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Research Article

A Sufficient Condition for Planar Graphs of Maximum Degree 6 to be Totally 7-Colorable

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A total k-coloring of a graph is an assignment of k colors to its vertices and edges such that no two adjacent or incident elements receive the same color. The total coloring conjecture (TCC) states that every simple graph G has a total $(\Delta(G)+2)$-coloring, where $\Delta(G)$ is the maximum degree of G. This conjecture has been confirmed for planar graphs with maximum degree at least 7 or at most 5, i.e., the only open case of TCC is that of maximum degree 6. It is known that every planar graph G of $\Delta(G)\geq 9$ or $\Delta(G)\in\{7,8\}$ with some restrictions has a total $(\Delta(G)+1)$-coloring. In particular, in (Shen and Wang, 2009), the authors proved that every planar graph with maximum degree 6 and without 4-cycles has a total 7-coloring. In this paper, we improve this result by showing that every diamond-free and house-free planar graph of maximum degree 6 is totally 7-colorable if every 6-vertex is not incident with two adjacent four cycles or three cycles of size p, q, ℓ for some $\{p, q, \ell\}\in\{\{3,4,4\},\{3,3,4\}\}$.

1. Introduction

Throughout the paper, we consider only simple, finite, and undirected planar graphs and follow [1] for terminologies and notations not defined here. Given a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. For a vertex $v\in V(G)$, we denote by $d_G(v)$ the degree of v in G and let $N_G(v)=\{u\,|\,uv\in E(G)\}$. A k-vertex, k^--vertex or k^+-vertex is a vertex of degree k, at most k, or at least k, respectively. For a planar graph G, we always assume that G is embedded in the plane, and denote by $F(G)$ the set of faces of G. The degree of a face $f\in F(G)$, denoted by $d_G(f)$, is the number of edges incident with f, where each cut-edge is counted twice. A face of degree k, at least k, or at most k is called a k-face, k^+-face, or k^--face. A k-face with consecutive vertices $v_1, v_2, \ldots, v_k$ along its boundary in some direction is often said to be a $(d_G(v_1), d_G(v_2), \ldots, d_G(v_k))$-face. Two faces are called adjacent if they are incident with a common edge.

A total k-coloring of a graph G is a coloring from $V(G)\cup E(G)$ to $\{1,2,\ldots,k\}$ such that no two adjacent or incident elements have the same color. A graph G is said to be totally k-colorable if it admits a total k-coloring. The total chromatic number of G, denoted by $\chi_t(G)$, is the smallest integer k such that G is totally k-colorable. The total coloring conjecture (TCC), which was proposed by Behzad [2] and Vizing [3] independently, states that every simple graph G is totally $(\Delta(G)+2)$-colorable, where $\Delta(G)$ is the maximum degree of G. TCC has been confirmed for graphs with the maximum degree at most 5 [4]. For planar graphs, it is known that every planar graph G with $\Delta(G)\geq 7$ is $(\Delta(G)+2)$-colorable [5]; in particular, if $\Delta(G)\geq 9$, then $\chi_t(G)=\Delta(G)+1$ [6, 7]. Therefore, the only open case of TCC for planar graphs is the ones with maximum degree 6. As for this special kind of planar graphs, the first work was conducted by Wang et al. [8], who verified TCC for planar graphs with maximum degree 6 and without cycles of length 4. Sun et al. [7] improved the result by proving that every planar graph G with maximum degree 6 is totally 8-colorable if G contains no adjacent triangles. In [9], Zhu and Xu further improved this result by showing that every planar graph with maximum degree 6 is totally 8-colorable if the graph does not contain any subgraph isomorphic to a 4-fan.

This paper mainly focuses on the sufficient conditions for planar graphs with maximum degree 6 to be totally 7-colorable. As for this topic, it has been proved that $\chi_t(G) = 7$ if G does not contain 5-cycles [10] or 4-cycles [11]. We will show that every planar graph G with $\Delta(G) = 6$ has a total 7-coloring if G contains no some forbidden 4-cycles, which improves the result of [11].

Theorem 1. *Suppose that G is a planar graph with $\Delta(G) = 6$. If G does not contain a subgraph isomorphic to a diamond or a house, as shown in Figure 1, and every 6-vertex in G is not incident with two adjacent 4-cycles or 3-cycles with sizes p, q, ℓ for some $\{p, q, \ell\} \in \{\{3, 4, 4\}, \{3, 4, 4\}\}$, then $\chi_t(G) = 7$.*

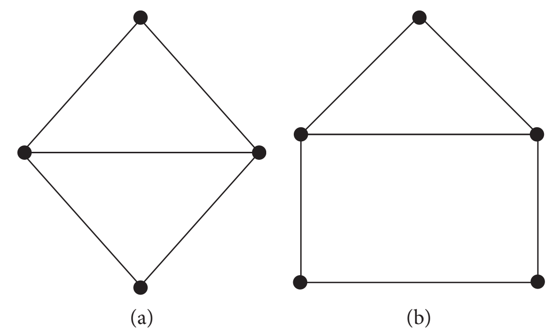


Figure 1: (a) Diamond. (b) House.

2. Reducible Configurations

Let H be a minimal counterexample to Theorem 1 in the sense that the quantity $|V(H)| + |E(H)|$ is minimum. That is, H satisfies the following properties:

(1) H is a planar graph of maximum degree 6

(2) H contains no subgraph isomorphic to a diamond or a house.

(3) Every 6-vertex of H is incident with neither two adjacent 4-cycles nor 3-cycles with sizes p, q, l for some $\{p, q, \ell\} \in \{\{3, 4, 4\}, \{3, 4, 4\}\}$

(4) H is not totally 7-colorable such that $|V(H)| + |E(H)|$ is minimum subject to (1), (2), and (3)

Notice that every planar graph with maximum degree 5 is totally 7-colorable [4]. Additionally, it is easy to check that every subgraph of H also possesses (2) and (3). Therefore, every proper subgraph of H has total 7-coloring ϕ using the color set $C = \{1, 2, \ldots, 7\}$.

For a vertex v, we use $C_\phi(v)$ to denote the set of colors appearing on v and its incident edges, and $\overline{C}_\phi(v) = (\{1, 2, \ldots, 7\} \backslash C_\phi(v))$. This section is devoted to investigating some structural information, which shows that certain configurations are reducible, i.e., they cannot occur in H.

Lemma 1

(1) Let uv be an edge of H such that $d_H(u) \le 3$ or $d_H(v) \le 3$. Then, $d_H(u) + d_H(v) \ge 8$

(2) The subgraph that is induced by all edges whose two ends are 2-vertex and 6-vertex, respectively, in H is a forest.

The proof of Lemma 1 can be found in [12].

For any component T of the forest stated in Lemma 1 (2), we can see that all leaves (i.e., 1-vertices) of T are 6-vertices. Therefore, T has maximum matching M that saturates every 2-vertex in T. For each 2-vertex v in T, we refer to the neighbor of v that is saturated by M as the master of v, see [13]. Clearly, for given M, each 6-vertex can be the master of at most one 2-vertex, and each 2-vertex has exactly one master.

The following result follows from Lemma 1 directly.

Lemma 2. *Every 4-face in H is incident with at most one 2-vertex.*

Lemma 3. *Let f be a 3-face incident with a 2-vertex. Then, every 6-vertex incident with f has only one neighbor of degree 2.*

Proof. Let v_1 be the 2-vertex incident with f, and v_2, v_3 be the two 6-vertex incident with f. We first show that the result holds for v_2 and then holds for v_3 analogously. Assume to the contrary that v_2 has another neighbor of degree 2, say $u(\neq v_1)$. Let ϕ be a total 7-coloring of $H - v_1v_2$ by the minimality of H. Erase the colors on v_1 and u. Without loss of generality, we assume $\overline{C}_\phi(v_2) = \{7\}$. If $\phi(v_1v_3) \neq 7$, then v_1v_2 can be properly colored with 7. Hence, H has a total 7-coloring by coloring v_1, u properly (since v_1, u are 2-vertices, there are at least three available colors for each of them) and a contradiction. So, we assume $\phi(v_1v_3) = 7$. Let $w = (N_H(u) \backslash \{v_2\})$. When $\phi(uw) \neq 7$, we can color v_1v_2 with $\phi(v_2u)$ and recolor v_2u with 7. When $\phi(uw) = 7$, let $\phi(v_2v_3) = x$ and $\phi(v_2u) = y$. We first exchange the colors of v_1v_3 and v_2v_3 and then color v_1v_2 with y and recolor v_2u with x. Therefore, we can obtain a 7-total-coloring of H by coloring v_1, u with two available colors. This contradicts the assumption of H. □

Lemma 4. *H has no (4, 4, 4)-face.*

Proof. Suppose that H has a (4, 4, 4)-face with three incident vertices v_1, v_2, and v_3. By the minimality of H, $H - \{v_1v_2, v_2v_3, v_3v_1\}$ has a total 7-coloring f. Erase the colors on v_i for $i = 1, 2, 3$. Clearly, each element in $\{v_1, v_2, v_3, v_1v_2, v_2v_3, v_3v_1\}$ has at least three available colors. Therefore, f can be extended to a total 7-coloring of H, a contradiction. □

Lemma 5. *H has no (3, 5, 3, 5)-face.*

Proof. Assume to the contrary that H has (3, 5, 3, 5)-face $f = v_1v_2v_3v_4$, where $d_H(v_1) = d_H(v_3) = 3$ and $d_H(v_2) = d_H(v_4) = 5$. By the minimality of H, $H - \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ has total 7-coloring ϕ. Erase the colors on v_1, v_3. Clearly, each edge of $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ has at least two available colors. Therefore, we can properly color edges v_1v_2, v_2v_3, v_3v_4 and v_4v_1. Additionally, since v_1 and v_3 are

3-vertices and they are not adjacent in H (because H contains no subgraph isomorphic to a diamond), we can properly color v_1 and v_3 with two available colors. Hence, we obtain a total 7-coloring of H and a contradiction. □

Lemma 6. *Every 6-vertex incident with a 2-vertex in H is adjacent to at most five 3^--vertices.*

Proof. Let v be a 6-vertex incident with a 2-vertex v_1 in H. Assume to the contrary that $N_H(v)$ contains six 3^--vertices. Let $N_H(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, where $d_H(v_1) = 2$, $d_H(v_i) \leq 3$ for $i = 2, 3, 4, 5, 6$. By the minimality of H, $H - \{vv_1\}$ has total 7-coloring ϕ. Without loss of generality, we assume $\overline{C}_\phi(v) = \{7\}$. Erase the colors on v_i for $i = 1, 2, 3, 4, 5, 6$. If 7 does not appear on the edges incident with v_1, then we can properly color vv_1 with 7. Otherwise, we can properly color vv_1 with $\phi(v)$ by recoloring v with 7. Additionally, since $v_1, v_2, v_3, v_4, v_5, v_6$ are 3^--vertices, there is at least one available color for each of them, and by Lemma 1 (1), $v_iv_j \notin E(H)$ for any $i, j \in \{1, 2, \ldots, 6\}$, $i \neq j$. Hence, we can obtain a 7-total-coloring of H and a contradiction. □

Lemma 7. *H contains no configurations depicted in Figure 2, where the vertices marked by • have no other neighbors in H.*

Proof. For configuration (1), by the minimality of H, $H - \{vv_1\}$ has 7-total-coloring ϕ. Without loss of generality, we assume that $C_\phi(v) = \{1, 2, 3, 4, 5, 6\}$. If $\phi(v_1u_1) \neq 7$ (or $\phi(v_2u_2) \neq 7$), then we can properly color vv_1 with 7 (or with $\phi(vv_2)$ by recoloring vv_2 with 7). If $\phi(v_4u) \neq 7$, then we can properly color vv_1 with $\phi(vv_4)$ by recoloring vv_4 with 7. So, we assume $\phi(u_1v_1) = \phi(u_2v_2) = \phi(uv_4) = 7$. Let $\phi(u_1u_2) = c_1$. Obviously, $c_1 \neq 7$. If $c_1 \neq \phi(vv_2)$, then we can recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_1, and then properly color vv_1 with 7. If $c_1 = \phi(vv_2)$, then $c_1 \neq \phi(vv_4)$. Therefore, we can safely interchange the colors of vv_2 and vv_4, recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_1, and then properly color vv_1 with 7. Thus, we obtain a 7-total-coloring of H and a contradiction.

For configuration (2), let ϕ be a 7-total-coloring of $H - \{vv_1\}$. Assume that $C_\phi(v) = \{1, 2, 3, 4, 5, 6\}$, where $\phi(vv_i) = i - 1$ for $i = 2, 3, 4, 5, 6$, and $\phi(v) = 6$. By a similar argument as in (1), we assume $\phi(u_1v_1) = \phi(u_2v_2) = \phi(u_3v_3) = \phi(uv_5) = 7$. Let $\phi(u_1u_2) = c_1$ and $\phi(v_2u_3) = c_2$. Obviously, $c_1, c_2 \neq 7$. First, if $c_1 \notin \{1, c_2\}$, then we can recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_1, and then properly color vv_1 with 7. Second, if $c_1 = 1$, then $c_1 \neq c_2$. When $c_2 \neq 4$, we can safely interchange the colors of vv_2 and vv_5, recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_1, and then properly color vv_1 with 7. When $c_2 = 4$, we can safely interchange the colors of vv_2 and vv_3, recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_1, and then properly color vv_1 with 7. Third, if $c_1 = c_2$, then $c_1 \neq 1$. When $c_1 \neq 2$, we can recolor u_1u_2 and v_2u_3 with 7, recolor u_1v_1, u_2v_2, and u_3v_3 with c_1, and then properly color vv_1 with 7. When $c_1 = 2$, we can safely interchange the colors of vv_3 and vv_5, recolor u_1u_2 and v_2u_3 with 7, recolor u_1v_1, u_2v_2, and u_3v_3 with c_1, and then properly color vv_1 with 7. Hence, we obtain a 7-total-coloring of H and a contradiction.

For configuration (3), let ϕ be a 7-total-coloring of $H - \{vv_1\}$. Assume that $C_\phi(v) = \{1, 2, 3, 4, 5, 6\}$, where $\phi(vv_i) = i - 1$ for $i = 2, 3, 4, 5, 6$, and $\phi(v) = 6$. By a similar argument as in (1), assume that $\phi(u_1v_1) = \phi(u_2v_2) = \phi(u_3v_3) = 7$. Let $\phi(u_1u_2) = c_1$ and $\phi(u_2v_3) = c_2$. Obviously, $c_1, c_2 \neq 7$ and $c_1 \neq c_2$. If $c_1 \neq 1$, then we can recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_1, and then properly color vv_1 with 7. If $c_1 = 1$, then $c_2 \neq 1$. Therefore, we can recolor u_1u_2 with 7, u_1v_1 with c_1, u_2v_2 with c_2, u_2v_3 with c_1, and then properly color vv_1 with 7. So, we obtain a 7-total-coloring of H and a contradiction. □

3. Discharging

In this section, to complete the proof of Theorem 1, we will use the discharging method to derive a contradiction. For a vertex v, we denote by $n_3(v)$ and $n_4(v)$ (or simply by n_3 and n_4) the number of 3-faces and 4-faces incident with v, respectively. For a face f, we denote by $m_2(f)$ and $m_3(f)$ (or simply by m_2 and m_3) the number of 2-vertices and 3-vertices incident with f, respectively.

According to Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, we have

$$\sum_{v \in V(H)} (d_H(v) - 4) + \sum_{f \in F(H)} (d_H(f) - 4) = -8 < 0. \quad (1)$$

Now, we define $c(x)$ to be the initial charge of $x \in V(H) \cup F(H)$. Let $c(x) = d_H(x) - 4$ for each $x \in V(H) \cup F(H)$. Obviously, $\sum_{x \in (V(H) \cup F(H))} ch(x) = -8 < 0$. Then, we apply the following rules to reassign the initial charge that leads to a new charge $c'(x)$. If we can show that $c'(x) \geq 0$ for each $x \in V(H) \cup F(H)$, then we obtain a contradiction and complete the proof. The discharging rules are defined as follows:

(R1): from each k-vertex to each of its incident k'-face f, transfer

- (1/3) if $k \in \{5, 6\}$, $k' = 3$, and f is a $(5^+, 5^+, 5^+)$-face
- (1/2) if $k \in \{5, 6\}$, $k' = 3$, and f is a $(4^-, 5^+, 5^+)$-face or $(4^-, 4^-, 5^+)$-face
- (1/5) if $k = 5$, $k' = 4$, and f is incident with a 2-vertex or 3-vertex

(R2): from each 6-vertex to each of its incident 4-face f, transfer

- (1/4) if f is a $(2, 6, 4^+, 6)$-face
- (5/12) if f is a $(2, 6, 3, 6)$-face
- (2/15) if f is a $(3, 5^+, 5^+, 5^+)$-face or $(3, 6, 4, 5)$-face
- (1/6) if f is a $(3, 6, 4, 6)$-face
- (1/3) if f is a $(3, 6, 3, 6)$-face
- (7/15) if f is a $(3, 6, 3, 5)$-face

(R3): from each 6-vertex u to each of its adjacent 2-vertex v, transfer

- (1/2) if v is incident with a 3-face
- (4/5) if v is not incident with a 3-face and u is a master of v
- (1/5) if v is not incident with a 3-face and u is not a master of v

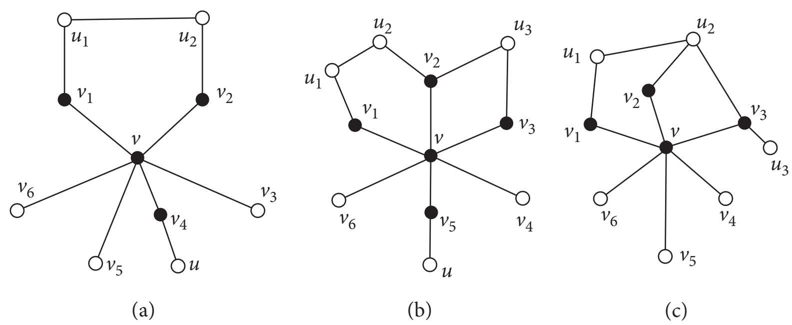


Figure 2: Three forbidden configurations in H.

(R4): from each 4-face to each of its adjacent k-vertex v, transfer

$(1/2)$ if $k = 2$ and v is not incident with a 3-face
$(1/3)$ if $k = 3$ and v is not incident with a 3-face

(R5): from each 5^+-face to each of its adjacent k-vertex v, transfer

1 if $k = 2$ and v is incident with a 3-face
$(1/2)$ if $k = 2$ and v is not incident with a 3-face
$(1/2)$ if $k = 3$ and v is incident with a 3-face
$(1/3)$ if $k = 3$ and v is not incident with a 3-face

(R6): each 5^+-face transfers $(1/6)$ to its adjacent $(4^-, 4^-, 5^+)$-face

(R7): every 4^+-face with positive charge after R1 to R6 transfers its remaining charges evenly among its incident 6-vertices.

The rest of this article is to check that $c'(x) \geq 0$ for every $x \in V(H) \cup F(H)$.

4. Final Charge of Faces

Let f be a face of H. Suppose that f is a 3-face. By Lemma 1 (1) and Lemma 4, it follows that f is incident with at most two 4^--vertices. If f is incident with at most one 4^--vertex, then by (R1), $c'(f) = 3 - 4 + 3 \times (1/3) = 0$ or $c'(f) = 3 - 4 + 2 \times (1/2) = 0$. If f is incident with two 4^--vertices, then by (R1) and (R6), $c'(f) = 3 - 4 + (1/2) + 3 \times (1/6) = 0$.

Suppose that f is a 4-face. Clearly, f is not adjacent to a 3-face since H does not contain any subgraph isomorphic to a house. If f is incident with neither a 2-vertex nor a 3-vertex, then $c'(f) = c(f) = 0$; if f is incident with a 2-vertex, then f is a $(2, 6, 3^+, 6)$-face by Lemma 1 (1) and Lemma 2, and the 2-vertex is not incident with any 3-face (since H contains no subgraph isomorphic to a diamond). So, by (R2), (R4), and (R7), $c'(f) = 2 \times (5/12) - (1/2) - (1/3) = 0$ (when f is incident with a 3-vertex) or $c'(f) = 2 \times (1/4) - (1/2) = 0$ (when f is not incident with any 3-vertex); if f is not incident with a 2-vertex but is incident with a 3-vertex, then f is either a $(3, 5^+, 4^+, 5^+)$-face or a 3, $(3, 5^+, 3, 6)$-face by Lemma 1 (1) and Lemma 5. For the former case, after (R1), (R2), and (R4), f has at least $3 \times (2/15) - (1/3) = (1/15)$ (when f is $(3, 5^+, 5^+, 5^+)$-face) or $(1/5) + (2/15) - (1/3) = 0$ (when f is $(3, 5^+, 4, 5)$-face) or $2 \times (1/6) - (1/3) = 0$ (when f is $(3, 6, 4, 6)$-face). Therefore, $c'(f) \geq 0$ by (R7). For the latter case, $c'(f) = (1/5) + (7/15) - 2 \times (1/3) = 0$ by (R1), (R2), (R4), and (R7) (when f is $(3, 5, 3, 6)$-face), or $c'(f) = (1/3) + (1/3) - 2 \times (1/3) = 0$ by (R2), (R4), and (R7) (when f is $(3, 6, 3, 6)$-face).

Suppose that f is a 5-face. Since H does not contain any subgraph isomorphic to a house, it has that every 2-vertex incident with it is not incident with a 3-face. Obviously, $m_2 + m_3 \leq 2$. If $m_2 + m_3 \leq 1$, then f has at least $5 - 4 - (1/2)(m_2 + m_3) - (1/6)(5 - 2(m_2 + m_3)) \geq 0$ after (R5) to (R6), and hence, $c'(f) \geq 0$ by R7. If $m_2 + m_3 = 2$, then f is not adjacent to any $(4^-, 4^-, 5^+)$-face. Hence, f has at least $5 - 4 - (1/2)(m_2 + m_3) = 0$ after (R5) to (R6), and $c'(f) \geq 0$.

Suppose that f is a 6-face. Then, at most one 2-vertex incident with f is incident with a 3-face. Otherwise, H contains a subgraph isomorphic to a house. By Lemmas 1 (1) and (2), it is easy to see that $m_2 \leq 2$ and $m_2 + m_3 \leq 3$. When $m_2 + m_3 \leq 2$, the number of $(4^-, 4^-, 5^+)$-faces adjacent to f is at most $6 - 2(m_2 + m_3)$. Therefore, f has at least $\min\{6 - 4 - 1 - (1/2) - 2 \times (1/6), 6 - 4 - 1 - 4 \times (1/6), 6 - 4 - 6 \times (1/6)\} = (1/6)$ after (R5) to (R6), and hence, $c'(f) \geq 0$ by R7. When $m_2 + m_3 = 3$, it follows that f is not adjacent to any $(4^-, 4^-, 5^+)$-faces. Therefore, f has at least $6 - 4 - 1 - 2 \times (1/2) = 0$ after (R5) to (R6), and $c'(f) \geq 0$.

For the convenience of proving $c'(v) \geq 0$ for every $v \in V(H)$, we first introduce the following lemma, which indicates that every 7^+-face has positive charges.

Lemma 8. *Let v be a 6-vertex. Then, v receives at least $(1/8)$ from each of its incident 7^+-face by (R7).*

Proof. Let f be a k-face incident with v, where k is an integer and $k \geq 7$. Clearly, the number of $(4^-, 4^-, 5^+)$-faces adjacent to f is at most $k - 2(m_2 + m_3)$.

Suppose $k \geq 8$. Then, f has at least $k - 4 - m_2 - (1/2) \times m_3 - (1/6) \times (k - 2 \times (m_2 + m_3)) = (5/6)k - 4 - (2/3)m_2 - (1/6)m_3$ charges after (R5) to (R6) since $0 \leq m_2 \leq \lceil k - 1/2 \rceil$ by Lemma 1 (2) and $m_2 + m_3 \leq \lceil k/2 \rceil$ by Lemma 1 (1). Therefore, v receives at least $((5/6)k - 4 - (2/3)m_2 - (1/6)m_3/k - m_2 - m_3) = (5/6) - (1/6) \cdot (24 - m_2 - 4m_3/k - m_2 - m_3) \geq (5/6) - (1/6) \times (17/4) = (1/8)$ when $m_2 = 3$, $m_3 = 1, k = 8$ from f.

Suppose $k = 7$. Clearly, $m_2 + m_3 \leq 3$. Particularly, in the case of $m_2 + m_3 = 3$, f is not adjacent to any $(4^-, 4^-, 5^+)$-face. First, when $m_2 = 3$, it has that f is incident with at most two 2-vertices that are incident with a 3-face (otherwise, there is a subgraph isomorphic to a house and a contradiction). Therefore, f has at least $7-4-2-(1/2)=(1/2)$ charges after (R5) to (R6). Second, when $m_2=2$, it follows that $m_3\leq 1$. If $m_3=0$, then f is adjacent to at most three $(4^-,4^-,5^+)$-faces (note that when f is adjacent to a $(4^-,4^-,5^+)$-face, f has to be incident with a 4-vertex. So, f is incident with at most four 6-vertices in this case). Therefore, f has at least $7-4-2-3\times(1/6)=(1/2)$ charges after (R5) to (R6). If $m_3=1$, then f is not adjacent to any $(4^-,4^-,5^+)$-face. Therefore, f has at least $7-4-2-(1/2)=(1/2)$ charges after (R5) to (R6). Third, when $m_2\leq 1$, it has that $m_3\leq 2$. In this case, we can see that f has at least $7-4-1-(1/2)-(1/2)=1$ charges after (R5) to (R6). All of the above show that f sends v at least $(1/4)\times(1/2)=(1/8)$ by (R7).

By Lemma 8, we can see that $c'(f)\geq 0$ for every 7^+-face f. □

4.1. Final Charge of Vertices. We start with an observation and a lemma.

Observation. Let v be a vertex of H. Since H has no subgraph isomorphic to a diamond, we have $n_3\leq\lceil(d_H(v)-1)/2\rceil$. Moreover, if v is a 6-vertex, then by the condition of Theorem 1, $n_4\leq 3$ and $n_3+n_4\leq 3$.

Lemma 9. *Suppose that v is a 6-vertex incident with three consecutive faces of size 4, 6, and 4, respectively, where the 6-face is denoted by f, see Figure 3(a). Then, by (R7), f gives v at least*

(1) $(1/8)$ if f is incident with at most two 3^--vertices

(2) $(1/6)$ if f is incident with three 3^--vertices and $d_H(v_1)=d_H(v_2)=2$ (see Figure 3(b))

(3) $(1/9)$ if f is incident with three 3^--vertices and $d_H(v_1)=d_H(v_2)=3$ (see Figure 3(c))

(4) $(2/9)$ if f is incident with three 3^--vertices and $d_H(v_1)\neq d_H(v_2)$

Proof. Since H contains no subgraph isomorphic to a diamond or a house, v_1 and v_2 are not incident with a 3-face if $d_H(v_1)\leq 3$ and $d_H(v_1)\leq 3$, and f is incident with at most one 2-vertex that is incident with a 3-face.

For (1), if $d_H(v_1)\leq 3$ and $d_H(v_2)\leq 3$, then f is adjacent to at most two $(4^-,4^-,5^+)$-faces. Therefore, f has at least $6-4-(1/2)-(1/2)-2\times(1/6)=(4/6)$ charges after (R5) to (R6), and v receives at least $(1/4)\times(4/6)=(1/6)$ from f.

If exact one of v_1 and v_2 is a 3^--vertex, say v_1, then we consider two cases. First, $m_2+m_3=2$. In this case, f is adjacent to at most one $(4^-,4^-,5^+)$-face. Particularly, if v_1 is a 2-vertex, denoted by $x=N_H(v_1)/\{v\}$, then x is not adjacent to another 2-vertex that is incident with a 3-face by Lemma 3. This implies that when f is incident with a 2-vertex that is incident with a 3-face, the 2-vertex is a neighbor of v_2, and so f is not incident with any $(4^-,4^-,5^+)$-face. Consequently, f has at least $\min\{6-4-(1/2)-1=(1/2)$ (when v_1 is a 2-vertex)and $6-4-(1/3)-1-(1/6)=(1/2)$ (when v_1 is a 3-vertex)$\}=(1/2)$ charges after (R5) to (R6), and v receives at least $(1/4)\times(1/2)=(1/8)$ from f. Second, $m_2+m_3=1$, i.e., f is incident with only one 3^--vertex v_1. Then, f is adjacent to at most three $(4^-,4^-,5^+)$-faces. Therefore, f has at least $6-4-(1/2)-3\times(1/6)=1$ charges after (R5) to (R6), and v receives at least $(1/4)$ from f.

If $d_H(v_1)\geq 4$ and $d_H(v_2)\geq 4$, then $m_2+m_3\leq 2$. When $m_2+m_3=2$, one can readily check that f is not adjacent to any $(4^-,4^-,5^+)$-face. Therefore, f has at least $6-4-(1/2)-1=(1/2)$ charges after (R5) to (R6), and v receives at least $(1/4)\times(1/2)=(1/8)$ from f. When $m_2+m_3=1$, it has that f is adjacent to at most two $(4^-,4^-,5^+)$-faces. Therefore, f has at least $6-4-1-2\times(1/6)=(2/3)$ charges after (R5) to (R6), and v receives at least $(1/5)\times(2/3)=(2/15)$ from f. When $m_2+m_3=0$, it has that f has at least $6-4-4\times(1/6)=(4/3)$ charges after (R5) to (R6), and v receives more than $(1/6)\times(4/3)=(2/9)$ from f.

For (2) and (3), it follows that f is not adjacent to any $(4^-,4^-,5^+)$-face. If $d_H(v_1)=d_H(v_2)=2$, then the other 3^--vertex incidents with f are a 3-vertex. Therefore, f has at least $6-4-(1/2)-(1/2)-(1/2)=(1/2)$ charges after (R5) to (R6), and v receives at least $(1/2)\times(1/3)=(1/6)$ from f. If $d_H(v_1)=d_H(v_2)=3$, then f has at least $6-4-(1/3)-(1/3)-1=(1/3)$ charges after (R5) to (R6), and v receives at least $(1/3)\times(1/3)=(1/9)$ from f.

For (4), it is clear that v_1 and v_2 are 3^--vertices. Without loss of generality, we assume $d_H(v_1)=2$ and $d_H(v_2)=3$, see Figure 3(d), where v_3 is another 3^--vertex incident with f. If $d_H(v_3)=2$, then v_3 is not incident with a 3-face by Lemma 3. Therefore, f has at least $6-4-(1/2)-(1/3)-(1/2)=(2/3)$ charges after (R5) to (R6), and v receives at least $(1/3)\times(2/3)=(2/9)$ from f.

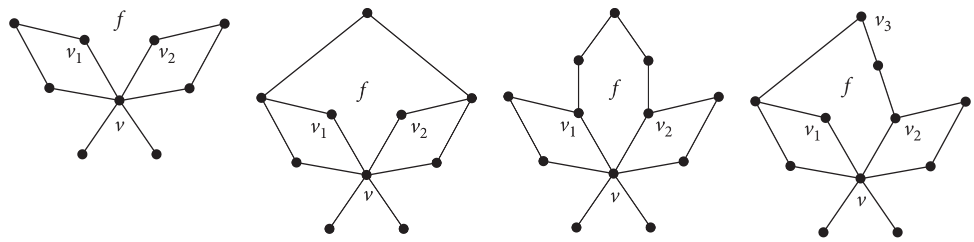
In the following, we turn to the proof of $c'(v)\geq 0$ for every $v\in V(H)$. Let $v\in V(H)$ be a vertex of H. By Lemma 1 (1), we have $d_H(v)\geq 2$.

Suppose that v is a 2-vertex. Then, v has two neighbors with degree 6 by Lemma 1 (1). If v is incident with a 3-face, then v is incident with a 5^+-face. So, v receives $(1/2)$ from each of its neighbors by (R3) and receives 1 from its incident 5^+-face. Hence, $c'(v)=2-4+2\times(1/2)+1=0$. If v is not incident with a 3-face, then by (R3), v receives $(4/5)$ from its master and $(1/5)$ from its other neighbor of degree 6 and receives $(1/2)$ from each of its adjacent 4^+-face by (R4) and (R5). Therefore, $c'(v)=2-4+(4/5)+(1/5)+2\times(1/2)=0$.

Suppose that v is a 3-vertex. If v is incident with a 3-face, then v is incident with two 5^+-faces since H does not contain any subgraph isomorphic to a house. So, by (R5), $c'(v)=3-4+2\times(1/2)=0$. If v is not incident with any 3-face, then v is incident with three 4^+-faces. Hence, by (R4) and (R5), $c'(v)=-1+3\times(1/3)=0$.

Suppose that v is a 4-vertex. By the discharging rules (R1) to (R7), we have $c'(v)=c(v)=0$.

Suppose that v is a 5-vertex. By the observation, we have $n_3\leq 2$. If $n_3=0$, then v is incident with at most five 4-faces. So, by (R1), $c'(v)\geq 1-5\times(1/5)=0$. If $n_3=1$, then $n_4\leq 2$ by the condition of Theorem 1. So, by (R1), $c'(v)\geq 1-(1/2)-2\times(1/5)=(1/10)=(1/10)$. If $n_3=2$, then $n_4=0$ by the same reason. So, by (R1), $c'(v)\geq 1-2\times(1/2)=0$.



(a) (b) (c) (d)

Figure 3: Cases of Lemma 9.

Suppose that $d_H(v) = 6$. By the observation, we have $n_3+n_4 \le 3$. Denote by t_2 the number of 2-vertices adjacent to v. Then, $t_2 \le 5$ by Lemma 6. When $t_2 = 0$, it is clear that $c'(v) \ge 6 - 4 - 3 \times (1/2) = (1/2)$ by (R1) and (R2). When $t_2 = 1$, denote by u the unique 2-vertex adjacent to v. First, $n_3 + n_4 \le 2$. Then, $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/2) = c'(v) \ge 6 - 4 - (1/2) - 3 \times (1/5) = 0$ by (R1), (R2), and (R3). Second, $n_3 + n_4 = 3$. In this case, by the condition of Theorem 1, either $n_3 = 3$ or $n_4 = 3$. For the former, we have $c'(v) \ge 6 - 4 - (1/2) - 3 \times (1/2) = 0$ by (R1), (R2), and (R3). For the latter case, if u is incident with a $(2, 6, 3, 6)$-face, then v is incident with at most one (3, 5, 3, 6)-face by Lemma 6. Therefore, $c'(v) \ge 6 - 4 - (4/5) - (5/12) - (7/15) - (1/6) = (3/20)$ by (R2) and (R3); if u is not incident with a $(2, 6, 3, 6)$-face, then v is incident with at most two (3, 5, 3, 6)-faces. Therefore, $c'(v) \ge 6 - 4 - (4/5) - (1/4) - (7/15) - (7/15) = (1/60)$ by (R2) and (R3). In what follows, we assume $t_2 \ge 2$, and then, by Lemma 3, we have that every 2-vertex is not incident with a 3-face. Thus, when $n_3+n_4 = 0$, $c'(v) \ge 6 - 4 - (4/5) - 4 \times (1/5) = (2/5)$ by (R3). Now, we further consider the following three cases:

Case 1. $n_3+n_4 = 1$. If $n_3 = 1$ and $n_4 = 0$, then $t_2 \le 4$ by Lemma 3. Therefore, $c'(v) \ge 6 - 4 - (4/5) - 3 \times (1/5) - (1/2) = (1/10)$ by (R1) and (R3). If $n_3 = 0$ and $n_4 = 1$, denoted by f the 4-face incident with v, then by Lemma 6 and (R2) and (R3), when f is a (2, 6, 3, 6)-face, it has that $t_2 \le 4$ and $c'(v) \ge 6 - 4 - (4/5) - 3 \times (1/5) - (5/12) = (11/60)$; when f is a $(2, 6, 4^+, 6)$-face, $t_2 \le 5$ and $c'(v) \ge 6 - 4 - (4/5) - 4 \times (1/5) - (1/4) = (3/20)$; when f is a $(3, 5^+, 3, 5^+)$-face, $t_2 \le 3$ and $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/5) - (7/15) = (1/3)$; when f is a $(3, 5^+, 4^+, 5^+)$-face, $t_2 \le 4$ and $c'(v) \ge 6 - 4 - (4/5) - 3 \times (1/5) - (1/6) = (13/30)$; and when f is a $(4^+, 4^+, 4^+, 4^+)$-face, $t_2 \le 4$ and $c'(v) \ge 6 - 4 - (4/5) - 3 \times (1/5) = (3/5)$.

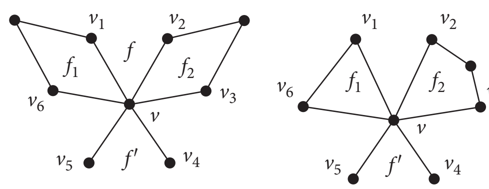
Case 2. $n_3 + n_4 = 2$. If $t_2 = 2$, then $c'(v) \ge 6 - 4 - (4/5) - (1/5) - (1/2) - (1/2) = 0$ by (R1), (R2), and (R3). If $t_2 \ge 3$, then we have $n_3 \le 1$ by Lemma 3 and the assumption that H contains no subgraph isomorphic to a diamond. In the following, we consider two subcases:

Case 2.1. $n_3 = 0$ and $n_4 = 2$. Let $N_H(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and f_1 and f_2 be the two 4-faces incident with v, see Figure 4(a). By Lemma 2, each of f_1 and f_2 is incident with at most one 2-vertex, so $t_2 \le 4$.

When $t_2 = 3$, it has that at least one of f_1, and f_2 is incident with a 2-vertex, say f_1. By Lemma 1 (1), it follows that f_1 is a $(2, 6, 3^+, 6)$-face. If f_1 is a $(2, 6, 4^+, 6)$-face, then $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/5) - (1/4) - (7/15) = (1/12)$ by (R2) and (R3). If f_1 is a (2, 6, 3, 6)-face, then we further consider the following two cases regarding to f_2. First, f_2 is incident with a 2-vertex, i.e., f_2 is a $(2, 6, 3^+, 6)$-face by Lemma 1 (1). If f_2 is a $(2, 6, 4^+, 6)$-face, then $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/5) - (1/4) - (5/12) = (2/15)$ by (R2) and (R3). Otherwise, f_2 is a (2, 6, 3, 6)-face. In this case, according to Lemmas 7 (1) and (2), we can deduce that f is a 6^+-face. Therefore, according to Lemmas 8 and 9, v receives at least (1/8) from f by (R7). So, $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/5) - 2 \times (5/12) + (1/8) = (11/120)$ by (R2), (R3), and (R7). Second, f_2 is not incident with any 2-vertex. Then, we consider $m_3(f_2)$, the number of 3-vertices incident with f_2. Obviously, $m_3(f_2) \le 1$ by Lemma 1 (1) and Lemma 6 (since $t_2 = 3$). Hence, $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/5) - (5/12) - (1/6) = (13/60)$ by (R2) and (R3).

When $t_2 = 4$, it follows that both f_1 and f_2 are $(2, 6, 3^+, 6)$-faces by Lemma 1 (1) and Lemma 2, and at most one of them is a $(2, 6, 3, 6)$-face by Lemma 6. Naturally, v_4 and v_5 are 2-vertex, and neither of them is incident with a 3-face. Denote by f the face incident with v, v_1, and v_2, see Figure 4(a). Then, $d_H(f') = 6$ by Lemma 7 (1). If $d_H(f') = 6$, then by (R7), it sends at least $\min\{(1/3) \times (6 - 4 - 2 \times (1/2) - (1/2)),\ (1/4) \times (6 - 4 - 2 \times (1/2))\} = (1/6)$ to v. If $d_H(f') \ge 7$, then it sends at least (1/8) to v by Lemma 8. Therefore, $c'(v) \ge 6 - 4 - (4/5) - 3 \times (1/5) - (5/12) - (1/4) + (1/8) = (7/120)$ by (R2) and (R3).

Case 2.2. $n_3 = 1$, and $n_4 = 1$. Denote by f_1 and f_2 the 3-face and 4-face incident with v and f' the face incident with v and not adjacent to f_1 or f_2, see Figure 4(b). By Lemmas 2 and 3, we can see that f_1 is not incident with any 2-vertex (since $t_2 \ge 3$), f_2 is incident with one 2-vertex, and v_4 and v_5 are 2-vertices. Obviously, v_4 and v_5 are not incident with any 3-face. Additionally, by Lemma 7 (1), we can see that $d_H(f') \ge 6$. Thus, with an analogous proof as above (Case 2.1), v can receive at least (1/8) from f' by (R7). Therefore, $c'(v) \ge 6 - 4 - (4/5) - 2 \times (1/5) - (1/2) - (5/12) + (1/8) = (1/120)$ by (R2) and (R3).



(a) (b)

Figure 4: Illustration for the proof of case 2.

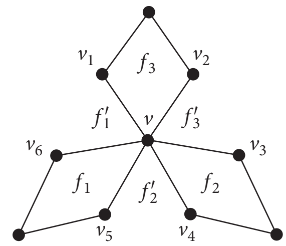


Figure 5: Illustration for the proof of case 3.

Case 3. $n_3 + n_4 = 3$. In this case, since we assume $t_2 \geq 2$, it has that $n_3 = 0$ and $n_4 = 3$ by the condition of Theorem 1 and Lemma 3. Denote by f_1, f_2, and f_3 the three 4-faces incident with v and f_1', f_2', and f_3' the three 5^+-faces incident with v, see Figure 5. By Lemma 2, it has that $t_2 \leq 3$.

Case 3.1. $t_2 = 2$. Without loss of generality, we assume f_1 and f_2 are incident with 2-vertices. Then, both of f_1 and f_2 are $(2, 6, 3^+, 6)$-faces by Lemma 1 (1). Now, we turn to considering f_3. First, f_3 is incident with two 3-vertices, i.e., both v_1 and v_2 are 3-vertices. Then, at most one of f_1 and f_2 is a $(2, 6, 3, 6)$-face. If both f_1 and f_2 are $(2, 6, 4^+, 6)$-face, then $c'(v) \geq 6 - 4 - (4/5) - (1/5) - (7/15) - (1/4) - (1/4) = (1/30)$ by (R2) and (R3). Otherwise, we assume f_1 is a $(2, 6, 3, 6)$-face. In this case, by (R7), when $d_H(v_5) = 3$, according to Lemmas 8 and 9, v can receive at least $(1/3) \times (5 - 4 - (1/3) - (1/2)) = (1/18)$ from f_i' for $i = 1, 2, 3$ if f_i' is a 5-face; when $d_H(v_5) = 2$, we have that $d_H(v_6) = 3$, and v can receive at least $(1/3) \times (5 - 4 - (1/3) - (1/3)) = (1/9)$ from f_1' if f_1' is a 5-face and at least $(1/3) \times (5 - 4 - (1/3) - (1/2)) = (1/18)$ from f_3' if f_3' is a 5-face. Additionally, by Lemmas 8 and 9, v can receive at least 1/9 from f_i' for $i = 1, 2, 3$ if f_i' is a 6^+-face. Therefore, $c'(v) \geq 6 - 4 - (4/5) - (1/5) - (7/15) - (5/12) - (1/4) + 3 \times (1/18) = (1/30)$ by (R2) and (R3). Second, f_3 is incident with at most one 3-vertices. Then, $c'(v) \geq 6 - 4 - (4/5) - (1/5) - (1/6) - (5/12) - (5/12) = 0$ by (R2) and (R3).

Case 3.2. $t_2 = 3$. Then, each of f_i is a $(2, 6, 3^+, 6)$-face, where $i \in \{1, 2, 3\}$. Particularly, by Lemma 6, at most two of them are $(2, 6, 3, 6)$-faces. When v is incident with at most one $(2, 6, 3, 6)$-face, we, without loss of generality, assume that f_3 is a $(2, 6, 3, 6)$-face, and by symmetry, let $d_H(v_1) = 3$ and $d_H(v_2) = 2$. If v_6 (or v_3) is a 2-vertex, then by Lemma 7 (2) (or Lemma 7 (1)), f_1' (or f_3') is a 6^+-face. If neither v_6 nor v_3 is a 2-vertex, then v_4 and v_5 are 2-vertices, and by Lemma 7 (1), f_2' is a 6^+-face. Therefore, by Lemmas 8 and 9, v can receive at least $(1/8)$ from f_i' for some $i \in \{1, 2, 3\}$ by (R7) (note that v is adjacent to at most one 3-vertex in this case). Hence, $c'(v) \geq 6 - 4 - (4/5) - 2 \times (1/5) - (1/4) - (5/12) - (1/4) + (1/8) = (1/120)$ by (R2) and (R3). When v is incident with two $(2, 6, 3, 6)$-faces, by symmetry, we assume f_1 and f_2 are $(2, 6, 3, 6)$-faces. Without loss of generality, we assume $d_H(v_1) = 2$ and then $d_H(v_2) \geq 4$.

First, if $d_H(v_5) = 3$ and $d_H(v_5) = 3$, then f_1' is a 6^+-face by Lemma 7 (1). When f_1' is a 6-face, f_1' gives v at least $(1/3) \times (6 - 4 - (1/2) - (1/2) - (1/2)) = (1/6)$; when f_1' is a k-face, where $k \geq 7$, by $(R2)$, $(R3)$, and $(R7)$, we can deduce that f_1' gives v at least $(k - 4 - 2\times\ (1/2) - (m_2(f_1') - 2) - (1/2) \times m_3(f_1') - (1/6) \times (k - 2m_2(f_1') - 2m_3(f_1'))/k - m_2(f_1') - m_3(f_1')) \geq\ (5/24)(k = 7, m_2(f_1') = 3, m_3(f_1') = 0)$.

Consider the face f_2'. If f_2' is a 6^+-face, then by $(R7)$, f_2' sends at least $(1/9)$ to v according to Lemmas 8 and 9. If f_2' is a 5-face, then by Lemmas 7 (1) and (2), we can deduce that $d_H(v_4) = 3$. So, f_2' sends at least $(1/3) \times (5 - 4 - (1/3) - (1/3)) = (1/9)$ to v by (R7).

Consider the face f_3'. If f_3' is a 6^+-face, then by $(R7)$, f_3' sends at least 1/8 to v according to Lemmas 8 and 9. If f_3' is a 5-face, then when $d_H(v_3) = 3$, f_3' sends at least $(1/3) \times (5 - 4 - (1/3) - (1/2)) = (1/18)$ to v; when $d_H(v_3) = 2$, let v_2, v, v_3, u_1, u_2 be the five vertices incident with f_3', where u_1 is adjacent to v_3 and u_2 is adjacent to v_2. Then, $d_H(u_1) = 6$ by Lemma 1 (1), and $d_H(u_2) \geq 3$ by Lemma 7 (3). If $d_H(u_2) \geq 4$, then v can receive at least $(1/3) \times (5 - 4 - (1/2) - 2 \times (1/6)) = (1/18)$ from f_3'. Otherwise, if $d_H(u_2) = 3$, then $d_H(v_2) \geq 5$, and f_3 is incident with only one 3^--vertex v_1. So, f_3 has at least $2 \times (1/4) + (1/5) - (1/2) = (1/5)$ after (R1), (R2), and (R4). So, f_3 gives v at least $(1/15)$ by (R7) in this case.

To sum up, we have $c'(v) \geq 6 - 4 - (4/5) - 2 \times (1/5) - (1/4) - (5/12) - (5/12) + (1/6) + (1/9) + (1/18) = (1/20)$ by (R2) and (R3).

Second, if $d_H(v_6) = 3$ and $d_H(v_5) = 2$, then both f_1' and f_2' are 6^+-faces by Lemmas 7 (1) and (2). So, each of f_1' and f_2'gives v at least $(1/8)$ by Lemmas 8 and 9. In addition, with the similar analysis as the above, v can receive at least either $(1/18)$ from f_3' or $(1/15)$ from f_3. Hence, $c'(v) \geq 6 - 4 - (4/5) - 2 \times (1/5) - (1/4) - (5/12) - (5/12) + (1/8) + (1/8) + (1/18) = (1/45)$ by (R2) and (R3).

In all cases, we have shown that $c'(x) \geq 0$ for every $x \in V(H) \cup F(H)$. Therefore, $\sum_{x \in V(H) \cup F(H)} c(x) = \sum_{x \in V(H) \cup F(H)} c'(x) \geq 0$, a contradiction. This completes the proof of Theorem 1. □

5. Conclusion

By using the "discharging" approach, we obtain a sufficient condition for a planar graph of maximum degree 6 to be totally 7-colorable. Since no planar graphs of maximum degree 6 that are not totally 7-colorable are found, it is widely believed that every planar graph of maximum degree 6 has a total 7-coloring [11]. Our result enhances the reliability of this conjecture. Nevertheless, to prove TCC for planar graphs, it still requires persistent efforts on the study of structures of planar graphs of maximum degree 6. As a future work, we would like to further explore the structural properties of this kind of graphs, as well as the possibility of applying them in the proof of TCC for planar graphs.

Data Availability

No external data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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