \xi a\xi l\xi l) \quad by\,(4)\\ &= (e\xi a\xi l\xi, al\xi l) \quad by\,(4)\\ &= (e\xi al\xi, al) = (e\xi ka\xi, \xi ka)\\ &= (e\xi ek\xi x, \xi ka) = (kx, \xi ka).\end{aligned} \tag{19}$$

Thus $(k, \xi k)(x, a) = (x, a)(l\xi, l)$. Similarly, for every $(l'\xi, l') \in \omega(h\xi, h)$, there exists $(k', \xi k') \in \omega(e, \xi e)$ such that $(x, a)(l'\xi, l') = (k', \xi k')(x, a)$. This shows that $L| \times |_\xi R$ is IC.

(iv) $L| \times |_\xi R$ satisfies the regularity condition.

If $(x, a) \in Reg(L| \times |_\xi R)$, then (y, b) exists such that $(x, a)(y, b)(x, a) = (x, a)$. Hence we have

$$\begin{aligned}&(x(a \circ y)y, a(a \circ y)b)(x, a) = (x, a) \Longrightarrow\\ &(x(a \circ y)y[a(a \circ y)b \circ x]x, a(a \circ y)\\ &\quad \cdot b[a(a \circ y)b \circ x]a) = (x, a) \Longrightarrow\\ &(x(a \circ y)y[(a \circ y)b\xi]^*(b \circ x)x, a(a \circ y)\end{aligned}$$

$$\begin{aligned}
&\cdot b\left[(a\circ y)\,b\xi\right]^{*}(b\circ x)\,a\big)=(x,a)\quad by\,(1)\Longrightarrow\\
&\left(x\,(a\circ y)\,y\,(b\circ x)\,x,a\,(a\circ y)\,b\,(b\circ x)\,a\right)=(x,a)\Longrightarrow\\
&x\,(a\circ y)\,y\,(b\circ x)\,x=x,\\
&a\,(a\circ y)\,b\,(b\circ x)\,a=a
\end{aligned}\tag{20}$$

whence $x\in Reg(L)$ and $a\in Reg(R)$. Since T is an adequate semigroup, Reg(T) is an inverse semigroup. If $(x,a)\in L|\times|_{\xi}R$, $x\in Reg(L)$, and $a\in Reg(R)$, then we can define $\beta=(\xi x\xi)^{-1}=(\xi a\xi)^{-1}\in T$. Then the first component of the product$(x,a)(\beta,\beta)(x,a)$ is as in (i),

$$\begin{aligned}
x\,(a\circ\beta)\,\beta\,(\beta\circ x)\,x&=e\xi ex\,(a\circ\beta)\,\beta\,(\beta\circ\xi)\,(\xi\circ x)\,x\\
&\qquad by\,(3)\\
&=ea\xi\,(a\circ\xi)\,(\xi\circ\beta)\,\xi\beta\xi\xi x\xi\\
&\qquad by\,(4)\\
&=e\xi a\xi\beta\xi x\xi\quad by\,(4)\\
&=e\xi x\xi=x
\end{aligned}\tag{21}$$

where $e\in E(R_x^*)$. In a similar way we can see that the second component of the product is a. Thus we see that $(x,a)(\beta,\beta)(x,a)=(x,a)$. Therefore,

$$\begin{aligned}
&(x,a)\in Reg\left(L|\times|_{\xi}R\right)\Longleftrightarrow\\
&x\in Reg\,(L)\,,\\
&a\in Reg\,(R)\,.
\end{aligned}\tag{22}$$

If $(x,a),(y,b)\in Reg(L|\times|_{\xi}R)$, then $x,y\in Reg(L)$ and $a,b\in Reg(R)$. Since L and R are both concordant, $x(a\circ y)y\in Reg(L)$ and $a(a\circ y)b\in Reg(R)$. Hence $(x,a)(y,b)=(x(a\circ y)y,a(a\circ y)b)\in Reg(L|\times|_{\xi}R)$. This proves that S satisfies the regularity condition. Thus S is a concordant semigroup.

(v) $\widehat{T}=\mathrm{diag}(T\times T)$ is a $*$-adequate submonoid of $L|\times|_{\xi}R$.

It is easy to see that $\widehat{T}$ is a subsemigroup of $L|\times|_{\xi}R$. Since $T=\xi L\xi=\xi R\xi$, by Lemma 4 and T is a $*$-adequate submonoid of both L and R. If $(\gamma,\gamma)\in\widehat{T}$, then, by (ii), $(\gamma^{+},\gamma^{+})\mathscr{R}^{*}(\gamma,\gamma)\mathscr{L}^{*}(\gamma^{*},\gamma^{*})$. Hence $\widehat{T}$ is abundant. The proof of the following results is essentially the same as that of [21, Theorem 17]: E($\widehat{T}$) is a semilattice and

$$\begin{aligned}
&(\gamma,\gamma)\in E\left(\widehat{T}\right)\Longleftrightarrow\\
&\gamma\in E\,(T)\,.
\end{aligned}\tag{23}$$

Therefore, $\widehat{T}$ is an adequate semigroup. Let $(x,a)\in E(L|\times|_{\xi}R)$ and $(\gamma,\gamma)\in E(\widehat{T})$ be such that $(x,a)\leq_{n}(\gamma,\gamma)$. Then $(x,a)(\gamma,\gamma)=(x,a)=(\gamma,\gamma)(x,a)$. Hence we have

$$\begin{aligned}
(x,a)\,(\gamma,\gamma)=(x,a)&\Longrightarrow\\
\left(x\,(a\circ\gamma)\,\gamma,a\,(a\circ\gamma)\,\gamma\right)=(x,a)&\Longrightarrow\\
\left(x\,(a\circ\xi)\,(\xi\circ\gamma)\,\gamma,a\,(a\circ\xi)\,(\xi\circ\gamma)\,\gamma\right)=(x,a)&\Longrightarrow\\
(x\xi\gamma\xi,\xi a\xi\gamma\xi)=(x,a)&\Longrightarrow\\
x\gamma=x&,\\
a\gamma=a&.
\end{aligned}\tag{24}$$

Similarly, we have $\gamma x=x$, $\gamma a=a$. Hence

$$\begin{aligned}
\gamma x=x&\Longrightarrow\\
\gamma\xi x=x&\Longrightarrow\\
\gamma a\xi=x&\Longrightarrow\\
a\xi=x&\Longrightarrow\\
\xi a\xi=x&
\end{aligned}\tag{25}$$

Similarly, $a=\xi x\xi$. Thus $a=x=\xi x\xi$ and so $(x,a)\in E(\widehat{T})$. Therefore, by [9, Lemma 1.6], $\widehat{T}$ is a $*$-adequate subsemigroup of $L|\times|_{\xi}R$. It is easy to see that (ξ,ξ) is an identity element for $\widehat{T}$.

(vi) $\widehat{T}$ is a quasi-ideal adequate transversal of $L|\times|_{\xi}R$.

Suppose that $(x,a)\in L|\times|_{\xi}R$ and $(e,\xi e)\mathscr{R}^{*}(x,a)\mathscr{L}^{*}(f\xi,f)$ where $e\in E(R_x^*)$, $f\in E(L_a^*)$. By the proof of (ii), $e\mathscr{L}\xi e=(\xi x\xi)^{+}$ and $f\mathscr{R}f\xi=(\xi x\xi)^{*}$. Hence $(e,\xi e)\mathscr{L}(\xi e,\xi e)$, $(f\xi,f)\mathscr{R}(f\xi,f\xi)$, and we have

$$\begin{aligned}
&(e,\xi e)\,(\xi x\xi,\xi x\xi)\,(f\xi,f)=(e\,(\xi e\circ\xi x\xi)\\
&\quad\cdot\xi x\xi,\xi e\,(\xi e\circ\xi x\xi)\,\xi x\xi)\,(f\xi,f)=(e\xi e\,(\xi e\circ\xi)\\
&\quad\cdot(\xi\circ\xi x\xi)\,\xi x\xi,\xi e\,(\xi e\circ\xi)\,(\xi\circ\xi x\xi)\,\xi x\xi)\,(f\xi,f)\\
&\quad=(e\xi e\xi x\xi,\xi e\xi x\xi)\,(f\xi,f)=(x,\xi x\xi)\,(f\xi,f)\\
&\quad=(x\,(\xi x\xi\circ f\xi)\,f\xi,\xi x\xi\,(\xi x\xi\circ f\xi)\,f)\\
&\quad=(e\xi x\xi\,(\xi x\xi\circ\xi)\,(\xi\circ f\xi)\,f\xi,\xi x\xi\,(\xi x\xi\circ\xi)\,(\xi\circ f\xi)\\
&\quad\cdot f\xi f)=(e\xi x\xi f\xi,\xi x\xi f\xi f)=(e\xi x\xi,a\xi f)=(x,a)\,.
\end{aligned}\tag{26}$$

Suppose that there exist $(\beta,\beta)\in\widehat{T},(e_1,f_1),(e_2,f_2)\in E(L|\times|_{\xi}R)$ such that $(x,a)=(e_1,f_1)(\beta,\beta)(e_2,f_2)$, $(e_1,f_1)\mathscr{L}(\beta,\beta)^{+}=(\beta^{+},\beta^{+})$, and $(e_2,f_2)\mathscr{R}(\beta,\beta)^{*}=(\beta^{*},\beta^{*})$. Then we have for $\alpha\in E(R_{e_1})$

$$\begin{aligned}
&(e_1,f_1)\,(\beta^{+},\beta^{+})=(e_1,f_1)\Longrightarrow\\
&\left(e_1\,(f_1\circ\beta^{+})\,\beta^{+},f_1\,(f_1\circ\beta^{+})\,\beta^{+}\right)=(e_1,f_1)\Longrightarrow\\
&\left(\alpha\xi e_1\,(f_1\circ\xi)\,(\xi\circ\beta^{+})\,\beta^{+},f_1\,(f_1\circ\xi)\,(\xi\circ\beta^{+})\,\beta^{+}\right)\\
&\quad=(e_1,f_1)\Longrightarrow\\
&\left(\alpha f_1\,(f_1\circ\xi)\,\xi\beta^{+}\xi,\xi f_1\xi\beta^{+}\xi\right)\Longrightarrow
\end{aligned}$$

$$(e_1\xi\beta^+\xi, \xi f_1\xi\beta^+\xi) = (e_1, f_1) \Longrightarrow$$
$$e_1\beta^+ = e_1,$$
$$f_1\beta^+ = f_1. \tag{27}$$

Similarly, $\beta^+e_1 = \beta^+$, $\beta^+f_1 = \beta^+$ and so $e_1e_1 = e_1\beta^+e_1 = e_1\beta^+ = e_1$, $f_1f_1 = f_1\beta^+f_1 = f_1\beta^+ = f_1 \in E(T)$. It follows that $\xi e_1\mathscr{L}e_1\mathscr{L}\beta^+$ and $f_1 = \beta^+$. Hence $f_1 = \xi e_1 = \beta^+$. In a similar way we have $e_2 = f_2\xi = \beta^*$ and $f_2 \in E(R_{\beta^*})$. Thus

$$\begin{aligned}
&(e_1, f_1)(\beta, \beta)(e_2, f_2) = ((e_1, \xi e_1)(\beta, \beta)(f_2\xi, f_2) \\
&\quad = (e_1(\xi e_1 \circ \beta)\beta, \xi e_1(\xi e_1 \circ \beta)\beta)(f_2\xi, f_2) \\
&\quad = (e_1\xi e_1(\xi e_1 \circ \xi)(\xi \circ \beta)\beta, \xi e_1(\xi e_1 \circ \xi)(\xi \circ \beta)\beta) \\
&\quad \cdot (f_2\xi, f_2) = (e_1\xi e_1\xi\beta\xi, \xi e_1\xi\beta\xi)(f_2\xi, f_2) \\
&\quad = (e_1\beta, \xi e_1\beta)(f_2\xi, f_2) = (e_1\beta(\xi e_1\beta \circ f_2\xi) \\
&\quad \cdot f_2\xi, \xi e_1\beta(\xi e_1\beta \circ f_2\xi)f_2) = (e_1\beta(\xi e_1\beta \circ \xi)(\xi \circ f_2\xi) \\
&\quad \cdot f_2\xi, \xi e_1\beta(\xi e_1\beta \circ \xi)(\xi \circ f_2\xi)f_2) \\
&\quad = (e_1\xi e_1\beta(\xi e_1\beta \circ \xi)\xi f_2\xi, \xi e_1\beta\xi(\xi \circ f_2\xi)f_2\xi f_2) \\
&\quad = (e_1\xi e_1\beta f_2\xi, \xi e_1\beta f_2\xi f_2) = (e_1\beta f_2\xi, \xi e_1\beta f_2).
\end{aligned} \tag{28}$$

It follows that $e_1\beta f_2\xi = x$ and so $\beta = \xi e_1\beta f_2\xi = \xi x\xi$. This shows that $\widehat{T}$ is an adequate transversal. That $\widehat{T}$ is a quasi-ideal holds by essentially the argument of [21, Theorem 17]

In summary, from the above and Theorem 18 we have that $L| \times |_\xi R$ is a locally adequate concordant semigroup with an adequate monoid transversal.

To show that every such semigroup is obtained in this way, let S be a locally adequate concordant semigroup with an adequate monoid transversal S°. Let the identity element of S° be ξ. By Theorem 25(3) we have $S^\circ = \xi S\xi$. Moreover, by Theorem 25(5), we see that $L = S\xi$ is a semigroup with right identity ξ, and $R = \xi S$ is a semigroup with left identity ξ. Thus ξ is a medial idempotent of both $S\xi$ and ξS. Moreover, $\xi L\xi = \xi S\xi = \xi R\xi$. Since $E(L) = I(S)$ and $E(R) = \Lambda(S)$, for every $x \in L$, $x\mathscr{R}^*e_x \in E(L)$ and $x\mathscr{L}^*f_x \in E(R)$. Hence $x = x\xi\mathscr{L}^*f_x\xi \in E(L)$ and so $f_x = f_x(f_x\xi) = f_x\xi \in T \subseteq L$. Hence L is abundant. Let $e \in E(L)$, $g \in E(S)$ be such that $g \leq_n e$. Then $g = ege = ege\xi \in L$. Hence E(L) is an order-ideal of $E(S)$ and so L is a $*$-subsemigroup of S. Let $x \in L$. Since S is IC, there is a bijection $\theta : < e_x > \longrightarrow < f_x >$ satisfying for all $y \in < e_x >, yx = x(y\theta)$. We have $< e_x > \subseteq e_xSe_x \subseteq L$, and $< f_x > \subseteq f_xSf_x \subseteq L$. Hence L is IC and so L is a concordant semigroup. Similarly, R is a concordant semigroup. We are therefore in the initial conditions of the first part. Consider therefore the mapping $\xi S \times S\xi \longrightarrow Reg(S^\circ) = Reg(\xi S\xi)$ given by

$$(a, x) \longmapsto a \circ x = f_ae_x. \tag{29}$$

To see that this satisfies property (1) above, observe that

$$\begin{aligned}
a(a \circ y)b \circ z &= af_ae_yb \circ z = ae_yb \circ z = f_{ae_yb}e_z \\
&= \left(f_ae_{e_yb}\overline{e_yb}\right)^* f_{e_yb}e_z \\
&= \left[f_ae_{e_y}\left(\overline{e_y}e_b\right)^+\overline{e_y}e_b\overline{b}\right]^*\left(\overline{e_y}e_b\overline{b}\right)^* f_be_z \\
&= \left(f_ae_y\overline{e_y}e_b\overline{b}\right)^*\left(\overline{e_y}e_b\overline{b}\right)^* f_be_z \\
&= \left(f_ae_yb\xi\right)^*\left(\xi e_yb\xi\right)^* f_be_z.
\end{aligned} \tag{30}$$

Since

$$\begin{aligned}
f_ae_yb\xi\left(\xi e_yb\xi\right)^* &= f_ae_y\xi e_yb\xi\left(\xi e_yb\xi\right)^* = f_ae_y\xi e_yb\xi \\
&= f_ae_yb\xi,
\end{aligned} \tag{31}$$

$(f_ae_yb\xi)^*(\xi e_yb\xi)^*\mathscr{L}^*f_ae_yb\xi(\xi e_yb\xi)^* = f_ae_yb\xi$. So $(f_ae_yb\xi)^*(\xi e_yb\xi)^* = (f_ae_yb\xi)^*$. Hence

$$\begin{aligned}
a(a \circ y)b \circ z &= \left(f_ae_yb\xi\right)^* f_be_z \\
&= \left[(a \circ y)b\xi\right]^*(b \circ z).
\end{aligned} \tag{32}$$

It is readily verified that (2)-(5) also hold.

Consider now the mapping $\vartheta : S \longrightarrow S\xi| \times |_\xi\xi S$ given by $\vartheta(x) = (x\xi, \xi x)$. For all $x, y \in S$, we have

$$\begin{aligned}
\vartheta(x)\vartheta(y) &= (x\xi, \xi x)(y\xi, \xi y) \\
&= (x\xi(\xi x \circ y\xi)y\xi, \xi x(\xi x \circ y\xi)\xi y) \\
&= \left(x\xi f_{\xi x}e_{y\xi}y\xi, \xi xf_{\xi x}e_{y\xi}\xi y\right) \\
&= \left(x\xi\left(\xi e_x\overline{x}\right)^* f_xy\xi, \xi xe_y\left(\overline{y}f_y\xi\right)^+\xi y\right) \\
&= \left(x\xi\left(\xi e_x\xi x\xi\right)^* f_xy\xi, \xi xe_y\left(\xi y\xi f_y\xi\right)^+\xi y\right) \\
&= \left(e_x\xi x\xi\left(\xi e_xx\xi\right)^* f_xy\xi, \xi xe_y\left(\xi y\xi\right)^+\xi y\xi f_y\right) \\
&= \left(e_x\xi x\xi\left(\xi x\xi\right)^* f_xy\xi, \xi xe_y\xi y\xi f_y\right) \\
&= \left(e_x\xi x\xi f_xy\xi, \xi xe_yyf_y\right) = (xy\xi, \xi xy) = \vartheta(xy)
\end{aligned} \tag{33}$$

and so ϑ is a morphism.

If $\vartheta(x) = \vartheta(y)$, then $x\xi = y\xi$ and $\xi x = \xi y$. Hence $\overline{x} = \xi x\xi = \xi y\xi = \overline{y}$. Thus $e_{x\xi} = e_x(\overline{x}f_x\xi)^+ = e_x(\xi x\xi f_x\xi)^+ = e_x(\xi xf_x\xi)^+ = e_x(\xi x\xi)^+ = e_x\overline{x}^+ = e_x$. Similarly, $f_{\xi x} = f_x$. Consequently, $x = e_x\overline{x}f_x = e_{x\xi}\overline{x}f_{\xi x} = e_{y\xi}\overline{y}f_{\xi y} = e_y\overline{y}f_y = y$. Thus ϑ is injective.

To see that ϑ is also subjective, let $(x\xi, \xi y) \in S\xi| \times |_\xi\xi S$. Then $\xi x\xi = \xi y\xi$, whence $\overline{x} = \overline{y}$. Now let $t = e_xy$. Then $t\xi = e_x\xi y\xi = e_x\xi x\xi = e_xx\xi = x\xi$ and $\xi t = \xi e_xy = \xi e_x\xi y = \overline{e_x}y = e_{\overline{x}}y = e_{\overline{y}}y = \overline{e_y}y = \xi e_y\xi y = \xi e_yy = \xi y$. Thus $\vartheta(t) = (t\xi, \xi t) = (x\xi, \xi y)$.

It follows from the above that $S \cong S\xi| \times |_\xi\xi S$. □

We can use the above theorem to establish a structure theorem for naturally ordered concordant semigroups with an adequate monoid transversal, which generalizes [21, Theorem 18].

Theorem 33. *Let S be a naturally ordered concordant semigroup with an adequate transversal S° that is a monoid with identity element ξ. Let $S\xi| \times |_\xi \xi S$ consist of the subset of the Cartesian ordered set $S\xi \times \xi S$ given by $S\xi| \times |_\xi \xi S = \{(x\xi, \xi x) \mid x \in S\}$ together with the multiplication $(x\xi, \xi x)(y\xi, \xi y) = (xy\xi, \xi xy)$. Then, $S\xi|\times|_\xi \xi S$ is an ordered concordant semigroup. Moreover, if either $\mathscr{L}^*$ or $\mathscr{R}^*$ is regular on S then there is an ordered semigroup isomorphism:*

$$S \cong S\xi \ | \times|_\xi \ \xi S. \tag{34}$$

Proof. Since S is locally adequate by Theorem 18, it follows from Theorems 25(5) and 32 that there is an algebraic isomorphism $\vartheta : S \longrightarrow S\xi| \times |_\xi \xi S$ given by $\vartheta(x) = (x\xi, \xi x)$. Suppose now that, for example, $\mathscr{R}^*$ is regular on S. If $\vartheta(x) \leq \vartheta(y)$, then $x\xi \leq y\xi$ and $\xi x \leq \xi y$. Hence, $x = e_x x = e_{x\xi} x = e_{x\xi}\xi x \leq e_{y\xi}\xi y = y$. Thus, $x \leq y \Longleftrightarrow \vartheta(x) = \vartheta(y)$. Therefore, the isomorphism ϑ is an order isomorphism. Similarly, if $\mathscr{L}^*$ is regular, then the isomorphism ϑ is also an order isomorphism. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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References

[1] T. S. Blyth and M. H. Almeida Santos, "A classification of inverse transverses," *Communications in Algebra*, vol. 29, pp. 611–624, 2001.

[2] T. Blyth and M. Almeida Santos, "Amenable orders associated with inverse transversals," *Journal of Algebra*, vol. 240, no. 1, pp. 143–164, 2001.

[3] T. S. Blyth and R. McFadden, "Regular semigroups with a multiplicative inverse transversal," *Proceedings of the Royal Society of Edinburgh*, vol. A91, pp. 107–122, 1981.

[4] T. Saito, "Quasi-orthodox semigroups with inverse transversals," *Semigroup Forum*, vol. 36, no. 1, pp. 47–54, 1987.

[5] A. El-Qallali, "Abundant semigroups with a multiplicative type A transversal," *Semigroup Forum*, vol. 47, no. 1, pp. 327–340, 1993.

[6] J. Al-Bar and J. Renshaw, "On adequate transversals," *Communications in Algebra*, vol. 37, no. 7, pp. 2309–2324, 2009.

[7] J. F. Chen, "Abundant semigroups with adequate transversals," *Semigroup Forum*, vol. 60, no. 1, pp. 67–79, 2000.

[8] A. El-Qallali and J. B. Fountain, "Quasi-adequate semigroups," *Proceedings of the Royal Society of Edinburgh*, vol. 91A, pp. 91–99, 1981.

[9] A. El-Qallali and J. B. Fountain, "Idempotent-connected abundant semigroups," *Proceedings of the Royal Society of Edinburgh*, vol. 91A, pp. 79–90, 1981.

[10] J. Fountain, "Adequate semigroups," *Proceedings of the Edinburgh Mathematical Society*, vol. 22, no. 2, pp. 113–125, 1979.

[11] J. Fountain, "Abundant semigroups," *Proceedings of the London Mathematical Society*, vol. 44, no. 3, pp. 109–129, 1982.

[12] X. Guo and L.-M. Wang, "Idempotent connected abundant semigroups which are disjoint union of quasi-ideal adequate transversals," *Communications in Algebra*, vol. 30, pp. 1779–1800, 2002.

[13] X. Guo and K. P. Shum, "Abundant semigroups with Q-adequate transversals and some of their special cases," *Algebra Colloquium*, vol. 14, pp. 687–704, 2007.

[14] X. J. Guo and K. P. Shum, "Right pp semigroups with right adequate transversal," in *Proceeeedings of the International conference in Algebra*, Advances in Algebra and combinatorics, pp. 309–319, World Scientific Inc, 2010.

[15] X. J. Guo, "Abundant semigroup with a multiplicative adequate transversal," *Acta Mathematica Sinica (Ser. B)*, vol. 18, pp. 229–244, 2002.

[16] X.-J. Guo and Y.-F. Luo, "The natural partial orders on abundant semigroups," *Advances in Mathematics*, vol. 34, pp. 297–308, 2005 (Chinese).

[17] X. J. Guo, "Amenably natural order abundant semigroups," *Advances in Mathematics*, vol. 30, pp. 156–164, 2001 (Chinese).

[18] M. V. Lawson, "The natural partial orders on an abundant semigroup," *Proceedings of the Edinburgh Mathematical Society*, vol. 30, pp. 169–186, 1987.

[19] M. V. Lawson, "Abundant Rees matrix semigroups," *Journal of the Australian Mathematical Society*, vol. 42A, pp. 132–142, 1987.

[20] M. Luo, "Relationship between the quasi-ideal adequate transversals of an abundant semigroup," *Semigroup Forum*, vol. 67, no. 3, pp. 411–418, 2003.

[21] T. S. Blyth and M. H. Almeida Santos, "Naturally ordered regular semigroups with an inverse monoid transversal," *Semigroup Forum*, vol. 76, no. 1, pp. 71–86, 2008.

[22] T. S. Blyth, *Lattices and Ordered Algebraic Structure*, Springer, London, UK, 2005.

[23] J. M. Howie, *Fundamentals of Semigroup Theory*, vol. 12 of *London Mathematical Society Monographs, New series*, Oxford Univ Press, New York, NY, USA, 1995.