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Review Article

Travelling Wave Solutions of Wu–Zhang System via Dynamic Analysis

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In this paper, based on the dynamical system method, we obtain the exact parametric expressions of the travelling wave solutions of the Wu–Zhang system. Our approach is much different from the existing literature studies on the Wu–Zhang system. Moreover, we also study the fractional derivative of the Wu–Zhang system. Finally, by comparison between the integer-order Wu–Zhang system and the fractional-order Wu–Zhang system, we see that the phase portrait, nonzero equilibrium points, and the corresponding exact travelling wave solutions all depend on the derivative order α. Phase portraits and simulations are given to show the validity of the obtained solutions.

1. Introduction

Recently, many authors made some efforts on nonlinear partial differential equations (NPDEs) (see [1–14]). Wu and Zhang [15] proposed the equation of the form

$$\begin{aligned} \epsilon_t + \epsilon\epsilon_x + u\epsilon_y + v_x &= 0, \\ u_t + \epsilon u_x + uu_y + v_y &= 0, \\ v_t + (\epsilon v)_x + (uv)_y + \frac{1}{3}\left(\epsilon_{xxx} + \epsilon_{xyy} + u_{xxy} + u_{yyy}\right) &= 0, \end{aligned} \tag{1}$$

where ϵ and u denote the surface velocity of water along the x-direction and the y-direction, respectively, and v means the elevation of water. Through some transformations, equation (1) reduces into the $(1+1)$-dimensional dispersive long wave equation (Wu–Zhang system) as follows:

$$\begin{aligned} u_t &= -uu_x - v_x, \\ v_t &= -vu_x - uv_x - \frac{1}{3}u_{xxx}. \end{aligned} \tag{2}$$

In fact, the Wu–Zhang system can describe the nonlinear water wave availability, and many engineers apply it in harbor and coastal design. For mathematical physics, one of the most important topics is to find the exact solutions. Many authors proposed various methods to solve the Wu–Zhang system numerically. We summarize as follows: the first integral method [16], extended tanh-function method [17], characteristic function method [18], modified Conte's invariant Painlevé expansion method and truncation of the WTC's approach [19–21], elliptic function rational expansion method [22], generalized extended tanh-function method [23], generalized extended rational expansion method [24], and so on.

However, different from the aforementioned methods [16–24], in this paper, we apply the dynamical system method to study the bifurcation and exact solutions of NPDEs. Dynamical system method is quite different from the mentioned methods, and it has many successful applications [25–39]. Another purpose is to study exact solutions of the fractional-order Wu–Zhang system [16]. It takes the form

$$D_t^\alpha u = -uu_x - v_x,$$
$$D_t^\alpha v = -vu_x - uv_x - \frac{1}{3}u_{xxx}, \tag{3}$$

and $t > 0$ and $0 < \alpha \leq 1$. So far, Khater et al. [40, 41] studied the fractional-order Wu–Zhang system by the numerical method and the modified auxiliary equation method. Inspired by [38], we study the fractional-order Wu–Zhang system via the extended three-step method.

The structure of this paper is as follows. In Section 2, we first consider the bifurcation of phase portraits for the integer-order Wu–Zhang system. Correspondingly, we calculate all possible exact parametric expressions of solutions to the integer-order Wu–Zhang system. In Section 3, we study the bifurcation of phase portraits and exact solutions of the fractional-order Wu–Zhang system. In Section 4, comparison of phase portraits and exact solutions between the integer-order and fractional-order Wu–Zhang system is presented. Finally, we end this paper with a conclusion.

2. Exact Solutions of Integer-Order Wu–Zhang System (2)

In this part, we first consider the exact solutions of integer-order Wu–Zhang system (2).

By the following transformations,

$$u(x,t) = u(\xi),$$
$$v(x,t) = v(\xi), \tag{4}$$
$$\xi = x - lt,$$

we have

$$\frac{\partial u(x,t)}{\partial t} = -lu_\xi,$$
$$\frac{\partial v(x,t)}{\partial t} = -lv_\xi, \tag{5}$$
$$\frac{\partial^3 u(x,t)}{\partial x^3} = u_{\xi\xi\xi},$$

where u_ξ and v_ξ denote the first-order derivative with respect to ξ, respectively, and $u_{\xi\xi\xi}$ means the third-order derivative with respect to ξ. We substitute (4) and (5) into system (2); then, (2) turns into

$$lu_\xi = uu_\xi + v_\xi,$$
$$lv_\xi = vu_\xi + uv_\xi + \frac{1}{3}u_{\xi\xi\xi}. \tag{6}$$

Then, by integration over both sides of the first equation (6) and setting constant of integration to zero, hence, we have $v(\xi) = lu(\xi) - (1/2)u^2(\xi)$. Substituting it into the second equation of (6), we have the following ordinary differential equation (ODE):

$$u_{\xi\xi\xi} - \frac{9}{2}u^2u_\xi + 9luu_\xi - 3l^2u_\xi = 0. \tag{7}$$

Integrating both sides on (7), it follows that

$$u_{\xi\xi} - \frac{3}{2}u^3 + \frac{9}{2}lu^2 - 3l^2u = c. \tag{8}$$

Obviously, equation (8) reduces to the planar dynamical system:

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{3}{2}u^3 - \frac{9}{2}lu^2 + 3l^2u + c. \tag{9}$$

Meanwhile, the first integral is written as

$$H(u,y) = \frac{1}{2}y^2 - \frac{3}{8}u^4 + \frac{3}{2}lu^3 - \frac{3}{2}l^2u^2 - cu = h. \tag{10}$$

We let $g(u) = (3/2)u^3 - (9/2)lu^2 + 3l^2u + c$; then, $g'(u) = (9/2)u^2 - 9lu + 3l^2$. Obviously, the roots of $g(u) = 0$ depend on the parameter group (l,c) taking different values. (i) If $c > (\sqrt{3}/3)l^3\,(l>0)$ or $c < -(\sqrt{3}/3)l^3\,(l>0)$ or $c > -(\sqrt{3}/3)l^3\,(l<0)$ or $c < (\sqrt{3}/3)l^3\,(l<0)$, then $g(u)$ has only one real root u_1; (ii) if $c = \pm(\sqrt{3}/3)l^3\,(l>0 \text{ or } l<0)$, then $g(u)$ has two real roots: u_2 and $u_3\,(u_3 < u_2)$; and (iii) if $-(\sqrt{3}/3)l^3 < c < (\sqrt{3}/3)l^3\,(l>0)$ or $(\sqrt{3}/3)l^3 < c < -(\sqrt{3}/3)l^3\,(l<0)$, then $g(u)$ has three real roots: u_4, u_5, and $u_6\,(u_6 < u_5 < u_4)$.

Furthermore, noting that

$$J(u_j, y_j) = \det M(u_j, y_j) = -g'(u) = -\left(\frac{9}{2}u^2 - 9lu + 3l^2\right), \tag{11}$$

where $M(u_j, y_j)$ is the coefficient matrix of the linearized system of (9) at an equilibrium point $E_j\,(j = 1,2,\ldots,6)$, $E_j = (u_j, 0)$. Using the theory of equilibrium points (see [42]), it is easy to compute that $J(u_1,0) < 0$ (saddle point) (see Figure 1(a)), $J(u_3,0) < 0$ (saddle point), $J(u_2,0) = 0$ (cusp point) (see Figure 1(b)) or $J(u_2,0) < 0$ (saddle point), $J(u_3,0) = 0$ (cusp point) (see Figure 1(c)), and $J(u_4,0) < 0$ (saddle point), $J(u_5,0) > 0$ (center point), $J(u_6,0) < 0$ (saddle point) (see Figure 1(d)). Then, we draw the bifurcation of phase portraits of system (9) by Maple (see Figure 1).

From (10), we set integral constant h be fixed, which yields

$$y^2 = \frac{3}{4}u^4 - 3lu^3 + 3l^2u^2 + 2cu + 2h = \frac{3}{4}G(u). \tag{12}$$

Then, we integrate over a branch of the curve from initial value $u(t_0) = u_0$. That is,

$$\xi = \int_{u_0}^{u} \sqrt{\frac{4}{3G(s)}}\,\mathrm{d}s. \tag{13}$$

Let $h_j = H(u_j, 0)\,(j = 1,2,\ldots,6)$; for the phase portraits, according to the parameters, we get $h_2 < h_3$ and $h_5 < h_4 < h_6$, respectively. Then, we just consider Figure 1(d) as follows:

(i) Firstly, if $h \in (h_5, h_4)$, we get the green curve. It corresponds to a family of periodic orbits of system (9) surrounding $E_5(u_5, 0)$. Then, $G(u) = (\lambda_1 -$

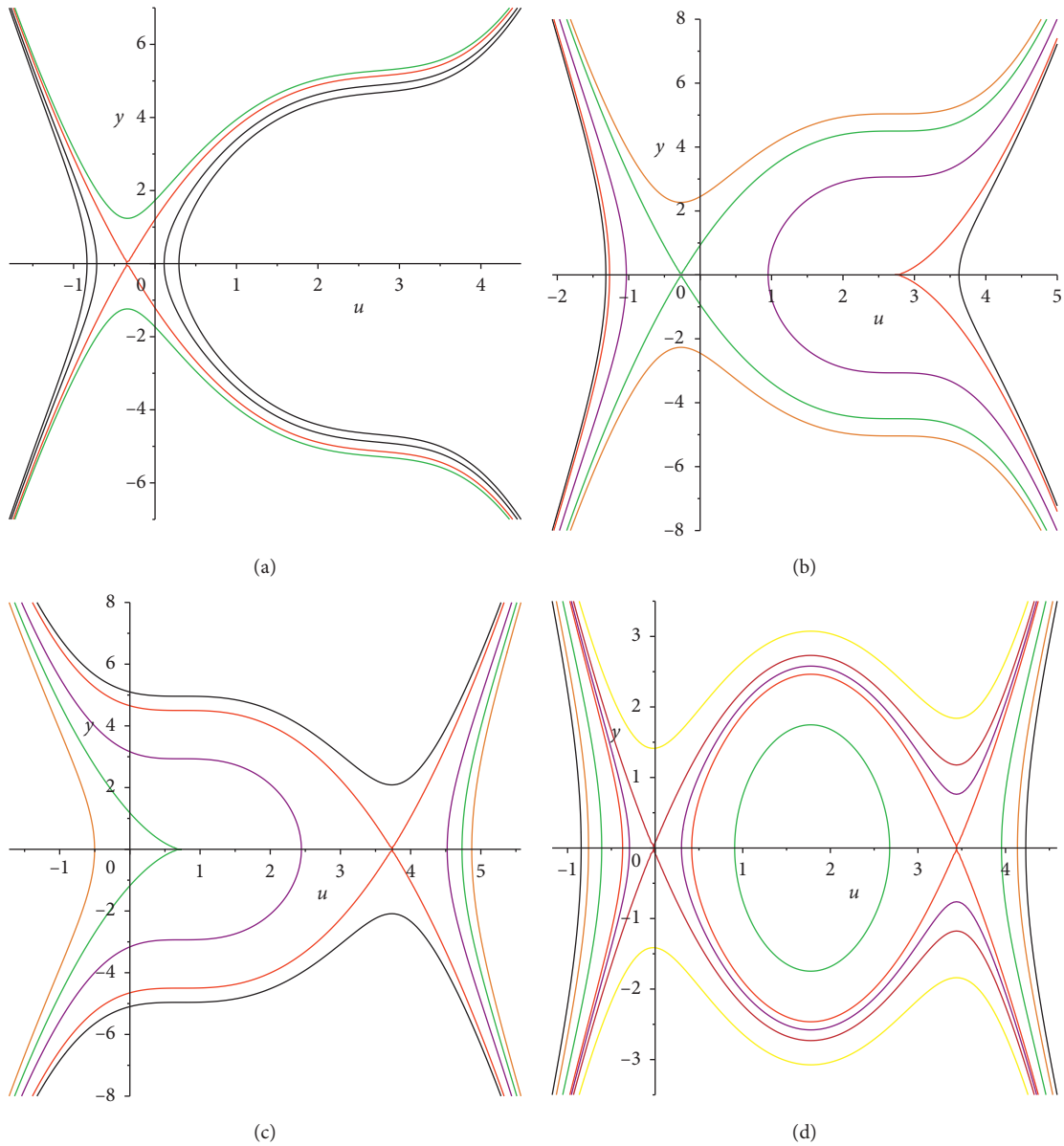


Figure 1: Bifurcation of phase portraits of system (9). (a) One equilibrium point. (b, c) Two equilibrium points. (d) Three equilibrium points. Parameters: (a) $c = 4$, $l = \sqrt{3}$, (b) $c = 3$, $l = \sqrt{3}$, (c) $c = -3$, $l = \sqrt{3}$, and (d) $c = 0.2$, $l = \sqrt{3}$.

$u)(\lambda_2 - u)(u - \lambda_3)(u - \lambda_4)$ (λ_1, λ_2, λ_3, and λ_4 are the points of intersection of the green curve with the u-axis in Figure 1(d)). Using the formula (see 254.00 in [43]), when $\lambda_4 < \lambda_3 \le u < \lambda_2 < \lambda_1$, then we have

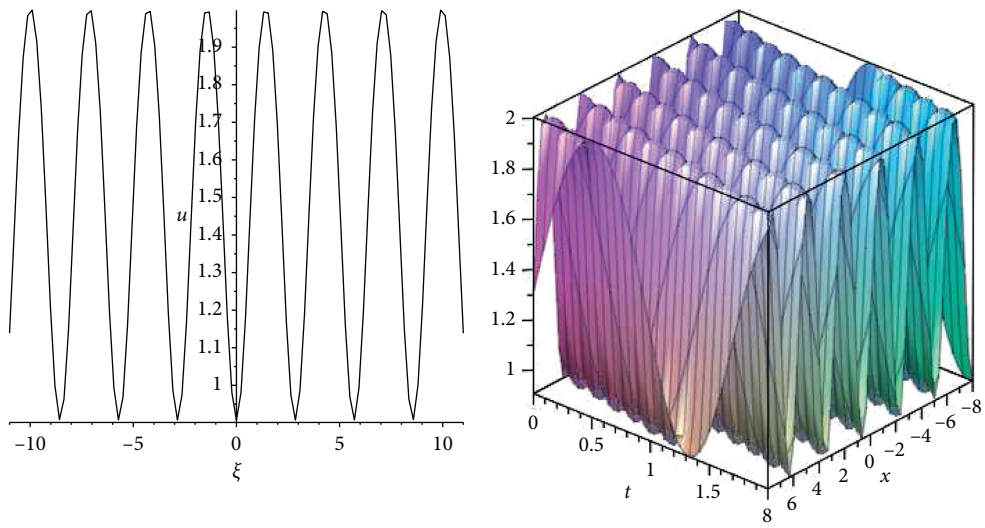
$$\int_{\lambda_3}^{u} \frac{ds}{\sqrt{(\lambda_1 - s)(\lambda_2 - s)(s - \lambda_3)(s - \lambda_4)}} = g sn^{-1}(\sin\varphi, k_1) = \frac{\sqrt{3}}{2}\xi, \tag{14}$$

where $g = 2/\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}$ and $k_1^2 = ((\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4))/((\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4))$. Consequently, on the basis of (14), $sn^{-1}(\sin\varphi, k_1) = (\sqrt{3}/2g)\xi$; by virtue of $\varphi = \sin^{-1}\sqrt{((\lambda_2 - \lambda_4)(u - \lambda_3))/((\lambda_2 - \lambda_3)(u - \lambda_4))}$, it leads us to $\sin\varphi = sn(\omega_1\xi, k_1) = \sqrt{((\lambda_2 - \lambda_4)(u - \lambda_3))/((\lambda_2 - \lambda_3)(u - \lambda_4))}$, where $\omega_1 = \sqrt{3(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}/4$.

Therefore, the parametric expression for the periodic orbit of system (9) is given by (see Figure 2(a))

$$u(\xi) = \frac{\lambda_4(\lambda_2 - \lambda_3)sn^2(\omega_1\xi, k_1) - \lambda_3(\lambda_2 - \lambda_4)}{(\lambda_2 - \lambda_3)sn^2(\omega_1\xi, k_1) - (\lambda_2 - \lambda_4)}, \tag{15}$$

where $sn(\xi, k)$ is the Jacobian elliptic function. Correspondingly, the exact periodic wave solutions of equation (2) (see Figure 2(b)) can be written as



(a) (b)

Figure 2: Periodic wave of (9) and exact periodic wave solutions of (2) when $c = 0.2$ and $l = \sqrt{3}$. (a) Periodic wave. (b) Exact periodic wave solutions.

$$u(x,t) = \frac{\lambda_4(\lambda_2-\lambda_3)sn^2(\omega_1(x-lt),k_1) - \lambda_3(\lambda_2-\lambda_4)}{(\lambda_2-\lambda_3)sn^2(\omega_1(x-lt),k_1) - (\lambda_2-\lambda_4)}. \tag{16}$$

(ii) Then, if $h = h_4$, we get the red level curve. Now, it corresponds to a homoclinic orbit surrounding $E_5(u_5, 0)$. Thus, $G(u) = (\lambda_1 - u)^2(u-\lambda_2)(u-\lambda_3)$ (λ_1, λ_2, and λ_3 are the points of intersection of the red curve with the u-axis in Figure 1(d)). When $\lambda_3 < \lambda_2 < u < \lambda_1$, then we have

$$\begin{aligned}&\int_{\lambda_2}^{u} \frac{ds}{\sqrt{(\lambda_1 - s)^2(s-\lambda_2)(s-\lambda_3)}}\\ &= \int_{\lambda_2}^{u} \frac{ds}{(\lambda_1 - s)\sqrt{(s-\lambda_2)(s-\lambda_3)}} = \frac{\sqrt{3}}{2}\xi.\end{aligned} \tag{17}$$

Let $v = \lambda_1 - u$; equation (17) turns into $\int_{\lambda_1-\lambda_2}^{\lambda_1-u} -dv/(v\sqrt{(\lambda_1-\lambda_2-v)(\lambda_1-\lambda_3-v)}) = (\sqrt{3}/2)\xi$, and by calculating, we have

$$v(\xi) = \frac{2(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}{(\lambda_2-\lambda_3)\cosh(\omega_2\xi) - (\lambda_2+\lambda_3-2\lambda_1)}. \tag{18}$$

Therefore, the exact parametric expression for the homoclinic orbit of system (9) is obtained by (see Figure 3(a))

$$u(\xi) = \lambda_1 - \frac{2(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}{(\lambda_2-\lambda_3)\cosh(\omega_2\xi) - (\lambda_2+\lambda_3-2\lambda_1)}, \tag{19}$$

where $\omega_2 = \sqrt{(3/4)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}$. Hence, correspondingly, we obtain solitary wave solutions of equation (2) as follows (see Figure 3(b)):

$$u(x,t) = \lambda_1 - \frac{2(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}{(\lambda_2-\lambda_3)\cosh(\omega_2(x-lt)) - (\lambda_2+\lambda_3-2\lambda_1)}. \tag{20}$$

Through the above analysis, we get the following theorems:

Theorem 1. *If the parameter group (l, c) satisfies $-(\sqrt{3}/3)l^3 < c < (\sqrt{3}/3)l^3\,(l>0)$ or $(\sqrt{3}/3)l^3 < c < -(\sqrt{3}/3)l^3\,(l<0)$ and level curves are defined by $h \in (h_5, h_4)$, then equation (2) has the periodic wave solutions with the exact parametric expression given by (16).*

Theorem 2. *If the parameter group (l, c) satisfies $-(\sqrt{3}/3)l^3 < c < (\sqrt{3}/3)l^3\,(l>0)$ or $(\sqrt{3}/3)l^3 < c < -(\sqrt{3}/3)l^3\,(l<0)$ and level curves are defined by $h = h_4$, then equation (2) has the solitary wave solutions with the exact parametric expression given by (20).*

3. Exact Solutions of Fractional-Order Wu–Zhang System (3)

Secondly, we consider the fractional-order system. Here, we use the conformable fractional derivative proposed by Khalil et al. [44]. Different from (4), taking $\xi = x - lt^\alpha$, we have

$$\begin{aligned}D_t^\alpha u(x,t) &= t^{1-\alpha}\frac{\partial u(x,t)}{\partial t} = t^{1-\alpha}\frac{du(\xi)}{dt} = t^{1-\alpha}u_\xi\frac{d\xi}{dt}\\ &= t^{1-\alpha}u_\xi(-l\alpha)t^{\alpha-1} = -l\alpha u_\xi,\end{aligned} \tag{21}$$

$$\begin{aligned}D_t^\alpha v(x,t) &= t^{1-\alpha}\frac{\partial v(x,t)}{\partial t} = t^{1-\alpha}\frac{dv(\xi)}{dt} = t^{1-\alpha}v_\xi\frac{d\xi}{dt}\\ &= t^{1-\alpha}v_\xi(-l\alpha)t^{\alpha-1} = -l\alpha v_\xi.\end{aligned} \tag{22}$$

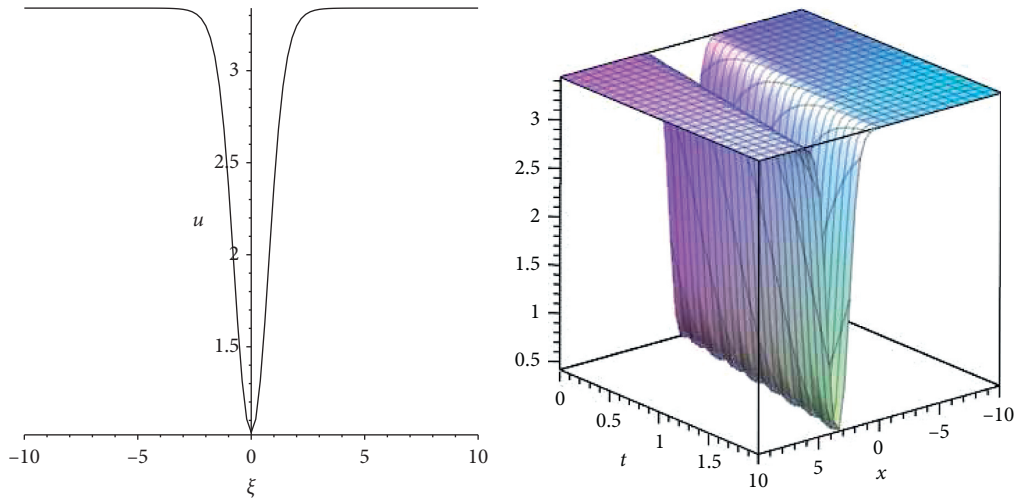


Figure 3: Solitary wave of (9) and exact solitary wave solutions of (2) when $c = 0.2$ and $l = \sqrt{3}$. (a) Solitary wave. (b) Solitary wave solutions.

Substituting (21) and (22) into system (3), then (3) turns into

$$\begin{aligned} l\alpha u_\xi &= uu_\xi + v_\xi, \\ l\alpha v_\xi &= vu_\xi + uv_\xi + \frac{1}{3}u_{\xi\xi\xi}. \end{aligned} \tag{23}$$

Similar discussion to the integer-order Wu–Zhang system leads us to

$$u_{\xi\xi} - \frac{3}{2}u^3 + \frac{9}{2}l\alpha u^2 - 3l^2\alpha^2 u = c. \tag{24}$$

Then, equation (24) reduces to

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\xi} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= \frac{3}{2}u^3 - \frac{9}{2}l\alpha u^2 + 3l^2\alpha^2 u + c. \end{aligned} \tag{25}$$

Meanwhile, the first integral is written as

$$H(u, y) = \frac{1}{2}y^2 - \frac{3}{8}u^4 + \frac{3}{2}l\alpha u^3 - \frac{3}{2}l^2\alpha^2 u^2 - cu = h. \tag{26}$$

It is not difficult to see that the first integral in the fractional-order case depends on the fractional α. Consequently, the equilibrium points $\widetilde{E}_i(\widetilde{u}_i, 0)\,(i = 1, 2, \ldots, 6)$ of system (25) change with the fractional α. Similarly, we let $\widetilde{g}(u) = (3/2)u^3 - (9/2)l\alpha u^2 + 3l^2\alpha^2 u + c$; then, $\widetilde{g}'(u) = (9/2)u^2 - 9l\alpha u + 3l^2\alpha^2$. The roots of $\widetilde{g}(u) = 0$ also depend on the parameter group (l, c, α) taking different values. (i) If $c > (\sqrt{3}/3)l^3\alpha^3\,(l > 0)$ or $c < -(\sqrt{3}/3)l^3\alpha^3\,(l > 0)$ or $c > -(\sqrt{3}/3)l^3\alpha^3\,(l < 0)$ or $c < (\sqrt{3}/3)l^3\alpha^3\,(l < 0)$, then $\widetilde{g}(u)$ has only one real root $\widetilde{u}_1$; (ii) if $c = \pm(\sqrt{3}/3)l^3\alpha^3\,(l > 0$ or $l < 0)$, then $\widetilde{g}(u)$ has two real roots: $\widetilde{u}_2$ and $\widetilde{u}_3$ $(\widetilde{u}_3 < \widetilde{u}_2)$; and (iii) if $-(\sqrt{3}/3)l^3\alpha^3 < c < (\sqrt{3}/3)l^3\alpha^3\,(l > 0)$ or $(\sqrt{3}/3)l^3\alpha^3 < c < -(\sqrt{3}/3)l^3\alpha^3\,(l < 0)$, then $\widetilde{g}(u)$ has three real roots: $\widetilde{u}_4$, $\widetilde{u}_5$, and $\widetilde{u}_6$ $(\widetilde{u}_6 < \widetilde{u}_5 < \widetilde{u}_4)$.

Then, as the same discussion as before, we set $\widetilde{h}_2 < \widetilde{h}_3$ and $\widetilde{h}_5 < \widetilde{h}_4 < \widetilde{h}_6$. Obviously, it has the similar representation of periodic orbits as (15) of the form

$$u(\xi) = \frac{\widetilde{\lambda}_4(\widetilde{\lambda}_2 - \widetilde{\lambda}_3)sn^2(\widetilde{\omega}_1\xi, \widetilde{k}_1) - \widetilde{\lambda}_3(\widetilde{\lambda}_2 - \widetilde{\lambda}_4)}{(\widetilde{\lambda}_2 - \widetilde{\lambda}_3)sn^2(\widetilde{\omega}_1\xi, \widetilde{k}_1) - (\widetilde{\lambda}_2 - \widetilde{\lambda}_4)}, \tag{27}$$

where $\widetilde{\omega}_1 = \sqrt{3(\widetilde{\lambda}_1 - \widetilde{\lambda}_3)(\widetilde{\lambda}_2 - \widetilde{\lambda}_4)}/2$ and $\widetilde{k}_1^2 = ((\widetilde{\lambda}_2 - \widetilde{\lambda}_3)(\widetilde{\lambda}_1 - \widetilde{\lambda}_4))/((\widetilde{\lambda}_1 - \widetilde{\lambda}_3)(\widetilde{\lambda}_2 - \widetilde{\lambda}_4))$, $\widetilde{\lambda}_4 < \widetilde{\lambda}_3 < \widetilde{\lambda}_2 < \widetilde{\lambda}_1$. Consequently, we have the exact periodic wave solutions of system (3):

$$u(x, t) = \frac{\widetilde{\lambda}_4(\widetilde{\lambda}_2 - \widetilde{\lambda}_3)sn^2(\widetilde{\omega}_1(x - lt^\alpha), \widetilde{k}_1) - \widetilde{\lambda}_3(\widetilde{\lambda}_2 - \widetilde{\lambda}_4)}{(\widetilde{\lambda}_2 - \widetilde{\lambda}_3)sn^2(\widetilde{\omega}_1(x - lt^\alpha), \widetilde{k}_1) - (\widetilde{\lambda}_2 - \widetilde{\lambda}_4)}. \tag{28}$$

We also get similar parametric representation for a homoclinic orbit of (25) when level curves are defined by $\widetilde{h} = \widetilde{h}_4$ as follows:

$$u(\xi) = \widetilde{\lambda}_1 - \frac{2(\widetilde{\lambda}_2 - \widetilde{\lambda}_1)(\widetilde{\lambda}_3 - \widetilde{\lambda}_1)}{(\widetilde{\lambda}_2 - \widetilde{\lambda}_3)\cosh(\widetilde{\omega}_2\xi) - (\widetilde{\lambda}_2 + \widetilde{\lambda}_3 - 2\widetilde{\lambda}_1)}, \tag{29}$$

where $\widetilde{\omega}_2 = \sqrt{(3/4)(\widetilde{\lambda}_2 - \widetilde{\lambda}_1)(\widetilde{\lambda}_3 - \widetilde{\lambda}_1)}$. Therefore, the exact solitary wave solutions of system (3) can be written as

$$u(x, t) = \widetilde{\lambda}_1 - \frac{2(\widetilde{\lambda}_2 - \widetilde{\lambda}_1)(\widetilde{\lambda}_3 - \widetilde{\lambda}_1)}{(\widetilde{\lambda}_2 - \widetilde{\lambda}_3)\cosh(\widetilde{\omega}_2(x - lt^\alpha)) - (\widetilde{\lambda}_2 + \widetilde{\lambda}_3 - 2\widetilde{\lambda}_1)}. \tag{30}$$

For the fractional-order situation, we have the similar theorems:

Theorem 3. *If the parameter group $(l, c, \alpha)\,(0 < \alpha < 1)$ satisfies $-(\sqrt{3}/3)l^3\alpha^3 < c < (\sqrt{3}/3)l^3\alpha^3\,(l > 0)$ or $(\sqrt{3}/3)l^3\alpha^3 < c < -(\sqrt{3}/3)l^3\alpha^3\,(l < 0)$ and level curves are defined by $\widetilde{h} \in (\widetilde{h}_5, \widetilde{h}_4)$, then equation (3) has the periodic wave solutions with the exact parametric expression given by (28).*

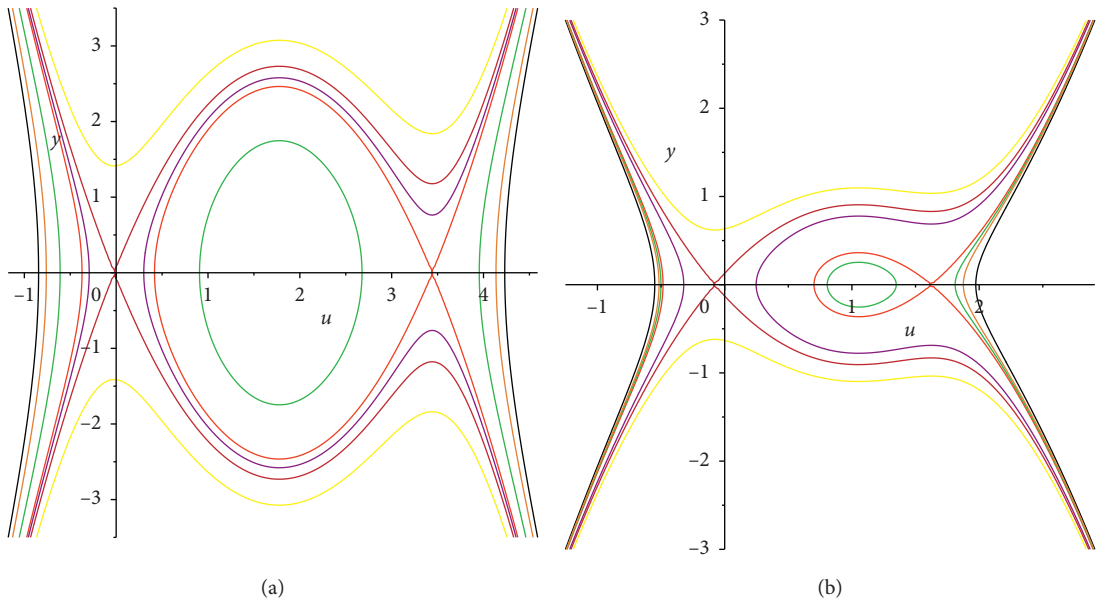


Figure 4: Phase portraits of system (9) and (25) when $c = 0.2$ and $l = \sqrt{3}$. (a) $\alpha = 1$, $E_4(u_4, 0)$, $E_5(u_5, 0)$, and $E_6(u_6, 0)$. (b) $\alpha = 1/2$, $\widetilde{E}_4(\widetilde{u}_4, 0)$, $\widetilde{E}_5(\widetilde{u}_5, 0)$, and $\widetilde{E}_6(\widetilde{u}_6, 0)$.

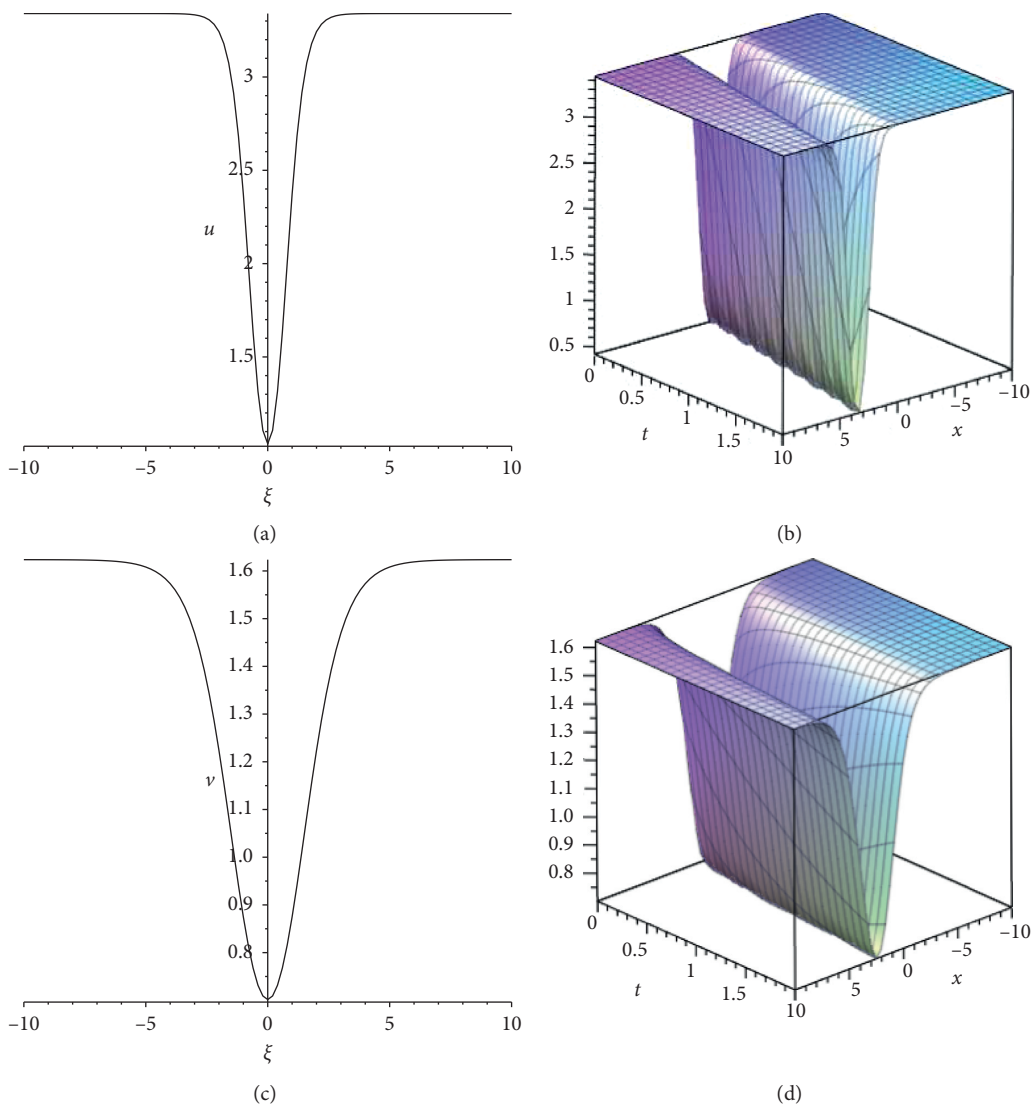


Figure 5: Comparison between the integer-order and fractional-order Wu–Zhang system when $c = 0.2$ and $l = \sqrt{3}$. (a) $\alpha = 1$ solitary wave. (b) $\alpha = 1$ exact solitary wave solutions. (c) $\alpha = 1/2$ solitary wave. (d) $\alpha = 1/2$ exact solitary wave solutions.

Theorem 4. *If the parameter group* $(l, c, \alpha)\,(0 < \alpha < 1)$ *satisfies* $-(\sqrt{3}/3)l^3\alpha^3 < c < (\sqrt{3}/3)l^3\alpha^3\,(l > 0)$ *or* $(\sqrt{3}/3)l^3\alpha^3 < c < -(\sqrt{3}/3)l^3\alpha^3\,(l < 0)$ *and level curves are defined by* $\tilde{h} = \tilde{h}_4$, *then equation (3) has the solitary wave solutions with the exact parametric expression given by (30).*

4. Comparison between the Integer-Order Wu–Zhang System and the Fractional-Order Wu–Zhang System

In this part, we compare the phase portraits and exact solutions of case (ii) for the integer-order and fractional-order Wu–Zhang system. Under the same parameters ($c = 0.2$ and $l = \sqrt{3}$), we take different derivative orders $\alpha = 1$ and $\alpha = 1/2$, respectively. According to different α, we obtain the different phase portraits of system (9) and system (25).

Obviously, we see that the nonzero equilibrium points are related to the derivative order α. The equilibrium points amount to $E_4(3.4414364477, 0)$, $E_5(1.0524405702, 0)$, and $E_6(-0.02180859881, 0)$ in Figure 4(a). Nevertheless, in Figure 4(b), the equilibrium points amount to $\tilde{E}_4(1.6236627178, 0)$, $\tilde{E}_5(1.0524405702, 0)$, and $\tilde{E}_6(-0.078027076777, 0)$. We find that the level curves defined by the first integral h (periodic orbits and homoclinic orbits) all rely on α. Thus, the corresponding exact parametric representations of the Wu–Zhang system also change along with α.

The exact solitary wave solution of the integer-order Wu–Zhang system is obtained by (see Figure 5(b))

$$u(x,t) = \lambda_1 - \frac{2(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}{(\lambda_2 - \lambda_3)\cosh(\omega_2(x - lt)) - (\lambda_2 + \lambda_3 - 2\lambda_1)}. \tag{31}$$

However, for the fractional-order Wu–Zhang system, the exact solitary wave solutions can be written as (see Figure 5(d))

$$u(x,t) = \tilde{\lambda}_1 - \frac{2(\tilde{\lambda}_2 - \tilde{\lambda}_1)(\tilde{\lambda}_3 - \tilde{\lambda}_1)}{(\tilde{\lambda}_2 - \tilde{\lambda}_3)\cosh(\tilde{\omega}_2(x - lt^{\alpha})) - (\tilde{\lambda}_2 + \tilde{\lambda}_3 - 2\tilde{\lambda}_1)}. \tag{32}$$

We compare Figure 5(a) with Figures 5(c) and 5(b) with Figure 5(d), respectively. Certainly, the height and opening size of the solitary wave are different. The height of the solitary wave of the integer-order Wu–Zhang system is 3.3 in Figures 5(a) and 5(b), while the height of the solitary wave of the fractional-order Wu–Zhang system is almost 1.65 in Figures 5(c) and 5(d). Meanwhile, opening size of the integer order is less than the fractional order. Thus, the height and opening size of the solitary wave all depend on the derivative order α.

5. Conclusion

This paper has studied the bifurcation and exact solutions of the Wu–Zhang system. We employ the dynamical system method to obtain the solitary wave solutions and periodic wave solutions. Moreover, we studied the integer-order and fractional-order Wu–Zhang system in a united way. We find that the bifurcation of phase portraits, nonzero equilibrium points, and exact solutions for the integer-order and fractional-order Wu–Zhang system all depend on the derivative order α.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

H. Zheng carried out the computations and figures in the proof. L. Guo helped to replot the figures and participated in the revision. Y. Xia conceived the study and designed and drafted the manuscript. Y. Bai participated in the discussion of the project. All authors read and approved the final manuscript.

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