

## Research Article

# A Novel Distribution and Optimization Procedure of Boundary Conditions to Enhance the Classical Perturbation Method Applied to Solve Some Relevant Heat Problems

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Received 10 June 2019; Accepted 27 August 2019; Published 24 January 2020

Academic Editor: Allan C. Peterson

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This work introduces a novel modification of classical perturbation method (PM), denominated Optimized Distribution of Boundary Conditions Perturbation Method (ODBCPM) with the purpose to improve the performance of PM in the solution of ordinary differential equations (ODEs). We will see that the main proposal of ODBCPM rests above all in the redistribution and optimization of the boundary conditions of the problem to be solved among the iterations of the proposed method. The solution of a couple of heat relevant problems indicates the potentiality of ODBCPM even for the case of large values of the perturbative parameter.

## 1. Introduction

One of the first methods proposed with the end to provide analytical approximate solutions to nonlinear differential equations was the classical perturbation method (PM). PM was proposed initially by S.D. Poisson, but it was Poincare who provided the main contribution to the method, building its theoretical bases, and applying widely it to various problems of nonlinear oscillators.

The main assumption of PM is that it is possible to express a nonlinear differential equation in terms of a linear and a nonlinear part [1–3] where, the nonlinear part is considered as a small perturbation through a small parameter (the perturbation parameter, which in principle should be much smaller than one). It is well known that this assumption is considered in principle, as a serious disadvantage of PM. Although

perturbation method generally provides better results for small parameter values, some articles have reported precise solutions in the application of PM to problems even for relatively large values of the perturbation parameter. Some of these works refer not only to the mere application of PM to particular problems but to modifications of the same and combinations of PM with other methods. Next, we briefly summarize some articles about the results obtained by PM and some of its modifications in the solution of some problems of interest.

Reference [4] applied PM to solve the complete elliptic integrals of the second kind and elliptic integrals of the first kind. Even though PM did not provide a satisfactory approximation for elliptic integral of the first kind, when it was coupled with Pade method, a good approximation was found. On the other hand, for the case of elliptic integrals of the second kind, PM obtained accurate approximations for both,

complete and incomplete integrals even for relatively big values of the perturbation parameter.

Article [5] employed PM to obtain a handy approximate solution to Gelfand's differential equation which governs combustible gas dynamics. In this case PM obtained results with good accuracy, even for relatively large values of the perturbation parameter.

Reference [6] proposed PM in order to provide a study of a nonlinear galactic model. Although it obtained accurate approximated solutions, the value of the perturbation parameter employed was indeed very small.

In [7] PM was used in order to obtain approximate solutions for the nonlinear differential equation that models the diffusion and reaction in porous catalysts. PM obtained good precision for this case study, regarding small values of the perturbation parameters employed.

In [8] this method was used with the purpose to get an approximate solution for Troesch differential equation. In addition, the coupling of Laplace transform and Padé transformation was introduced in order to deal with the truncated series obtained by the PM. In both cases, handy accurate approximations were obtained, even for the case of relatively large values of the perturbation parameter.

Article [9] employed PM in order to get an approximate solution to the steady state nonlinear reaction diffusion equation containing a nonlinear term related to Michaelis–Menten equation. In this article PM obtained accurate approximate solutions, even for relatively large values of the perturbation parameter.

Reference [10] proposed Perturbation Method with the purpose to provide an approximate solution for the problem of a symmetric axis Newtonian fluid squeezed between two large parallel plates. We noted that the proposed solutions were both, handy and accurate even for large values of the perturbation parameter and therefore that PM was efficient for this case study.

In [11] PM was employed to get approximate solutions for the case of differential equations related with heat transfer phenomena. This paper presented two case studies. The first one, showed the potential of PM. It obtained a solution with good accuracy despite that the presented case study, employed a big value of the perturbation parameter. The second case study proposed a problem with mixed boundary conditions. In this case PM obtained just an acceptable precision, even though the proposed problem used a small value of the perturbation parameter. In particular, PM was not so efficient to model the initially unknown left boundary.

Article [12] presented the Enhanced Perturbation Method (EPM) as a novel modification to the PM, with the purpose to solve nonlinear differential equations depending on a perturbative parameter. The method works applying differential operators on both sides of a nonlinear differential equation, in order to get higher order equations in the successive stages and, therefore, additional adjustment parameters, which are determined, so that we get an analytical approximate solution for the nonlinear differential equation to solve. This work, compared PM and EPM methods through a couple of case studies. In spite of PM solutions provided solutions of good accuracy, even for big values of the perturbation parameter,

EPM resulted clearly better, as larger values of the parameter were considered. In particular, the Gelfand's equation was studied for both methods, and from [5] we know that third order approximation for PM obtained good precision even for values of the perturbation parameter of  $\varepsilon = 1$  and  $\varepsilon = 1.5$ . Nevertheless, this work showed that PM third order approximation did not result adequate for  $\varepsilon = 2.5$ , while that EPM first order approximation was very accurate for this case study.

Reference [13] presented the Laplace transform–perturbation method (LT–PM) as a novel modification to the PM, with the purpose to solve nonlinear differential equations depending on a perturbative parameter with boundary conditions defined on finite intervals. The article showed that LT–PM has potential to find multiple solutions for nonlinear problems and enhances the application of PM, in some cases of mixed and Neumann boundary conditions, where PM is unsuitable to provide results.

This brief compilation of works on PM suggests that although the method usually works well in the expected case of small values of the perturbation parameter, it is also efficient in some problems with large parameters. It is also clear that there are case studies for which, although PM provides approximate solutions with good accuracy for large values of the already mentioned perturbative parameter, its performance increases significantly by modifying it as it occurred in [4, 12, 13], where the proposed methods obtained good approximations for larger values of the parameter. In a sequence, [13] showed that the proposed modification of perturbation method, denominated LT–PM was able to get accurate approximations, where PM resulted unable to provide any solution.

Since perturbative problems are common in science and engineering, the foregoing observations indicate that PM and especially its modifications is a subject that should be further investigated. The appearance of more sophisticated methods has been justified on the basis of the assumed limitation that PM is a method applicable to the case of problems that are weakly perturbed by the nonlinear term. Indeed, such as it was already mentioned, this is not always the case, therefore this article proposes to modify the usual way of using the perturbation method, through a method which we will call, from here on Optimized Distribution of Boundary Conditions Perturbation Method (or initial conditions, as the case may be) (ODBCPM) with the purpose to improve the performance of PM by means of a procedure which in principle should provide optimal results for both, small and large  $\varepsilon$  values. Thus, unlike the aforementioned comments about PM, from which it is deduced that does not exist certainty about the results obtained with PM when it is applied a nonlinear problem, the ODBCPM proposal consists in providing a method focus to nonlinear perturbative problems, which as we will see after, systematically offers results based on an optimization criterion. In order to grasp the motivation of ODBCPM in the context of the above considerations it is necessary provide some basic ideas of PM and some mathematical results which will turn out useful later, when two heat related problems are presented as case studies. The main idea is to note that the usual procedure of PM, to introduce the boundary conditions of the problem to be solved from the first iteration is not always convenient and that redistributing them in the iterative

process can result in a better approximation, even for large values of the perturbative parameter.

The rest of this work is organized as follows. Section 2 introduces the basic idea of PM. In Section 3, we provide two relevant mathematical results involved with ordinary differential equations which are important for this work. Section 4 provides the basic idea of perturbation method with optimized distribution of boundary conditions (ODBCPM) while Section 5 discusses the least square method as a useful tool to optimize the approximated solutions which emanate from ODBCPM. Section 6 presents the application of the proposed method for the case of two relevant heat problems, while Section 7 provides a detailed discussion of the relevant aspects of this work and the main obtained results. Finally, the conclusions of the work are exposed in Section 8.

## 2. Basic Idea of Classical Perturbation Method (PM)

We will assume that the ordinary differential equation of one dimensional nonlinear system can be written in the form [1, 2, 4–14].

$$L(x) + \varepsilon N(x) = 0, \quad (1)$$

where  $x$  is a function of one variable  $x = x(t)$ ,  $L(x)$  is a linear operator which, in general, contains derivatives in terms of  $t$ ,  $N(x)$  is a nonlinear operator, and  $\varepsilon$  is a small parameter.

Considering that the nonlinear term in (1) is a small perturbation and assuming that the solution for (1) can be written as

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots \quad (2)$$

After substituting (2) into (1) and equating terms having identical powers of  $\varepsilon$ , we get a number of differential equations that can be integrated, recursively, to find successively the functions:  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ ,  $\dots$  in accordance with the following general scheme:

$$\begin{aligned} L(x_0) &= 0, x_0(A) = a, x_0'(A) = b, \\ x_0(B) &= c, x_0'(B) = d, \\ L(x_1) + N(x_0) &= 0, x_1(A) = 0, x_1'(A) = 0, \\ x_1(B) &= 0, x_1'(B) = 0, \\ L(x_2) + N(x_0, x_1) &= 0, x_2(A) = 0, x_2'(A) = 0, \\ x_2(B) &= 0, x_2'(B) = 0, \\ L(x_j) + N(x_0, x_1, \dots, x_{j-2}, x_{j-1}) &= 0, x_j(A) = 0, x_j'(A) = 0, \\ x_j(B) &= 0, x_j'(B) = 0, \dots \end{aligned} \quad (3)$$

where  $t = A$  and  $t = B$ , denote the boundary conditions of the problem.

It is clear that  $x_0(t)$  is the first approximation for the solution of (1) that satisfies its boundary conditions, and  $x_1(t)$ ,  $x_2(t)$ ,  $x_3 \dots$  are the higher order approximations, whose values have to be zero if they are evaluated for the same boundary values in accordance with the classical procedure implemented by PM. On the other hand, for the case of problems with initial conditions, the changes in scheme (3) would be the expected.

## 3. Mathematical Results

Next, relevant results are introduced in order to provide a stronger mathematical fundamental to the proposed case studies for this work.

**3.1. Poincaré–Lyapunov Theorem.** We present briefly the version for a scalar equation of Poincaré–Lyapunov theorem, which is a basic result in the study of stability of equilibrium points [14].

Regarding the differential equation

$$\frac{dy}{dx} = Ay + g(y), \quad y(0) = c, \quad (4)$$

where,  $A$  is a constant and  $g(y)$  is a nonlinear function. It is possible to furnish the following sufficient condition.

**Theorem 1.** *If*

- (1) *the solution of the linear equation  $dy/dx = Ay$ , approaches to  $y \rightarrow \text{zero}$  as  $x \rightarrow \infty$ ,*
- (2) *the initial value,  $c$  is sufficiently close to the origin,*
- (3) *the power series of  $g(y)$  lacks of constant and linear terms,*

*then the solution of (4) approaches to  $y \rightarrow \text{zero}$  as  $x \rightarrow \infty$  (see the second case study and Discussion Section).*

**3.2. Theorems of Existence and Uniqueness**

**Theorem 2.** *Suppose that  $D = \{(t, y) / a \leq t \leq b, -\infty < y < \infty\}$ , and that  $f(t, y)$  is a continuous function in  $D$ . If  $f$  satisfies a Lipschitz condition in  $D$  for the variable  $y$  [15], then the initial value problem, expressed as.  $y'(t) = f(t, y)$ ,  $a \leq t \leq b$ ,  $y(a) = A$ , has a unique solution  $y(t)$  for  $a \leq t \leq b$ .*

**Theorem 3.** *Suppose that the function  $F$  defined in accordance with the general boundary value problem [15].*

$$y'' = F(x, y, y'), \quad \alpha \leq x \leq \beta, \quad y(\alpha) = a, \quad y(\beta) = b, \quad (5)$$

*is continuous in the set*

$$S = \{(x, y, y') \mid \alpha \leq x \leq \beta, -\infty < y < \infty, -\infty < y' < \infty\}, \quad (6)$$

*and that the partial derivatives  $F_y$  and  $F_{y'}$  are also continuous in  $S$ . Then, if.*

1.  $\partial F(x, y, y') / \partial y > 0$ , for all  $(x, y, y') \in S$ , and
2. *a constant  $N$  exist, defined in such a way that,*

$$|\partial F(x, y, y') / \partial y'| \leq N, \quad \text{for all } (x, y, y') \in S, \quad (7)$$

*then the boundary value problem (5) has a unique solution (see the first case study and Discussion Section).*

### 4. Main Idea of ODBCPM

The distribution of the boundary conditions among the iterations of a perturbative iterative method is found in [16] where this idea was used to improve the application of the rational homotopy perturbation method (RHPM) to problems containing two-point Neuman boundary conditions and using the Least Square Method proposed to optimize the boundary conditions. However, the main differences among this work and the published one in [16] are: (1) ODBCPM introduces a general procedure to distribute the boundary conditions among all the iterations, while [16] does not provides such mechanisms, (2) On one hand, this work proposes a modification for the well-known and studied, perturbation method in order to distribute effectively all the boundary conditions of each iteration of the method to satisfy the boundary conditions of the proposed nonlinear problem. On the other hand, [16] employed a new method denominated RHPM which is still under development. Finally, both methods employ the Least Square Method in order to optimize the obtained solutions using the distributed boundary conditions.

Essentially, ODBCPM repeats the same steps of PM and the relevant changes consist, on one hand, in the proposal of distributing the values of the boundary conditions of the problem (or at least, one of them) among the different orders of the iterative process, and on the other hand, the optimization of the above mentioned values, with the purpose to get a better approximation than the one obtained by PM.

We assume again a perturbative equation as (1).

$$L(x) + \epsilon N(x) = 0, \quad x(A) = a, \quad x(B) = f, \quad (8)$$

where, to be more specific, we will assume for the rest of this section that  $L(x)$  is a linear operator of second order (for the case where  $L(x)$  be an operator of first order, the arguments that follow are immediately transferred to the only initial condition of the problem to solve).

Next we propose to distribute, for instance, the right boundary conditions of (8) among the different orders of the iterative process, in such a way that in accordance with the propose method, in order to find successively the functions:  $x_0(t), x_1(t), x_2(t), \dots$  mentioned in Section 2, we will use the following procedure.

$$\begin{aligned} L(x_0) &= 0, \quad x_0(A) = a, \quad x_0(B) = b, \\ L(x_1) + N(x_0) &= 0, \quad x_1(A) = 0, \quad x_1(B) = c, \\ L(x_2) + N(x_0, x_1) &= 0, \quad x_2(A) = 0, \quad x_2(B) = d, \\ \dots L(x_j) + N(x_0, x_1, \dots, x_{j-2}, x_{j-1}) &= 0, \quad x_j(A) = 0, \quad x_j(B) = e, \\ &\dots \end{aligned} \quad (9)$$

where  $t = A$  and  $t = B$ , delimit the boundary conditions of the problem and the right boundary conditions for each iteration are related with the right boundary condition of the problem  $x(B) = f$ , in accordance with the relation

$$b + \epsilon c + \epsilon^2 d + \dots + \epsilon^j e + \dots = f. \quad (10)$$

In each iteration, two integration constants are generated, one of which is used so that the corresponding solution satisfies the boundary condition that is handled according to PM, while the

other one is expressed in terms of  $b, c, d, \dots, e$ . Next, the solutions that emanates from (9) are substituted into (2) in order to get an approximate solution for the nonlinear problem to solve. In this point (10) is substituted into the ODBCPM solution, with the purpose that it satisfies the right boundary condition of (8). Then, the proposed solution will depend on some of the boundary conditions  $b, c, d, \dots, e$  and the goal is to optimize their values in order that the approximated solution obtained in this way provides an accurate analytical approximate solution.

Of course, it is also possible to distribute the two boundary conditions instead of just one, but the mathematical procedure involved in the different iterations could become too cumbersome. In the same way, the process to optimize the values of boundary conditions for each iteration, which will be explained below, may become computationally more complicated. Thus, for the sake of simplicity, for the case of second order ODEs with boundary or initial conditions, ODBCPM procedure is preferably applied to one of the conditions, or to the only initial condition for the case of first order differential equations.

### 5. Approximations by Employing Least Square Method

With the end to optimize the values of the boundary conditions  $b, c, d, \dots, e$  in order to obtain accurate approximate solutions we will employ the following procedure reported in [15–17] (see below).

Let  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_i, y_i)$  be the data set coordinates where  $i = 0, 1, 2, \dots, n$ , and the adjustment curve  $y = f(x_i, a_0, a_1, a_2, \dots, a_j)$ ; where  $a_j$  (for  $j = 0, 1, 2, \dots$ ) are adjustment parameters (which for our case study, they correspond to some of the initially unknown boundary conditions  $b, c, d, \dots, e$ ). The Least Squares method aims to minimize the sum of the squares from vertical distances of  $y_i$  values to the corresponding  $f(x_i, a_0, a_1, a_2, \dots, a_j)$ , thus obtaining a function  $S(a_0, a_1, a_2, \dots, a_j)$  which defines the quadratic error, expressed by

$$S(a_0, a_1, a_2, \dots, a_j) = \sum_{i=1}^n ((y_i) - f(x_i, a_0, a_1, a_2, \dots, a_j))^2. \quad (11)$$

In order to minimize the error with respect to the sample set, we calculate the partial derivatives with respect to each of the adjustment parameters, in order to create a nonlinear equation system based on the criterion of relative extremes for functions of several variables, which is given by

$$\begin{aligned} \frac{\partial S}{\partial a_0} &= \frac{\partial}{\partial a_0} \left( \sum_{i=1}^n ((y_i) - f(x_i, a_0, a_1, a_2, \dots, a_j))^2 \right) = 0, \\ \frac{\partial S}{\partial a_1} &= \frac{\partial}{\partial a_1} \left( \sum_{i=1}^n ((y_i) - f(x_i, a_0, a_1, a_2, \dots, a_j))^2 \right) = 0, \\ \frac{\partial S}{\partial a_2} &= \frac{\partial}{\partial a_2} \left( \sum_{i=1}^n ((y_i) - f(x_i, a_0, a_1, a_2, \dots, a_j))^2 \right) = 0, \\ &\vdots \\ \frac{\partial S}{\partial a_n} &= \frac{\partial S}{\partial a_n} \left( \sum_{i=1}^n ((y_i) - f(x_i, a_0, a_1, a_2, \dots, a_j))^2 \right) = 0. \end{aligned} \quad (12)$$

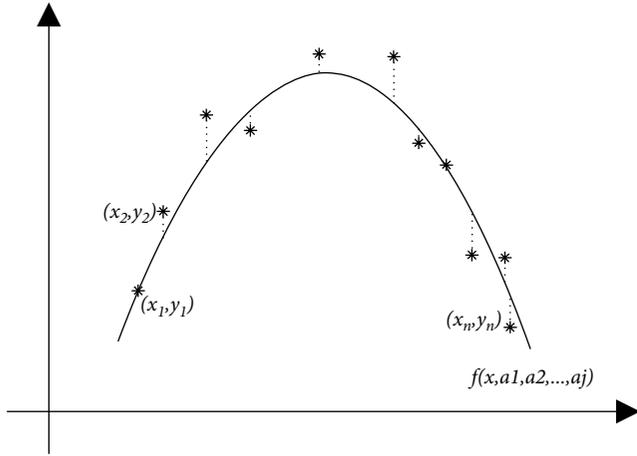


FIGURE 1: Least square method.

After solving (12) for adjustment constants  $a_p$ , the model with the best adjustment to the data set is obtained. From Figure 1, we note the data set is represented by asterisks for each data  $(x_i, y_i)$  where  $i = 1, 2, 3, \dots$ . In this fashion, the best adjusting data function is obtained and it is represented by a solid black line.

## 6. Applications of ODBCPM for the Case of Two Relevant Heat Problems

As first case study, we will consider the following nonlinear problem with mixed boundary conditions, which consists in determining the temperature distribution in a uniformly thick rectangular fin radiation to free space with nonlinearity of higher order [11, 18, 19].

$$\frac{d^2 y(x)}{dx^2} - \epsilon y^4(x) = 0, \quad 0 \leq x \leq 1, \quad y'(0) = 0, y(1) = 1, \quad (13)$$

where  $y(x)$  is a dimensionless variable directly related with the temperature of the fin radiating above mentioned (in fact, from [18] we note that the temperature  $T$  used in the problem under study, which gives rise to (13), is expressed in Kelvin degrees and given that  $y = T/T_a$ , where  $T_a$  is the temperature in Kelvin degrees of the right end of the fin, then we will assume that  $y > 0$  for  $0 \leq x \leq 1$ ) while  $\epsilon$  is a positive parameter of the problem, which is related directly with the emissivity and inversely with the thermal conductivity of the fin [18, 19].

In order to frame the problem (13) from a more formal mathematical point of view, next we will analyse the feasibility of guaranteeing the existence and possibly the uniqueness of the solution of (13) in accordance with the Theorem 3. As it will be seen, we will complement the mathematical arguments with the physical ones in order to facilitate the discussion. As a matter of fact, we will be able only to ensure, at least the existence of one solution.

Since  $\epsilon > 0$ , then from (13) we deduce that  $d^2 y/dx^2 > 0$ , in the interval of interest and for the same reason  $y$  has to be concave upwards. Besides, mathematically the function defined by  $F_1(x, y, y') = \epsilon y^4$ , is clearly continuous in the set

$$S = \{(x, y, y') \mid 0 \leq x \leq 1, -\infty < y < \infty, -\infty < y' < \infty\}. \quad (14)$$

Since that, as it was previously mentioned for physical reasons  $y > 0$ , we will redefine  $F(x, y, y')$  by convenience as follows:

$$\begin{aligned} F(x, y, y') &= \epsilon y^4, \quad y < 0, \\ F(x, y, y') &= -\epsilon y^4, \quad y \leq 0, \end{aligned} \quad (15)$$

without causing significant alterations in the approach of the problem (where the relevant part corresponds to  $y > 0$ ), for instance (15) is continuous in  $S$  (see (14)).

From (15),  $F_x$  and  $F_y$  are continuous in  $S$  and although (13) provides only the derivative of  $y$  at  $x = 0$ , it is clear from the previously explained that the starting point(s)  $y(0)$ , must be at some point in the interval  $(0, 1)$  (note that  $y'(0) = 0$ ). Since  $\partial F(x, y, y')/\partial y > 0$ , for all  $(x, y, y') \in S$ ; we note that this inequality is satisfied except in  $y = 0$  (see Discussion section) and clearly exist not only one, but an infinity of constants  $N$ , such that satisfy  $|\partial F(x, y, y')/\partial y'| \leq N$ , for  $(x, y, y') \in S$ , then we would have concluded from Theorem 3 that the boundary value problem (13) has a unique solution, unless because it is not possible to ensure only one value  $y(0)$  corresponding to  $y'(0) = 0$ , for  $y \in (0, 1)$ . If it was the case that exists more than one value of  $y(0)(\alpha_1, \alpha_2, \dots, \alpha_n)$  we could apply the results obtained above separately for each pair of boundary conditions  $\{y(0) = \alpha_1, y(1) = 1\}$ ,  $\{y(0) = \alpha_2, y(1) = 1\}$ , ...  $\{y(0) = \alpha_n, y(1) = 1\}$  and ensures that the problem (13) has  $n$  solutions. Thus, this discussion guarantees that the problem has at least one solution; from the latter, and the fact that (13) defines a perturbative problem, this increases the possibility that PM provides an approximate solution, at least for small values of the parameter. However, reference [11] shows that even for the relatively small value of  $\epsilon = 0.30$ , the fifth order approximation of PM was not able to accurately model the unknown left boundary and therefore it might be necessary to consider even higher order approximations though PM was used to model just a perturbative case.

In accordance with ODBCPM procedure, next we will propose redistribute the right boundary condition of (13) among the different iterations, while the left boundary will be handed in accordance with the known algorithm of classical PM (see Section 2). As a matter of fact, we will obtain better results than the one obtained in [11], even regarding larger values of the perturbative parameter.

At the first iteration, we consider in accordance with ODBCPM.

$$\epsilon^0) \quad y_0'' = 0, y_0'(0) = 0, y_0(1) = \alpha, \quad (16)$$

the solution of the above equation is given by

$$y_0(x) = c_1 x + c_2, \quad (17)$$

where  $c_1$  and  $c_2$  are integration constants, which are determined applying the boundary conditions, so that we obtain

$$y_0(x) = \alpha. \quad (18)$$

The second iteration is given as

$$\varepsilon^1) \quad y_1''(x) - y_0^4(x) = 0, \quad y_1'(0) = 0, \quad y_1(1) = \beta, \quad (19)$$

after substituting (18) into (19), the solution of the above equation is given by

$$y_1(x) = \frac{\alpha^4 x^2}{2} + c_3 x + c_4, \quad (20)$$

we note that  $c_3$  and  $c_4$  are integration constants, which are determined after applying the boundary conditions from (19), in such a way that we get

$$y_1(x) = \frac{\alpha^4 x^2}{2} - \frac{\alpha^4}{2} + \beta. \quad (21)$$

On the other hand, the third iteration is expressed as follows:

$$\varepsilon^2) \quad y_2''(x) - 4y_0^3(x)y_1(x) = 0, \quad y_2'(0) = 0, \quad y_2(1) = \gamma, \quad (22)$$

after substituting (18) and (21), and integrating twice (22), we get

$$y_2(x) = \frac{\alpha^7 x^4}{6} - \alpha^7 x^2 + 2\beta\alpha^3 x^2 + c_5 x + c_6. \quad (23)$$

In the same way, applying the boundary conditions from (22) we rewrite (23) in the form.

$$y_2(x) = \frac{\alpha^7(x^4 - 1)}{6} + \alpha^7(1 - x^2) + 2\beta\alpha^3(x^2 - 1) + \gamma. \quad (24)$$

The fourth iteration is given by

$$\varepsilon^2) \quad y_3''(x) - 6y_0^2(x)y_1^2(x) - 4y_0^3(x)y_2(x) = 0, \quad y_3'(0) = 0, \quad y_3(1) = \Gamma, \quad (25)$$

The integration of (25), after using the previous approximations (18), (21) and (24) yields to the following result for  $y_3$ .

$$y_3(x) = x^2 \left( \frac{29\alpha^{10}}{12} - 7\alpha^6\beta + 3\alpha^2\beta^2 + 2\alpha^3\gamma \right) + \frac{65\alpha^{10}x^6}{900} + \left[ \frac{-7\alpha^{10}}{12} + \frac{7\alpha^6\beta}{6} \right] + c_7 x + c_8. \quad (26)$$

Applying the boundary conditions from (25), it is possible to evaluate the constants of integration  $c_7$  and  $c_8$ , in such a way that it is possible to rewrite (26) as follows:

$$y_3(x) = \Gamma + \frac{65\alpha^{10}(x^6 - 1)}{900} + \left( \frac{-7\alpha^{10}}{12} + \frac{7\alpha^6\beta}{6} \right) (x^4 - 1) + \left( \frac{29\alpha^{10}}{12} - 7\alpha^6\beta + 3\alpha^2\beta^2 + 2\alpha^3\gamma \right) (x^2 - 1). \quad (27)$$

A third order approximation for (13) is obtained, by substituting (18), (21), (24) and (27) into

$$y(x) = \sum_{n=0}^3 \varepsilon^n y_n(x). \quad (28)$$

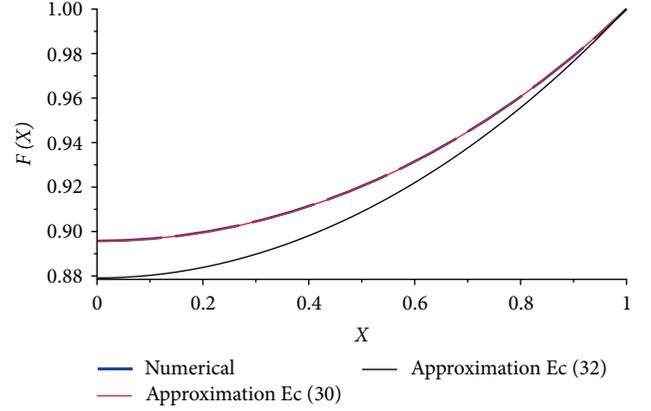


FIGURE 2: Comparison between numerical solution of (13) and approximations (30) and (32) for  $\varepsilon = 0.30$ .

Next, we will apply the general boundary condition (10) in the form  $\Gamma\varepsilon^3 = 1 - \alpha - \varepsilon\beta - \gamma\varepsilon^2$ , which is employed to ensure that  $y(1) = 1$ .

The explicit approximate solution that rises of the above mentioned procedure is given by

$$y(x) = 1 + \frac{\varepsilon\alpha^4(x^2 - 1)}{2} + \varepsilon^2 \left[ \frac{\alpha^7(x^4 - 1)}{6} + (-\alpha^7 + 2\alpha^3\beta)(x^2 - 1) \right] + \varepsilon^3 \left[ \frac{65\alpha^{10}(x^6 - 1)}{900} + \left( -\frac{7\alpha^{10}}{12} + \frac{7\alpha^6\beta}{6} \right) (x^4 - 1) + \left( \frac{29}{12}\alpha^{10} - 7\alpha^6\beta + 3\alpha^2\beta^2 + 2\alpha^3\gamma \right) (x^2 - 1) \right]. \quad (29)$$

Note that from (29) it is obtained  $y(1) = 1$  as it has to be (see right boundary condition of (13)).

As a case study we propose the values of  $\varepsilon = 0.30$  and  $\varepsilon = 2$ . In order to get that (29) corresponds to an accurate approximate solution of (13), we will identify the conditions  $\alpha, \beta$ , and  $\gamma$  for both cases, by using the Least Square Method, which is detailed explained in Section 5, resulting in the following handy expressions.

$$y(x) = 0.895783918821769 + 0.0961177191079606x^2 + 0.00821629349970925x^4 \quad (\varepsilon = 0.30), \quad (30)$$

$$y(x) = 0.694822163554290 + 0.219330073635731x^2 + 0.0873444721669237x^4 \quad (\varepsilon = 2). \quad (31)$$

Next, with the purpose to compare, we provide the following five order approximation obtained by PM for (13), where was used the value  $\varepsilon = 0.30$  (see Discussion Section) [11].

$$y(x) = \frac{70329323}{80000000} + \frac{1898907}{16000000}x^2 - \frac{393}{8000000}x^4 + \frac{3939}{16000000}x^6 - \frac{3933}{16000000}x^8 + \frac{2787}{80000000}x^{10}. \quad (32)$$

Although, from Figures 2 and 3 it is clear the accuracy of the proposed solutions, we will see that it is possible to quantify it, by using the square residual error (SRE) (see Discussion section).

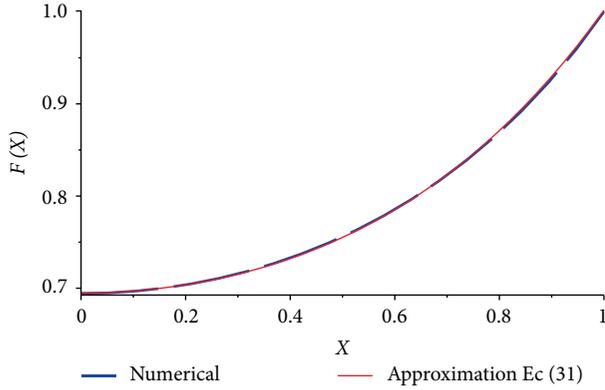


FIGURE 3: Comparison between numerical solution of (13) and approximation (31) for  $\epsilon = 2$ .

As second case study, we propose the following nonlinear first order problem which models the cooling of a lumped system, of volume  $V$ , surface area  $A$  and density  $\rho$  with variable specific heat  $c$ , depending linearly of temperature, exposed to a convective environment at a constant temperature  $T_a$  [17].

$$\frac{dy(x)}{dx} + \epsilon y(x) \frac{dy(x)}{dx} + y(x) = 0, y(0) = 1, x \in [0, \infty), \tag{33}$$

where  $y(x)$  is a dimensionless variable expressed as a linear function of the system temperature  $T$  in accordance with  $y = T - T_a / T_i - T_a$ , where  $T_i$  is the initial temperature of the system under study,  $x$  is a quantity directly proportional to time  $t$ , while  $\epsilon$  is a parameter of the problem directly related with the difference between the system initial temperature and the environment temperature [17].

Next, we will see that it is possible to predict an asymptotic behavior of (33), by employing Theorem 1. In a sequence from the application of Theorem 2 we will conclude that (33) possess a unique solution.

With the end to use Theorem 1, we begin rewriting (33) as follows:

$$y' = -y + (y - y(1 + \epsilon y))^{-1}, \tag{34}$$

in such way that the solution of  $y' = -y$ ; which satisfies the initial condition of (33) is

$$y = e^{-x}, \tag{35}$$

and clearly  $y \rightarrow 0$ , as  $x \rightarrow \infty$ .

Besides, the power series of  $g(y) = y - y(1 + \epsilon y)^{-1}$

$$g(y) = y - y(1 + \epsilon y)^{-1} = \epsilon y^2 - \epsilon^2 y^3 + \epsilon^3 y^4 - \dots, \tag{36}$$

lacks of constant and linear terms, therefore in accordance with Theorem 1, the solution of (33) would approach to  $y \rightarrow 0$  as  $x \rightarrow \infty$  assuming that the point of coordinates  $(0, 1)$  is sufficiently close to the origin.

Thus, starting from reasonable assumptions, we expect that the asymptotic solution of (33) goes as  $y \rightarrow 0$ . As a matter of fact, we will see that, indeed, PM solutions satisfy this asymptotic requirement, for the examples considered;

nevertheless it results that Perturbation Method will not describe correctly the intermediate part of the solution. In contrast, ODBCPM solutions are accurate for the whole domain of the problem  $x \in [0, \infty)$ , including of course the intermediate part. Incidentally we will see that point  $(0, 1)$  is sufficiently close to the origin, for the purposes of the application of the Poincaré–Lyapunov theorem.

With the purpose to show that (33) has a unique solution, we have to satisfy the content of Theorem 2.

First, we express (33) in the form

$$y' = \frac{-y}{1 + \epsilon y}. \tag{37}$$

In such a way that, in accordance with Theorem 2, it is recognized

$$f_1(x, y) = \frac{-y}{1 + \epsilon y}. \tag{38}$$

Let  $D = \{(x, y) / 0 \leq x \leq b, -\infty < y < \infty\}$  be for some arbitrary value  $b > 0$ .

From the previously mentioned, it is expected that  $y \rightarrow 0$  as  $x \rightarrow \infty$  and from the initial condition,  $y(0) = 1$ . Physically, this means that the system starts with an initial temperature  $T_i > T_a$ , and as the time goes,  $T \rightarrow T_a$  asymptotically in such a way that for finite  $x$ ;  $y > 0$ , or  $y \geq 0$ , considering the interval  $x \in [0, \infty)$ . In a sequence,  $y$  has to be a monotonically decreasing function, with increasing values of  $x$ . Therefore, we may redefine by convenience to  $f_1(x, y)$  as follows

$$\begin{aligned} f(x, y) &= \frac{-1}{1 + \epsilon}, & y > 1, \\ f(x, y) &= \frac{-y}{1 + \epsilon y}, & 0 \leq y \leq 1, \\ f(x, y) &= 0, & y < 0, \end{aligned} \tag{39}$$

without causing significant alterations in the approach of the problem (by physical reasons the relevant interval is  $0 \leq y \leq 1$ ).

Next, we will see that  $f$  expressed by (39) satisfies a Lipschitz condition in  $D$  for the variable  $y$  [15].

Next, let us consider the following arguments for  $f(x, y)$ .

The mean value theorem, establishes that for a interval  $(y_1, y_2)$  there exists a  $\xi$  such that [16].

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} = \frac{\partial f(x, \xi)}{\partial y}, \tag{40}$$

where the points  $(x, y_1), (x, y_2) \in D$ .

By using (39), we conclude that

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} = \frac{-1}{(1 + \epsilon \xi)^2}, 0 \leq y \leq 1, \tag{41}$$

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} = 0, y < 0. \tag{42}$$

$$\frac{f(x, y_2) - f(x, y_1)}{y_2 - y_1} = 0, y > 1.$$

and therefore,

$$|f(x, y_2) - f(x, y_1)| = \frac{|y_2 - y_1|}{(1 + \varepsilon\xi)^2}, 0 \leq y \leq 1, \quad (43)$$

$$\begin{aligned} |f(x, y_2) - f(x, y_1)| &= 0, y < 0. \\ |f(x, y_2) - f(x, y_1)| &= 0, y > 1. \end{aligned} \quad (44)$$

Then, (43) and (44) indicate that effectively  $f$  satisfies a Lipschitz condition in  $D$  for the variable  $y$  with the Lipschitz constant  $L = 1$  (that is:  $|f(x, y_2) - f(x, y_1)| \leq |y_2 - y_1|$ ) [15]. Also, since  $f$  is continuous at  $0 \leq x \leq b$  (for some arbitrary value  $b > 0$ ) and  $-\infty < y < \infty$ , the Theorem 2 indicates that the initial value problem has a unique solution.

Next, we will compare PM and ODBCPM approximate solutions for (33), adopting  $\varepsilon$  as a perturbative parameter.

We will distribute the initial condition of (33) among the different iterations in order to seek a good approximation, even for large values of the perturbative parameter.

At the first iteration we consider.

$$\varepsilon^0) \quad y_0' + y_0 = 0, \quad y_0(0) = k_1, \quad (45)$$

after separating the variables and integrating, the solution of the above equation is given by

$$y_0(x) = c_1 e^{-x}, \quad (46)$$

where  $c_1$  is an integration constant, which is determined applying the initial condition from (45), so that we obtain

$$y_0(x) = k_1 e^{-x}. \quad (47)$$

The second iteration is expressed by

$$\varepsilon^1) \quad y_1'(x) + y_1(x) + y_0(x)y_0'(x) = 0, \quad y_1(0) = k_2. \quad (48)$$

After substituting (47) into (48), and solving the indicated differential equation, we get.

$$y_1(x) = -k_1^2 e^{-2x} + c_2 e^{-x}, \quad (49)$$

we note that  $c_2$  is an integration constant, which is determined in terms of  $k_1$  and  $k_2$ , after applying the initial condition of (48), in such a way that we obtain

$$y_1(x) = -k_1^2 e^{-2x} + k_2 e^{-x} + k_1^2 e^{-x}. \quad (50)$$

A third iteration is given by

$$\varepsilon^2) \quad y_2'(x) + y_2(x) + y_0(x)y_1'(x) + y_1(x)y_1'(x) = 0, \quad y_2(0) = k_4. \quad (51)$$

After substituting (47) and (50) into (51), and performing the indicated linear ODE we get

$$y_2(x) = \frac{3k_1^3 e^{-3x}}{2} - k_3 e^{-2x} + c_3 e^{-x}, \quad (52)$$

where, for the sake of simplicity we define

$$k_3 = 2k_1 k_2 + 2k_1^3. \quad (53)$$

We note that  $c_3$  is an integration constant, which is determined after applying the initial condition of (51), so that we get

$$y_2(x) = \frac{3k_1^3 e^{-3x}}{2} - k_3 e^{-2x} + \left[ k_4 - \frac{3k_1^3}{2} + k_3 \right] e^{-x}. \quad (54)$$

On the other hand, with the purpose to get a better approximate solution, we consider the fourth iteration given by

$$\begin{aligned} \varepsilon^3) \quad & y_3'(x) + y_3(x) + y_2(x)y_2'(x) + y_0(x)y_2'(x) + y_1(x)y_1'(x), \\ & y_3(0) = k_8. \end{aligned} \quad (55)$$

After substituting (47), (50), and (54) into (55), we obtain the following elementary linear first order ODE.

$$y_3'(x) + y_3(x) = 8k_1^4 e^{-4x} - k_6 e^{-3x} + k_7 e^{-2x}, \quad (56)$$

where we have defined for simplicity:

$$\begin{aligned} k_5 &= k_4 - \frac{3}{2}k_1^3 + k_3, \quad k_6 = 3k_3 k_1 + 3k_1^2 k_2 + 3k_1^4, \\ k_7 &= 2k_1 k_5 + k_2^2 + 2k_1^2 k_2 + k_1^4. \end{aligned} \quad (57)$$

After solving the indicated elementary linear differential equation (56), we get

$$y_3(x) = \frac{-8k_1^4 e^{-4x}}{3} + \frac{k_6 e^{-3x}}{2} - k_7 e^{-2x} + c_4 e^{-x}. \quad (58)$$

In the previous equation,  $c_4$  is an integration constant, which is determined after applying the initial condition of (55), so that we obtain

$$\begin{aligned} y_3(x) &= \frac{-8k_1^4 e^{-4x}}{3} + \frac{k_6 e^{-3x}}{2} - k_7 e^{-2x} + k_8 e^{-x} \\ &+ \frac{8k_1^4 e^{-x}}{3} - \frac{k_6 e^{-x}}{2} + k_7 e^{-x}. \end{aligned} \quad (59)$$

Thus, a third order approximation for (33) is obtained, by substituting (47), (50), (54) and (59) into

$$y(x) = \sum_{n=0}^3 \varepsilon^n y_n(x). \quad (60)$$

Next, we will apply the general initial condition  $k_1 + \varepsilon k_2 + \varepsilon^2 k_4 + \varepsilon^3 k_8 = 1$ , with the purpose to eliminate for example the term  $k_8 \varepsilon^3$  of (60) and above all, to ensure that (60) satisfies the boundary condition of the problem  $y(0) = 1$ . In this way, we express (60) in the following form:

$$\begin{aligned} y(x) &= e^{-x} + \varepsilon k_1^2 (e^{-x} - e^{-2x}) \\ &+ \varepsilon^2 \left[ \frac{3k_1^3 (e^{-3x} - e^{-x})}{2} + k_3 (e^{-x} - e^{-2x}) \right] \\ &+ \varepsilon^3 \left[ \frac{8k_1^4 (e^{-x} - e^{-4x})}{3} + \frac{k_6 (e^{-3x} - e^{-x})}{2} + k_7 (e^{-x} - e^{-2x}) \right]. \end{aligned} \quad (61)$$

We note, from (61) that  $y(0) = 1$  (see (33)) while (53) and (57) show that it is possible to express (61) in terms of the values of  $k_1$ ,  $k_2$ , and  $k_4$ .

As a case study we propose the values of  $\varepsilon = 1.5$  and  $\varepsilon = 1.7$ . In order to get that (61) corresponds to an accurate approximate solution for (33), we will optimize the values of the conditions  $k_1$ ,  $k_2$ , and  $k_4$ , for both cases, by using the Least Square Method (see Section 5), resulting the following handy expressions (see (53) and (57)).

$$\begin{aligned} y(x) &= 3.553097345e^{-x} - 7.6742857143097345e^{-2x} \\ &+ 8.759493671e^{-3x} - 3.638297872e^{-4x} (\varepsilon = 1.5), \end{aligned} \quad (62)$$

$$y(x) = 3.986425339e^{-x} - 9.4119850195e^{-2x} + 11.12765957e^{-3x} - 4.702127660e^{-4x} \quad (\varepsilon = 1.7). \quad (63)$$

With the purpose to compare, we provide the following five order approximation obtained by PM for (33), where there were used the same values of  $\varepsilon = 1.5$  and  $\varepsilon = 1.7$  (see Discussion section).

$$y(x) = 7.5625e^{-x} - 12.75e^{-2x} + 15.1875e^{-3x} - 9e^{-4x} \quad (\varepsilon = 1.5), \quad (64)$$

$$y(x) = 9.876833333e^{-x} - 17.306e^{-2x} + 21.5305e^{-3x} - 13.10133333e^{-4x} \quad (\varepsilon = 1.7). \quad (65)$$

(equivalently, the boundary conditions by iteration of classical PM correspond to substitute the values  $k_1 = 1, k_2 = 0, k_3 = 2, k_4 = 0, k_5 = 1/2, k_6 = 7, k_8 = 0$  into (61)). From Figures 5 and 6 is clear that the solutions derived from this set of values (64) and (65) are not accurate. Conversely, although from the same Figures, it is clear the accuracy of the proposed solutions (62) and (63); we will quantify it, by using the square residual error (SRE) (see Discussion section).

## 7. Discussion

As it is known, the question respect what conditions have to satisfy the series (2) to converge and represent a solution of (1) is indeed complicated. As a matter of fact, there is not a large collection of rigorously established results in this topic and the known results depend heavily upon the particular nature of the operators  $L(x)$  and  $N(x)$  [14] (see (1)). For this reason, instead of trying to aboard the proposal of improving PM approximate solutions from the formal view of the analysis we proposed a modification of PM regarding that for practical applications in sciences and engineering, many times is not necessary the rigorous knowledge of the convergence of (2), but an approximate solution with a finite number of terms of the mentioned series solution but with good accuracy. Therefore, this work presented the Optimized Distribution of Boundary Conditions Perturbation Method (or initial conditions, as the case may be) (ODBCPM), which unlike the usual procedure used by the classical PM in the solution of nonlinear problems, the presented work distributes the boundary conditions of the problem (or at least one of them) among the different orders of the iterative process with the end to find their values which optimize the accuracy of the proposed approximated solution. The accuracy of the analysed case studies suggests that the usual choice of PM, which introduces from the first iteration the initial or boundary conditions of the problem to be solved, is only one possibility among many others, and this choice obeys, above all, more to reasons of computational convenience than a real necessity. As a matter of fact, it is well known that PM many times calculates its different approximate orders with relatively little effort and even by hand, without the support of a computational software, which does not necessarily occur when ODBCPM is implemented. From the above, it is reasonable to consider PM as a particular case of the proposed method ODBCPM, which,

although simple computationally, not necessarily provides the best approximate solution, such as it was showed with our case studies. With the purpose to optimize the values of the boundary conditions  $b, c, d, \dots, e$  introduced in the different iterations by the ODBCPM algorithm; we employed the Least Square Method, which aims to minimize the sum of the squares of vertical distance from the exact  $y_i$  values of the proposed approximate solution emanating from ODBCPM (that is, the quadratic error). Thus, the proposed procedure consists in minimizing the quadratic error, by differentiating it respect to each of the adjustment constants (in this case, the values of the initially unknown boundary conditions  $b, c, d, \dots, e$ ), and equating the previous results to zero in order to create a nonlinear equation system by them (12) (see Section 5). Finally, the solutions  $b, c, d, \dots, e$ , emanating from (12), provide the approximate solution with the best adjustment to the data set.

Such as it was mentioned, PM is conceived as a valid method, above all, for small values of a perturbative parameter, which is considered as a drawback for the method; nevertheless, some articles have reported precise solutions in the application of PM to problems even for relatively large values of the perturbation parameter. These works refer not only to the mere application of perturbation method to particular problems, but to modifications of the same one, and combinations of PM with other methods (see Section 1). From the case studies of this work, it is suggested that problems for which PM method does not work properly, ODBCPM could yield useful, including for the cases where PM is totally inappropriate from the beginning. From the above mentioned, we could conjecture that, for the cases where perturbation method provides good approximations, despite the large values of perturbation parameter, the values of its boundary (or initial) conditions expressed for (3) result appropriated from the point of view expressed for ODBCPM in this article. Although the distribution of boundary conditions among the iterations of the process does not possess some special meaning; from the mathematical point of view, the procedure explained for ODBCPM is formally correct, since finally, the proposed approximated solution satisfies the boundary conditions of the problem to solve. In a sequence, it is reasonable to expect that, if the values of the boundary conditions  $b, c, d, \dots, e$  are chosen following an optimization procedure like Least Square Method, already explained in this work, then the approximate solution which emanates from ODBCPM iterative procedure will result more accurate than the one of PM. In order to show the potentiality of the proposed method we considered as case studies two relevant heat problems. First, we considered the nonlinear problem with mixed boundary conditions, which consists in determining the temperature distribution in a uniformly thick rectangular fin radiation to free space (see Section 6) [11, 18]. With the end to frame the problem (13), from a more formal mathematical point of view, we analysed the feasibility of guaranteeing the existence and possibly the uniqueness of the solution of (13) in accordance with Theorem 3. With the purpose to allow the application of Theorem 3, we redefined  $F_1 = \varepsilon y^4$  from (13) like (15). Given that  $y > 0$  by physical reasons, then this extension did not cause significant alterations in the original problem (13). As a matter of fact, Although this scheme

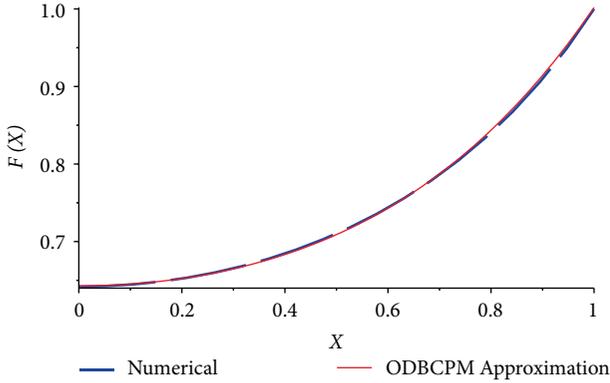


FIGURE 4: Comparison between numerical solution of (13) and approximate solution (29) for  $\epsilon = 3$ .

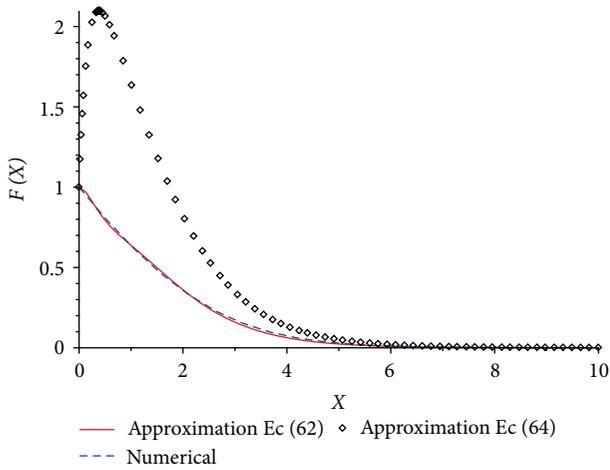


FIGURE 5: Comparison between numerical solution of (33) and approximations (62) and (64) for  $\epsilon = 1.5$ .

proved effectiveness, the requirement  $\partial F/\partial y > 0$  was not satisfied in  $y = 0$ . A possible way to remedy this, would be to replace (15) by  $F(x, y, z) = \epsilon y^4 + \epsilon \delta y (y < 0)$ ,  $F(x, y, z) = -\epsilon y^4 + \epsilon \delta y (y \leq 0)$ , where  $\delta > 0$  is a quantity as small as required. For this choice it is not hard to show that the requirements of Theorem 3 are satisfied. In a sequence, the solution of (13); after considering the substitution of  $\epsilon y^4 \rightarrow \epsilon y^4 + \epsilon \delta y$  clearly does not change appreciably in the limit of values  $\delta \rightarrow 0$  (The inclusion of the term  $y\epsilon\delta$  would yield in the inclusion of additional terms in (29), containing factors of  $\delta$ -powers, which result much smaller than the other in the limit  $\delta \rightarrow 0$ ). Since we guarantee that the problem has at least one solution, and given that (13) is an equation of perturbative character, we could assume that the PM method would work correctly, at least for small values of the perturbative parameter. We noted that this problem was studied already by PM method in [11] and this article showed that even for the relatively small value of  $\epsilon = 0.30$ , the fifth order approximation of PM resulted inappropriate to model the initially unknown left boundary and therefore it could be needed even higher order approximations just to study a perturbative case (that is,  $\epsilon$  small). In accordance with ODBCPM, we proposed to redistribute the right boundary condition of (13), while the left boundary was handed in

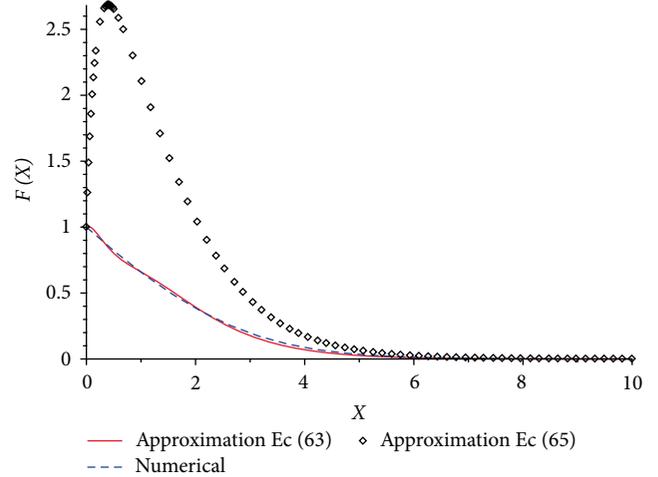


FIGURE 6: Comparison between numerical solution of (33) and approximations (63) and (65) for  $\epsilon = 1.7$ .

accordance with the classical PM procedure. We proposed a third order approximation for (13), expressed for (28). After applying the boundary condition  $\alpha + \epsilon\beta + \gamma\epsilon^2 + \Gamma\epsilon^3 = 1$ , which ensures that the proposed boundaries  $\alpha, \beta, \gamma$ , and  $\Gamma$ , satisfy in set, the right boundary condition of the problem; so that it was possible to express (28) explicitly as (29). Thus, we obtained a third order analytic approximate solution for (13) in terms of  $\alpha, \beta$ , and  $\gamma$ . As a case study we proposed the values of  $\epsilon = 0.30$  and  $\epsilon = 2$ , from where we obtained the approximate solutions (30) and (31). In particular we worked the case  $\epsilon = 0.30$  in order to compare ODBCPM solution with that obtained by PM. With this purpose we provided the five order approximate solution (32) obtained by using PM to problem (13) [11], where it was used precisely the value  $\epsilon = 0.30$ . From Figure 2 of [11] (see also Figure 2 of this paper), we deduce at simple sight, that (32) has just an acceptable precision, taking up that problems with mixed boundary conditions, have the added difficulty of not provide one of the endpoints of the interval (in this case  $y(0)$ ). As a matter of fact, such as it was previously mentioned, it is clear that PM method is not adequate to model the left end point, and for obtaining a better result, it would have required possibly, higher order approximations. On the other hand, Figure 2, shows the comparison between numerical and ODBCPM (30) solutions. From this we note that both are indeed overlapping, including the initially unknown left endpoint. Next, we proposed as a case study the value of  $\epsilon = 2$ , which is even more difficult to model than the former, since this perturbative parameter is indeed big. Even so, from the Figure 3, we deduced that numerical and ODBCPM (31) solutions are indeed overlapped; including the difficult to model left endpoint (another relevant point is that unlike (32), the expressions (30) and (31) are not only precise but handy and therefore ideal for applications). Although from Figures 2 and 3 it is clear the accuracy of our approximations, it is possible to introduce the square residual error (SRE) in order to quantify the precision of ODBCPM. The SRE is a positive number, representative of the committed error using (30) and (31). The main reason to employ SRE is that it is reliable and independent of numerical simulations [15, 17, 18]; whereby this concept

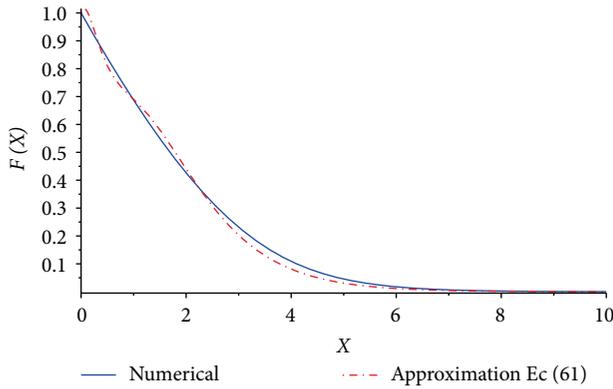


FIGURE 7: Comparison between numerical solution of (33) and approximate solution (61) for  $\varepsilon = 2$ .

is useful even when numerical solution for a given problem is not available. To obtain SRE, we just substitute the ODBCPM solution into the left hand side of the differential equation to solve, expressed in the form (8) (that is, written with zero on the differential equation right side) with the purpose to get the so-called residual  $R$ . Finally, the SRE is obtained from  $\int_{t_0}^{t_1} R^2(t)dt$ , where  $t_0$  and  $t_1$  denotes the end points for the interval of interest. Thus, the SRE values of (30) and (31) turned out to be 0.000005545845393 and 0.01553905469 respectively, which it confirms the accuracy of the proposed solutions. The fact that SRE for the second case study resulted larger is because problem (13) for the value  $\varepsilon = 2$ , is highly nonlinear and significantly more difficult to model. Nevertheless, it is possible to improve the accuracy of the ODBCPM proposed solution, regarding more terms emanating from iterative procedure (9). What is more, Figure 4 shows that (29) even has a good precision for the case highly no perturbative  $\varepsilon = 3$ , which contributes to show the potentiality of the proposed method. As second case study, we studied the nonlinear first order problem which models the cooling of a lumped system, of volume  $V$ , surface area  $A$  and density  $\rho$  with variable specific heat  $c$ , which depends linearly of temperature (33) [17]. Physically, we emphasize the relevance of the parameter for this case study. Thus, small values of  $\varepsilon$  are related with the case of small difference between the system initial temperature and the environment temperature, in such a way that small values of heat flux are expected from the system to the environment. In this case the main contribution comes from the linear part of (33). An interesting case study occurs if the temperature difference above mentioned was not necessarily small and corresponded to bigger values of the parameter. From (33), it is clear that the nonlinear term becomes important and its contribution yields in big values of heat flux from the system to the environment, which are more complicated to model and emphasizes the importance of having methods able to model mathematically the two mentioned cases for  $\varepsilon$ . Unlike the first case study, this problem consists in solving a nonlinear first order ODE, defined in an infinite interval. It is also possible to motivate the solution of this problem from a mathematical point of view. Starting from the powerful Poincaré–Lyapunov theorem, we anticipated that the solution of (33) would approach to  $y \rightarrow zero$  as  $x \rightarrow \infty$ . In a sequence, redefining (38) as (39) we

were able to show that (33) satisfies the requirements of Theorem 2, whereby we knew that the solution for this problem was unique. We distributed the initial condition of (33) among the first four iterations; with better results than PM, even for relatively large values of the perturbative parameter. Thus, a third order approximation for (33) was obtained, by substituting (47), (50), (54) and (59) into (60), where the assumed values for the initial conditions by each iteration are related for  $k_1 + \varepsilon k_2 + \varepsilon^2 k_4 + \varepsilon^3 k_8 = 1$ . We already note that, unlike the previous example, this is a problem of initial conditions defined in an infinite interval, what denotes the potentiality and versatility of ODBCPM method. Again, Least Square Method was employed to optimize the values of the conditions  $k_1, k_2$ , and  $k_4$ , for case studies,  $\varepsilon = 1.5$  and  $\varepsilon = 1.7$ . Figures 5 and 6; compare the ODBCPM solutions (62), (63) for the above mentioned parameters, with their corresponding PM solutions, (64) and (65) respectively. Clearly, ODBCPM provided accurate solutions, which is verified for the SRE values of (62) and (63) (the SRE for these cases is given for the improper integral  $\lim_{b \rightarrow \infty} \int_0^b R^2(t)dt$ , where the residual  $R$  was defined previously). We noted that values of 0.0684261 and 0.1472697 are notably precise, taking into account that in this case the SRE is measured along the infinite interval  $[0, \infty)$ , which confirms the good performance of the proposed method, while from Figures 5 and 6, it results evident that PM method simply is not adequate to model (33). In a sequence, Figure 7 shows that approximate solution (61) even has a good precision for the case highly no perturbative  $\varepsilon = 2$ , which contributes to show the potential of the proposed method.

## 8. Conclusions

This work proposed the ODBCPM, which works in a similar way to PM and the relevant change consists in the distribution of the boundary conditions of the problem (at least one of them) among the different orders of the iterative process. With this purpose, ODBCPM establishes a set of boundary conditions for each iteration so that the proposed approximate solution satisfies the boundary conditions of the problem to solve. From the explanation in this work, one can consider the boundary or initial conditions of PM as one particular case among an infinity of possibilities, and the objective of ODBCPM is to determine the best one of them based on the criterion that this election corresponds to that, which provides the most accurate approximate solution. This article proposed to optimize the election of the above mentioned conditions following the Least Square Method. On the other hand, the proposed cases study showed the possibility of getting accurate approximate solutions, even considering the case of large values of the parameter  $\varepsilon$ , which from the application point of view, it results to be advantageous. Thus, unlike the comments about PM in the introduction and the case studies of this article, from which we noted that does not exist security about the results obtained for PM; ODBCPM was focused with the purpose to provide a method to find analytical approximate solutions to perturbative problems (1) that systematically offered the best possible results in accordance with the Least Square Method. Finally, this work indicates that when PM fails, it is even possible to obtain a

better result by choosing other set of boundary conditions in accordance with ODBCPM algorithm before resorting to other sophisticated methods of the literature.

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that there are no conflicts of interests regarding the publication of this paper.

### Acknowledgments

The authors would like to thank Rogelio Alejandro Callejas-Molina and Roberto Ruiz-Gomez for their contribution to this project.

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