Existence and Uniqueness of Positive Solutions for a New System of Fractional Differential Equations

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By virtue of a recent existing fixed point theorem of increasing θ−(h,e)-concave operator by Zhai and Wang, we consider the existence and uniqueness of positive solutions for a new system of Caputo-type fractional differential equations with Riemann–Stieltjes integral boundary conditions.

1. Introduction

In this paper, we consider the following nonlinear Caputo-type fractional system:

\[
\begin{align*}
\mathcal{D}_0^\theta x(t) + f_1(t, x(t), y(t)) &= a_1(t), & t &\in [0, 1], \\
\mathcal{D}_0^\theta y(t) + f_2(t, x(t), y(t)) &= a_2(t), & t &\in [0, 1], \\
x(0) &= x''(0) = 0, \\
y(0) &= y''(0) = 0, \\
y(1) &= \int_0^1 y(t)\,dA_2(t), \\
x(1) &= \int_0^1 x(t)\,dA_1(t),
\end{align*}
\]

where \(\theta \in (2, 3), \theta_2 \in (2, 3); \mathcal{D}_0^\theta \) and \(\mathcal{D}_0^\theta \), are the Caputo fractional derivative; \(f_1 \) and \(f_2 \): \([0, 1] \times (-\infty, +\infty) \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)\) are continuous; \(a_1, a_2 \): \([0, 1] \rightarrow [0, +\infty)\) are continuous; and \(A_1 \) and \(A_2 \) are bounded variation functions with positive measures with \(B_1 = \int_0^1 t\,dA_1(t) < 1, B_2 = \int_0^1 tdA_2(t) < 1\).

In recent decades, fractional-order calculus has been widely used in engineering, biology, physics, and so on. Based on it, many scholars have been interested in the study of the existence of nontrivial solutions for various fractional boundary value problems. For some recent works, we can refer to [1–30] and the references therein. For example, in [10], by virtue of Guo–Krasnosel’skii fixed point theorem, Ma and Cui studied the following fractional boundary value problem:

\[
\begin{align*}
\mathcal{D}_0^\mu p(y) + \mu f(y, p(y)) &= 0, & y &\in [0, 1], \\
p(0) &= p''(0) = 0, \\
p(1) &= \int_0^1 p(y)\,dA(y),
\end{align*}
\]

where \(f \in C([0, 1] \times [0, +\infty), (0, +\infty)), \mathcal{D}_0^\mu \) is the Caputo fractional derivative, \(\theta \in (2, 3), \) and \(\mu > 0\) is a parameter. In the case that the parameter \(\mu \) satisfied some conditions, the existence of positive solutions for the boundary value problem (2) was proved. Meanwhile, many scholars considered some various fractional systems, such as [16–30] and the reference therein. For instance, in [16], the authors obtained the existence of two positive solutions for a nonlinear Caputo-type fractional system by virtue of fixed point index theory. In [17], by using monotone iterative approach, the authors investigated the iterative positive solutions of a system of fractional Riemann–Liouville-type equations with four-point boundary conditions. In [19], by using Banach’s contraction principle, the authors studied the uniqueness of solution for a system of Hadamard-type fractional differential equations with integral boundary...
conditions. In [21], by using Guo–Krasnosel’skii fixed point theorem, the authors studied the existence of positive solutions for an infinite system of fractional Caputo-type differential equations.

In [22], Zhai and Wang introduced a new concept of \( \varphi - (h, e) \)-concave operator and obtained fixed point theorems of increasing \( \varphi - (h, e) \)-concave operator. In recent years, by using the fixed point theorems of \( \varphi - (h, e) \)-concave operator, Zhai and Jiang investigated the uniqueness of positive solutions for a Riemann–Liouville-type fractional system with integral boundary conditions in [23]; Zhai and Wang considered the uniqueness of positive solutions for a system of Hadamard fractional differential equations with integral equations in [24]; Zhai and Zhu considered the uniqueness of positive solutions for a system of Riemann–Liouville fractional differential equations in [25].

Inspired by [10, 22–25], we introduce a new system of nonlinear Caputo-type fractional differential equations (1). There are few papers about the application of \( \varphi - (h, e) \)-concave operator in nonlinear Caputo-type fractional boundary value problems. So, in this paper, we use the recent fixed point theorems of \( \varphi - (h, e) \)-concave operator by Zhai and Wang to study system (1). The result of the existence and uniqueness of positive solutions for system (1) is obtained.

2. Preliminaries

In this section, we briefly introduce Caputo’s fractional derivative and the fixed point theorem of \( \varphi - (h, e) \)-concave operator. For details, we can refer to the literature [1, 22]. And we give some lemmas about the relevant Green’s functions.

**Definition 1** (see [1]). For a function \( x \in C^n[0, +\infty) \), we define Caputo’s fractional derivative of order \( \theta > 0 \) as follows:

\[
\begin{align*}
\mathcal{D}_0^\theta x(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} x^{(n)}(s) \text{d}s, \quad n-1 < \theta < n,
\end{align*}
\]

where \( n \) is the smallest integer greater than or equal to \( \theta \).

By [10], the following lemmas are listed.

**Lemma 1** (see [10]). Let \( u \in C[0, 1] \) and \( \theta_1, \theta_2 \in (2, 3) \). Then, the linear Caputo fractional differential equation

\[
\begin{align*}
\mathcal{D}_{0+}^\theta p(t) + u(t) = 0, & \quad t \in [0, 1], \\
p(0) = p''(0) = 0, & \\
p(1) = \int_0^1 p(t) \text{d}A_i(t)
\end{align*}
\]

has a unique solution

\[
p(t) = \int_0^1 K_i(t, s)u(s) \text{d}s,
\]

where

\[
K_i(t, s) = \frac{1}{\Gamma(\theta_i)} \begin{cases} 
\frac{t}{1-B_i} \left[ (1-s)^{\theta_i-1} - \int_s^t (t-s)^{\theta_i-1} \text{d}A_i(t) \right] - (t-s)^{\theta_i-1}, & 0 \leq s \leq t \leq 1, \\
\frac{t}{1-B_i} \left[ (1-s)^{\theta_i-1} - \int_s^t (t-s)^{\theta_i-1} \text{d}A_i(t) \right], & 0 \leq t \leq s \leq 1,
\end{cases}
\]

and \( B_i = \int_0^1 t \text{d}A_i(t) < 1, \quad i = 1, 2. \)

**Lemma 2.** The above Green’s function \( K_i(t, s)(i = 1, 2) \) has the following properties:

(i) \( K_i(t, s) \geq 0 \) and \( K_i(t, s) \) is continuous on \([0, 1] \times [0, 1] \).

(ii) \( (t(1-s)^{\theta_i-1}/(\Gamma(\theta_i)(1-B_i))) \int_s^t (t-s)^{\theta_i-1} \text{d}A_i(t) \leq K_i(t, s) \leq ((t(1-s)^{\theta_i-1}/(\Gamma(\theta_i)(1-B_i))) \int_0^1 (t-s)^{\theta_i-1} \text{d}A_i(t)), \quad t, s \in [0, 1]. \)

**Proof.** By (6), we know that \( K_i(t, s) \geq 0 \) and \( K_i(t, s) \) is continuous on \([0, 1] \times [0, 1] \).

By (6), we easily know that

\[
K_i(t, s) \leq \frac{t}{\Gamma(\theta_i)(1-B_i)}
\]

From (6), when \( 0 \leq s \leq t \leq 1 \), we have

\[
\int_s^t (t-s)^{\theta_i-1} \text{d}A_i(t) \geq \int_s^t (t-ts)^{\theta_i-1} \text{d}A_i(t)
\]
Let \( 1 \leq x - t \) be a real Banach space and \((x, y) \) is induced by \( (1 - s)_{\varepsilon} = (1 - s)^{\varepsilon - 1} - \left( \int_s^1 (t - ts)^{\varepsilon - 1} dA_t (t) \right) \) 

\[
\geq (1 - s)^{\varepsilon - 1} (t - t^{\varepsilon - 1}) + \frac{t}{1 - B_t} \int_0^1 (t - t^{\varepsilon - 1}) dA_t (t)
\]

(8)

When \( t \leq s \), we have

\[
\frac{t}{1 - B_t} \left[ (1 - s)^{\varepsilon - 1} - \int_s^1 (t - ts)^{\varepsilon - 1} dA_t (t) \right]
\]

\[
\geq \frac{t}{1 - B_t} \left[ (1 - s)^{\varepsilon - 1} - \int_s^1 (t - ts)^{\varepsilon - 1} dA_t (t) \right]
\]

(9)

By (8) and (9), we have

\[
K_1 (t, s) \geq \left( \frac{t (1 - s)^{\varepsilon - 1}}{1 - B_t} \right) \int_0^1 (t - t^{\varepsilon - 1}) dA_t (t).
\]

(10)

Let \( E \) be a real Banach space and \( P \subset E \) be a cone. A partial order on \( E \) is induced by \( P \). For any \( x, y \in E \), the notation \( x \sim y \) denotes that there are \( \lambda > 0 \) and \( \mu > 0 \) such that \( \lambda x \leq y \leq \mu x \). Take \( h > \theta \) (i.e., \( h \geq \theta \) and \( h \neq \theta \)); let \( P_h = \{ x \in E \mid x \sim h \} \); then, \( P_h \subset P \). Choose \( e \in P \) with \( \theta \leq e \leq h \), and let \( P_{h,e} = \{ x \in E \mid x + e \in P_h \} \).

\[
\text{Definition 2 (see [22]). Let } T \colon P_{h,e} \rightarrow E \text{ be an operator. If for any } x \in P_{h,e} \text{ and } \eta \in (0, 1), \text{ there exists } \varphi (\eta) > \eta \text{ such that}
\]

\[
T (\eta x + (\eta - 1) e) \geq \varphi (\eta) T x + (\varphi (\eta) - 1) e.
\]

(11)

Then, \( T \) is called a \( \varphi - (h,e) \)-concave operator.

\[
\text{Lemma 3 (see [22]). Let } P \text{ be a normal cone. Suppose that } T \text{ is an increasing } \varphi - (h,e) \text{-concave operator and } Th \in \text{P}_{h,e}; \text{ then, } T \text{ has a unique fixed point } u \in P_{h,e}. \text{ For any } u_0 \in P_{h,e}, \text{ the sequence } u_n = T u_{n-1}, n = 1, 2, \ldots, \text{ and then } \| u_n - u \| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

3. Main Results

Let \( E = C[0, 1] \) with the norm \( \| x \| = \max_{0 \leq t \leq 1} | x (t) | \). Let \( E = \mathbb{R} \times E \) with the norm \( \| x, y \|_E = \max \{ \| x \|, \| y \| \} \). Let \( P = \{ x \in E \mid x (t) \geq 0, t \in [0, 1] \} \). Then, \( P \) is a normal cone. Let \( \mathbb{P} = \mathbb{R} \times \mathbb{P} \). It is obvious that \( P \) is a normal cone of \( E \). We define the following partial order on \( E \): \( (x_1, y_1) \leq (x_2, y_2) \iff x_1 (t) \leq x_2 (t), y_1 (t) \leq y_2 (t), t \in [0, 1] \). For the detailed knowledge about the cone, we can refer to [31].

Define the following operators \( T_1 \colon E \rightarrow E \), \( T_2 \colon E \rightarrow E \) and \( T \colon E \rightarrow E \):

\[
T_1 (x, y) (t) = \int_0^t K_1 (t, s) f_1 (s, x (s), y (s)) ds - \int_0^t K_1 (t, s) a_1 (s) ds, \quad t \in [0, 1],
\]

\[
T_2 (x, y) (t) = \int_0^t K_2 (t, s) f_2 (s, x (s), y (s)) ds - \int_0^t K_2 (t, s) a_2 (s) ds, \quad t \in [0, 1],
\]

\[
T (x, y) (t) = (T_1 (x, y), T_2 (x, y)) (t), \quad (x, y) \in E, t \in [0, 1],
\]

(12)

where \( K_i (t, s) (i = 1, 2) \) is defined by (6).

By [10], we easily know that fixed points of the operator \( T \) are solutions of system (1).

Let \( e_1 (t) = \int_0^1 K_1 (t, s) a_1 (s) ds \), \( e_2 (t) = \int_0^1 K_2 (t, s) a_2 (s) ds \), \( h_1 (t) = C_1 t \), \( h_2 (t) = C_2 t \), \( t \in [0, 1] \),

(13)

where
\[ C_1 > \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} a_1(s)ds, \]
\[ C_2 > \frac{1}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1} a_2(s)ds. \]

Let
\[ P_h = \{ x \in E: x \sim h \}, \]
\[ P_{h,1} = \{ x \in E: x \sim h_1 \}, \]
\[ P_{h,2} = \{ x \in E: x \sim h_2 \}, \]
\[ P_{h,1,2} = \{ x \in E: x \sim e \}. \]

Set \( h(t) = (h_1(t), h_2(t)) \) and \( e(t) = (e_1(t), e_2(t)) \). By [29], we have
\[ P_{h,1} \times P_{h,2}, \]
\[ P_{h,1,2} \times P_{h,2}. \]

Let \( M_1 = \max_{x \in [0,1]} e_1(t), M_2 = \max_{x \in [0,1]} e_2(t). \) For convenience, the following conditions are given.

(i) \( (S_1) \) \( f_1: [0,1] \times [-M_1, +\infty) \times [-M_2, +\infty) \rightarrow (-\infty, +\infty) \) is increasing about the second and third variables; \( f_2: [0,1] \times [-M_1, +\infty) \times [-M_2, +\infty) \rightarrow (-\infty, +\infty) \) is increasing about the second and third variables.

(ii) \( (S_2) \) For \( \eta \in (0,1) \), there exists \( \varphi(\eta) > \eta \) such that
\[ f_1(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) \geq \varphi(\eta) f_1(t, u_1, u_2), \]
\[ f_2(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) \geq \varphi(\eta) f_2(t, u_1, u_2), \]
\[ t \in [0,1], u_1, u_2 \in (-\infty, +\infty), v_1 \in [0, M_1], v_2 \in [0, M_2]. \]

By Lemma 2, we have
\[ \begin{align*}
&\frac{t(1-s)^{\theta_1-1}}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (t - s)^{\theta_1-1} dA_i(t) \leq K_i(t, s) \\
&\leq \frac{t(1-s)^{\theta_2-1}}{\Gamma(\theta_2)(1-B_2)} \\
&\quad t, s \in [0,1], i = 1, 2.
\end{align*} \]

From (21), for \( t \in [0,1] \), we have
\[ e_1(t) = \int_0^1 K_1(t, s) a_1(s)ds \leq \frac{t}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} a_1(s)ds \leq C_1 t = h_1(t), \]
\[ e_2(t) = \int_0^1 K_2(t, s) a_2(s)ds \leq \frac{t}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1} a_2(s)ds \leq C_2 t = h_2(t). \]

Thus, we obtain that
\[ 0 \leq e_1 \leq h_1, \]
\[ 0 \leq e_2 \leq h_2. \]

In the following, we divide three parts to prove this theorem. Firstly, we prove that \( T: P_{h,e} \rightarrow E \) is a \( \varphi \) - \( (h,e) \) concave operator. By \( (S_2) \), for \( \eta \in (0,1) \) and \( \eta \in (0,1) \), we have
\[ (S_3) \] \( f_1(t, 0, 0) \geq 0, f_2(t, 0, 0) \geq 0 \) and \( f_1(t, 0, 0) = 0, f_2(t, 0, 0) = 0, \forall t \in [0,1]. \)
\[ T_1(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t) \]
\[ = \int_0^1 K_1(t, s) f_1(s, \eta x(s) + (\eta - 1)e_1(s), \eta y(s) + (\eta - 1)e_2(s))ds - \int_0^1 K_1(t, s)a_1(s)ds \]
\[ \geq \int_0^1 K_1(t, s)\varphi(\eta)f_1(s, x(s), y(s))ds - \int_0^1 K_1(t, s)a_1(s)ds \]
\[ = \varphi(\eta)\int_0^1 K_1(t, s)f_1(s, x(s), y(s))ds - e_1(t) \]
\[ = \varphi(\eta)\bigg[ \int_0^1 K_1(t, s)f_1(s, x(s), y(s))ds - \int_0^1 K_1(t, s)a_1(s)ds \bigg] + (\varphi(\eta) - 1)e_1(t) \]
\[ = \varphi(\eta)T_1(x, y)(t) + (\varphi(\eta) - 1)e_1(t). \] 

Similarly, by \((S_4)\), for \(\eta \in (0, 1)\) and \(\eta \in (0, 1)\), we obtain

\[ T_2(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t) \]
\[ = \int_0^1 K_2(t, s) f_2(s, \eta x(s) + (\eta - 1)e_1(s), \eta y(s) + (\eta - 1)e_2(s))ds - \int_0^1 K_2(t, s)a_2(s)ds \]
\[ \geq \int_0^1 K_2(t, s)\varphi(\eta)f_2(s, x(s), y(s))ds - \int_0^1 K_2(t, s)a_2(s)ds \]
\[ = \varphi(\eta)\int_0^1 K_2(t, s)f_2(s, x(s), y(s))ds - e_2(t) \]
\[ = \varphi(\eta)\bigg[ \int_0^1 K_2(t, s)f_2(s, x(s), y(s))ds - \int_0^1 K_2(t, s)a_2(s)ds \bigg] + (\varphi(\eta) - 1)e_2(t) \]
\[ = \varphi(\eta)T_2(x, y)(t) + (\varphi(\eta) - 1)e_2(t). \]

By (24) and (25), for \((x, y) \in P_{h,c}\), \(\eta \in (0, 1)\), and \(t \in [0, 1]\), we have

\[ T(\eta(x, y) + (\eta - 1)e)(t) \]
\[ = T(\eta(x, y) + (\eta - 1)(e_1, e_2))(t) \]
\[ = T(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t) \]
\[ = (T_1(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t), T_2(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t)) \]
\[ \geq (\varphi(\eta)T_1(x, y)(t) + (\varphi(\eta) - 1)e_1(t), \varphi(\eta)T_2(x, y)(t) + (\varphi(\eta) - 1)e_2(t)) \]
\[ = \varphi(\eta)(T_1(x, y)(t), T_2(x, y)(t)) + (\varphi(\eta) - 1)(e_1(t), e_2(t)) \]
\[ = \varphi(\eta)T(x, y)(t) + (\varphi(\eta) - 1)e(t). \]

Namely, for \((x, y) \in P_{h,c}\) and \(\eta \in (0, 1)\), we obtain that

\[ T(\eta(x, y) + (\eta - 1)e)(t) \geq \varphi(\eta)T(x, y) + (\varphi(\eta) - 1)e. \] 

(27)

Secondly, we show that \(T: P_{h,c} \rightarrow E\) is an increasing operator. By the definition of \(P_{h,c}\), for \((x, y) \in P_{h,c}\), we get that \((x, y) + e \in P_{h}\), i.e., \((x + e_1, y + e_2) \in \overline{P}_h \times \overline{P}_h\). By the definitions of \(\overline{P}_h\) and \(\overline{P}_h\), we obtain that there exist \(\eta_1 > 0\) and \(\eta_2 > 0\) such that

\[ x(t) + e_1(t) \geq \eta_1 h_1(t), \]
\[ y(t) + e_2(t) \geq \eta_2 h_2(t), \quad t \in [0, 1]. \]

(28)

So,

\[ x(t) \geq \eta_1 h_1(t) - e_1(t) \geq - e_1(t) \geq - M_1, \]
\[ y(t) \geq \eta_2 h_2(t) - e_2(t) \geq - e_2(t) \geq - M_2, \quad t \in [0, 1]. \]

(29)
From (S₁), we easily know that the operators \( T_1 \) and \( T_2 \) are increasing, so it is obvious that \( T: P_{h,e} \to E \) is increasing.

In the end, we prove \( Th \in P_{h,e} \). By [22], we know that \( P_{h,e} = \overline{P}_{h,\varepsilon_1} \times \overline{P}_{h,\varepsilon_2} \). Since \( (T\phi)(t) = T(h_1, h_2)(t) = (T_1(h_1, h_2)(t), T_2(h_1, h_2)(t)) \), we need to prove \( T_1(h_1, h_2) \in \overline{P}_{h,\varepsilon_1}, T_2(h_1, h_2) \in \overline{P}_{h,\varepsilon_2} \). In the following, by the definitions of \( \overline{P}_{h,\varepsilon_1} \) and \( \overline{P}_{h,\varepsilon_2} \), we prove \( T_1(h_1, h_2) + e_1 \in \overline{P}_{h,\varepsilon_1}, T_2(h_1, h_2) + e_2 \in \overline{P}_{h,\varepsilon_2} \), respectively. From (S₁) and (S₃), combine (21), and we have

\[
T_1(h_1, h_2)(t) + e_1(t) = \int_0^1 K_1(t, s)f_1(s, h_1(s), h_2(s))ds + e_1(t)
\]

\[
\leq \frac{t}{\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, h_1(s), h_2(s))ds
\]

\[
\leq \frac{t}{\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, C_1, C_2)ds
\]

\[
= \frac{h_1(t)}{C_1\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, C_1, C_2)ds,
\]

\[
T_1(h_1, h_2)(t) + e_1(t) = \int_0^1 K_1(t, s)f_1(s, h_1(s), h_2(s))ds
\]

\[
\geq \frac{t}{\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, h_1(s), h_2(s))ds \int_0^1 (1 - t)^{\theta_1 - 1}dA_1(t)
\]

\[
\geq \frac{t}{\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, 0, 0)ds \int_0^1 (1 - t)^{\theta_1 - 1}dA_1(t)
\]

\[
= \frac{h_1(t)}{C_1\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, 0, 0)ds \int_0^1 (1 - t)^{\theta_1 - 1}dA_1(t).
\]

Let

\[
\lambda_1 = \frac{1}{C_1\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, 0, 0)ds
\]

\[
\lambda_2 = \frac{1}{C_1\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1}f_1(s, C_1, C_2)ds.
\]

Obviously, \( \lambda_1 > 0, \lambda_2 > 0 \). So, we have \( \lambda_1 h_1(t) \leq T_1(h_1, h_2)(t) + e_1(t) \leq \lambda_2 h_2(t) \). And \( T_1(h_1, h_2) + e_1 \in \overline{P}_{h,\varepsilon_1} \). Similarly, we have \( T_2(h_1, h_2) + e_2 \in \overline{P}_{h,\varepsilon_2} \). So, \( Th \in P_{h,e} \) is proved.

Therefore, by Lemma 3, the operator \( T \) has a unique fixed point \((x^*, y^*) \in P_{h,e}\). For any given \((x_0, y_0) \in P_{h,e}\), define the sequences:

\[
x_{n+1}(t) = \int_0^1 K_1(t, s)f_1(s, x_n(s), y_n(s))ds
\]

\[
y_{n+1}(t) = \int_0^1 K_2(t, s)f_2(s, x_n(s), y_n(s))ds
\]

where \( n = 0, 1, 2, \ldots \).
4. Application

Example 1. We study the following fractional system with integral boundary conditions:

\[
\begin{aligned}
&cD^{(5/2)}_0 x(t) + \left( \frac{2835 \sqrt{\pi}}{944} x + 1 \right)^{(1/5)} + \left( \frac{16}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/5)} t^{(1/5)} \\
&+ \left( \frac{2835 \sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} + \left( \frac{32}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/5)} t^{(1/5)} = t, \quad t \in [0, 1], \\
&cD^{(5/2)}_0 y(t) + \left( \frac{2835 \sqrt{\pi}}{944} x + 1 \right)^{(1/3)} + \left( \frac{16}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/3)} t^{(1/3)} = 2t, \quad t \in [0, 1], \\
x(0) = x''(0) = 0, \\
x(1) = \frac{1}{2} \int_0^1 x(t) \, dt, \\
y(0) = y''(0) = 0, \\
y(1) = \frac{1}{2} \int_0^1 y(t) \, dt,
\end{aligned}
\]

where \( \theta_1 = \theta_2 = (5/2); A_1(t) = A_2(t) = (1/2)t; B_1 = B_2 = (1/4); a_1(t) = t; a_2(t) = 2t; \) and

\[
\begin{aligned}
f_1(t, x, y) &= \left( \frac{2835 \sqrt{\pi}}{944} x + 1 \right)^{(1/5)} + \left( \frac{16}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/5)} + \left( \frac{2835 \sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} + \left( \frac{32}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/5)} t^{(1/5)}, \\
f_2(t, x, y) &= \left( \frac{2835 \sqrt{\pi}}{944} x + 1 \right)^{(1/3)} + \left( \frac{16}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/3)} + \left( \frac{2835 \sqrt{\pi}}{1888} y + 1 \right)^{(1/3)} + \left( \frac{32}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right)^{(1/3)} t^{(1/3)}.
\end{aligned}
\]
Obviously,

\[
\begin{align*}
    f_1(t, 0, 0) &= \left[ \frac{16}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} + \left[ \frac{32}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)}, \\
    f_2(t, 0, 0) &= \left[ \frac{16}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} + \left[ \frac{32}{105 \sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)},
\end{align*}
\]

(36)

and

\[ f_1(t, 0, 0) \equiv 0, \quad f_2(t, 0, 0) \geq 0, \quad f_1(t, 0, 0) \geq 0, \quad f_2(t, 0, 0) \geq 0. \]

Green’s functions are as follows:

\[
K_1(t - s) = K_2(t - s) = K(t - s) = \frac{1}{\Gamma(5/2)} \begin{cases} 
4t/3 \left( (1-s)^{(3/2)} - \frac{1}{5} (1-s)^{(5/2)} \right) - (t-s)^{(3/2)}, & 0 \leq s \leq t \leq 1, \\
4t/3 \left( (1-s)^{(3/2)} - \frac{1}{5} (1-s)^{(5/2)} \right), & 0 \leq s \leq t \leq 1.
\end{cases}
\]

Then, we have

\[
e_1(t) = \int_0^1 K_1(t, s)a_1(s)ds
\]

\[
= \frac{4t}{3\Gamma(5/2)} \left[ \frac{2}{5} \left( (1-t)^{(5/2)} - \frac{4}{35} (1-t)^{(7/2)} + \frac{4}{35} (1-t)^{(7/2)} + \frac{4}{315} (1-t)^{(9/2)} - \frac{4}{315} \right) + \frac{1}{\Gamma(5/2)} \frac{4}{35} \right] + \frac{1}{\Gamma(5/2)} \frac{4}{35}
\]

\[
+ \frac{4t}{3\Gamma(5/2)} \left[ \frac{2}{5} \left( (1-t)^{(5/2)} - \frac{2}{35} (1-t)^{(7/2)} - \frac{2}{35} t (1-t)^{(7/2)} - \frac{4}{315} (1-t)^{(9/2)} \right) \right]
\]

\[
= \frac{16t}{105 \sqrt{\pi}} \left( \frac{32}{27} + t^{(5/2)} \right),
\]

\[
e_2(t) = \int_0^1 K_2(t, s)a_2(s)ds = \frac{8t}{3\Gamma(5/2)} \left[ \frac{2}{5} \left( (1-s)^{(5/2)} - \frac{1}{5} (1-s)^{(5/2)} \right) - \frac{2}{\Gamma(5/2)} \right] + \frac{8t}{3\Gamma(5/2)} \left[ \frac{1}{5} \left( (1-s)^{(5/2)} - \frac{1}{5} (1-s)^{(5/2)} \right) \right]
\]
\[
\begin{align*}
&= \frac{8t}{3\Gamma(5/2)} \left[ \frac{2}{5} t^{(5/2)} - \frac{4}{35} t^{(7/2)} + \frac{4}{35} + \frac{2}{35} (1-t)^{(7/2)} + \frac{4}{315} (1-t)^{(9/2)} - \frac{4}{315} \right] + \frac{2}{3}\left( \frac{4}{35} \right)^t \\
&+ \frac{8t}{3\Gamma(5/2)} \left[ \frac{2}{5} t^{(5/2)} - \frac{4}{35} t^{(7/2)} - \frac{2}{35} t (1-t)^{(7/2)} - \frac{4}{315} (1-t)^{(9/2)} \right] \\
&= \frac{32t}{105\sqrt{\pi}} \left( \frac{32}{27} + t^{(5/2)} \right),
\end{align*}
\]

\[
M_1 = \max_{t \in [0,1]} e_1(t) = \max_{t \in [0,1]} \frac{16t}{105\sqrt{\pi}} \left( \frac{32}{27} + t^{(5/2)} \right) = \frac{944}{2835\sqrt{\pi}}
\]

\[
M_2 = \max_{t \in [0,1]} e_2(t) = \max_{t \in [0,1]} \frac{32t}{105\sqrt{\pi}} \left( \frac{32}{27} + t^{(5/2)} \right) = \frac{1888}{2835\sqrt{\pi}}
\]

Let \( h_1(t) = C_1 t, h_2(t) = C_2 t, \) where

\[
C_1 = \frac{4}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds = \frac{4}{3\Gamma(5/2)} \frac{4}{35} = \frac{16}{105\Gamma(5/2)},
\]

\[
C_2 = \frac{8}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds = \frac{8}{3\Gamma(5/2)} \frac{4}{35} = \frac{32}{105\Gamma(5/2)}.
\]

We have

\[
e_1(t) = \frac{16t}{105\sqrt{\pi}} \left( \frac{32}{27} + t^{(5/2)} \right) \leq \frac{4t}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds
\]

\[
= \frac{16t}{105\Gamma(5/2)} \leq C_1 t = h_1(t),
\]

\[
e_2(t) = \frac{32t}{105\sqrt{\pi}} \left( \frac{32}{27} + t^{(5/2)} \right) \leq \frac{8t}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds
\]

\[
= \frac{32t}{105\Gamma(5/2)} \leq C_2 t = h_2(t).
\]

By (34) and (35), we have

\[
f_1(t, x, y) = \left( \frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/5)} \left[ \frac{16}{105\sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} + \left( \frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} \left[ \frac{32}{105\sqrt{\pi}} \left( t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)}
\]

\[
= \left( \frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/5)} \left( e_1(t)^{(1/5)} + \frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} \left( e_2(t)^{(1/5)} \right)
\]

\[
= \left( \frac{2835\sqrt{\pi}}{944} x e_1(t) + e_1(t)^{(1/5)} \right) + \left( \frac{2835\sqrt{\pi}}{1888} y e_2(t) + e_2(t)^{(1/5)} \right),
\]
\[ f_2(t, x, y) = \left( \frac{2835 \sqrt{\pi}}{944} x + 1 \right)^{(1/3)} \left[ \frac{16}{105 \sqrt{\pi}} (t^{(5/2)} + \frac{32}{27}) \right]^{(1/3)} + \left( \frac{2835 \sqrt{\pi}}{1888} y + 1 \right)^{(1/3)} \left[ \frac{32}{105 \sqrt{\pi}} (t^{(5/2)} + \frac{32}{27}) \right]^{(1/3)} \]

\[ = \left( \frac{2835 \sqrt{\pi}}{944} x + 1 \right)^{(1/3)} (e_1(t))^{(1/3)} + \left( \frac{2835 \sqrt{\pi}}{1888} y + 1 \right)^{(1/3)} (e_2(t))^{(1/3)} \]

\[ = \left( \frac{2835 \sqrt{\pi}}{944} xe_1(t) + e_1(t) \right)^{(1/3)} + \left( \frac{2835 \sqrt{\pi}}{1888} ye_2(t) + e_2(t) \right)^{(1/3)}. \]

(41)

Take \( \varphi(\eta) = \eta^{(1/3)} \); then, \( \varphi(\eta) > \eta, \eta \in (0, 1) \). For \( \eta \in (0, 1), \quad u_1, u_2 \in (-\infty, +\infty), \quad \nu_1 \in [0, M_1], \quad \text{and} \quad \nu_2 \in [0, M_2], \quad \text{we have} \)

\[ f_1(t, \eta u_1 + (\eta - 1)\nu_1, \eta u_2 + (\eta - 1)\nu_2) = \left[ \frac{2835 \sqrt{\pi}}{944} e_1(t) \left[ \eta u_1 + (\eta - 1)\nu_1 \right] + e_1(t) \right]^{(1/5)} \]

\[ + \left[ \frac{2835 \sqrt{\pi}}{1888} e_2(t) \left[ \eta u_2 + (\eta - 1)\nu_2 \right] + e_2(t) \right]^{(1/5)} \]

\[ = \eta^{(1/5)} \left[ \frac{2835 \sqrt{\pi}}{944} e_1(t) u_1 + \frac{2835 \sqrt{\pi}}{944} \left( -\frac{1}{\eta} \right) e_1(t) \nu_1 + \frac{1}{\eta} e_1(t) \right]^{(1/5)} \]

\[ + \eta^{(1/5)} \left[ \frac{2835 \sqrt{\pi}}{1888} e_2(t) u_2 + \frac{2835 \sqrt{\pi}}{1888} \left( -\frac{1}{\eta} \right) e_2(t) \nu_2 + \frac{1}{\eta} e_2(t) \right]^{(1/5)} \]

\[ \geq \eta^{(1/5)} \left[ \frac{2835 \sqrt{\pi}}{944} e_1(t) u_1 + \left( -\frac{1}{\eta} \right) e_1(t) + \frac{1}{\eta} e_1(t) \right]^{(1/5)} \]

\[ + \eta^{(1/5)} \left[ \frac{2835 \sqrt{\pi}}{1888} e_2(t) u_2 + \left( -\frac{1}{\eta} \right) e_2(t) + \frac{1}{\eta} e_2(t) \right]^{(1/5)} \]

\[ = \eta^{(1/5)} \left[ \frac{2835 \sqrt{\pi}}{944} e_1(t) u_1 + e_1(t) \right]^{(1/5)} + \eta^{(1/5)} \left[ \frac{2835 \sqrt{\pi}}{1888} e_2(t) u_2 + e_2(t) \right]^{(1/5)} \]
\[ f_2(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) = \eta^{(1/3)} f_1(t, u_1, u_2) \geq \varphi(\eta) f_1(t, u_1, u_2), \]

\[ f_2(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) = \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{944} e_1(t) [\eta u_1 + (\eta - 1)v_1] + \frac{e_1(t)}{\eta} \right\}^{(1/3)} \]

\[ = \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{944} e_1(t) [u_1 + \left( 1 - \frac{1}{\eta} \right) v_1] + \frac{1}{\eta} e_1(t) \right\}^{(1/3)} \]

\[ + \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{1888} e_2(t) [u_2 + \left( 1 - \frac{1}{\eta} \right) v_2] + \frac{1}{\eta} e_2(t) \right\}^{(1/3)} \]

\[ = \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{944} e_1(t) u_1 + \frac{2835 \sqrt{\pi}}{944} \left( 1 - \frac{1}{\eta} \right) e_1(t) v_1 + \frac{1}{\eta} e_1(t) \right\}^{(1/3)} \]

\[ + \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{1888} e_2(t) u_2 + \frac{2835 \sqrt{\pi}}{1888} \left( 1 - \frac{1}{\eta} \right) e_2(t) v_2 + \frac{1}{\eta} e_2(t) \right\}^{(1/3)} \]

\[ \geq \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{944} e_1(t) u_1 + \left( 1 - \frac{1}{\eta} \right) e_1(t) v_1 + \frac{1}{\eta} e_1(t) \right\}^{(1/3)} \]

\[ + \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{1888} e_2(t) u_2 + \left( 1 - \frac{1}{\eta} \right) e_2(t) v_2 + \frac{1}{\eta} e_2(t) \right\}^{(1/3)} \]

\[ = \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{944} e_1(t) u_1 + e_1(t) \right\}^{(1/3)} + \eta^{(1/3)} \left\{ \frac{2835 \sqrt{\pi}}{1888} e_2(t) u_2 + e_2(t) \right\}^{(1/3)} \]

\[ = \eta^{(1/3)} f_2(t, u_1, u_2) \geq \varphi(\eta) f_2(t, u_1, u_2). \]
where \( n = 0, 1, 2, \ldots \), \( K(t, s) \) is defined by (37), \( x_{n+1}(t) \rightarrow x^{*} (n \rightarrow \infty) \), and \( y_{n+1}(t) \rightarrow y^{*} (n \rightarrow \infty) \).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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**References**


