Research Article

The Oscillatory of Linear Conformable Fractional Differential Equations of Kamenev Type

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In this paper, the oscillatory of the Kamenev-type linear conformable fractional differential equations in the form of $(p(t)y^{(\alpha+1)}(t))^{(\alpha)} + y^{(\alpha+1)}(t) + q(t)y(t) = 0$ is studied, where $t \geq t_0$ and $0 < \alpha \leq 1$. By employing a generalized Riccati transformation technique and integral average method, we obtain some oscillation criteria for the equation. We also give some examples to illustrate the significance of our results.

1. Introduction

The fractional derivative originated in the 17th century and formed a relatively complete form in the 19th century. In the existing literature, we can find many definitions of the fractional integral and fractional differential given by these scholars, such as Riemann–Liouville, Caputo, Weyl, Hadamard, and Chen. In the past decades, fractional differential equations have been widely used in viscoelasticity, electrical networks, signal processing, systems identification, and other natural phenomena and physical problems, fluid flow, rheology, and many other fields. There are many literature studies that have investigated the existence, uniqueness, and stability of fractional differential equations and a series of conclusions about the qualitative of solutions, see [1–9] and their references.

In 2014, a new definition, which is called the conformable derivative, was proposed in the literature [10], and the properties and calculations of conformable derivatives have been studied in the literature [11–14]. It almost satisfied all the properties of the normal derivative: the derivative of constant is zero, the rules for the product and quotient of two functions, and the chain rule. Because it combines the best properties, many research studies have increased their research on it in recent years. Although there is much about the oscillations of fractional differential equations (see

[15–28] and the references therein), there is little about the conformable fractional differential equations of Kamenev type.

In 1978, Kamenev [7] studied the oscillation of the following:

$$(p(t)y^{(\alpha)}(t))^{(\alpha)} + q(t)y(t) = 0, \quad t \geq t_0, \quad (1)$$

where $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$, $0 < \alpha \leq 1$ and q might change signs. And the above equation is oscillation under the condition

$$\limsup \frac{1}{t^m} \int_{t_0}^t (t-s)^m q(s) ds = \infty, \quad m > 1. \quad (2)$$

In 2019, Shao and Zhaowen [14] established new oscillation criteria of Kamenev type for linear conformable of the above equation, where $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$, $0 < \alpha \leq 1$ and q might change signs.

Inspired by the above, by extending the order of the equation containing conformable fractional derivative and increasing the number of terms, we make it have new application value.

In this paper, we discuss the oscillatory behavior about linear conformable fractional differential equations of Kamenev type of the following:

$$(p(t)y^{(\alpha+1)}(t))^{(\alpha)} + y^{(\alpha+1)}(t) + q(t)y(t) = 0, \quad t \geq t_0, \quad (3)$$

where $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), R)$, $0 < \alpha \leq 1$.

In this paper, we always assume that the following conditions are in force:

(C₁): there exist a constant $k \in \mathbb{N}^+$ such that $(y^{(\alpha)}(t)/y^{(\alpha+1)}(t)) \geq k$, $t \geq t_0$.

(C₂): $H(t, s)$ is a continuous function and satisfies the following:

- (1) $H(t, t) = 0$, $H(t, s) > 0$ for $t > s \geq t_0$.
- (2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable, where $D_0 = \{(t, s): t > s \geq t_0\}$.
- (3) Let $h: D_0 \rightarrow \mathbb{R}^+$ be a continuous function and satisfy

$$H_s(t, s) = -h(t, s)\sqrt{H(t, s)}. \quad (4)$$

2. Main Results

First, we present the definition and properties of conformable fractional derivatives and integral.

(D₁): the left conformable fractional derivative starting from t_0 of a function $f: [t_0, \infty) \rightarrow \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by

$$(T_{t_0}^\alpha f)(t) = f^{(\alpha)}(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon(t - t_0)^{1-\alpha}) - f(t)}{\epsilon}, \quad (5)$$

where $t_0 = 0$, we write T_α . If $(T_{t_0}^\alpha f)(t)$ exists on (t_0, t_1) , then

$$(T_{t_0}^\alpha f)(t_0) = \lim_{t \rightarrow t_0^+} f^{(\alpha)}(t). \quad (6)$$

If f is differentiable, then

$$(T_{t_0}^\alpha f)(t) = f^{(\alpha)}(t) = (t - t_0)^{1-\alpha} f'(t). \quad (7)$$

(D₂): if functions $f, g \in C[(0, \infty), \mathbb{R}]$ are α -differential for $t_0 > 0$, $\alpha \in (0, 1]$ and continuous at t_0 .

(1) For all real constants a and b , we can get

$$T_{t_0}^\alpha (af + bg) = aT_{t_0}^\alpha (f) + bT_{t_0}^\alpha (g)$$

(2) $T_{t_0}^\alpha (fg) = fT_{t_0}^\alpha (g) + gT_{t_0}^\alpha (f)$

(3) For all p , $T_{t_0}^\alpha (t^p) = pt^{p-\alpha}$

(4) $T_{t_0}^\alpha (f/g) = (gT_{t_0}^\alpha (f) - fT_{t_0}^\alpha (g))/g^2$, for $g \neq 0$

(5) When c is a constant, we have $T_{t_0}^\alpha (c) = 0$

(D₃): let $f, g: [t_0, t_1] \rightarrow \mathbb{R}$ be two functions, and fg is differentiable. Then,

$$\int_{t_0}^{t_1} f(s)g^{(\alpha)}(s)d_{t_0}^\alpha s = f(s)g(s)|_{t_0}^{t_1} - \int_{t_0}^{t_1} g(s)f^{(\alpha)}(s)d_{t_0}^\alpha s. \quad (8)$$

Theorem 1. Suppose that (C₁) and (C₂) hold, and there exists a function $r(t) \in C([t_0, \infty), \mathbb{R}^+)$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0+1}^t H(t, s) \left[r(s)q(s) - \frac{1}{64k} \frac{z^2(s)}{H(t, s)} \right] d_{t_0}^\alpha s = \infty, \quad (9)$$

then equation (3) is oscillatory, where $M_+(s) = \max\{0, r^{(\alpha)}(s)p(s) - r(s)\}$,

$$Z(s) = (r(s)p(s))^{-(3/2)} \frac{M_+(s)H(t, s)}{(s - t_0)^{1-\alpha}h(t, s)}. \quad (10)$$

Proof. On the contrary, we suppose that equation (3) has a nonoscillation solution $y(t) \neq 0$, $t \geq t_0$. Defining $V(t)$ as a generalized Riccati substitution,

$$V(t) = r(t) \frac{p(t)y^{(\alpha+1)}(t)}{y(t)}. \quad (11)$$

From (C₁), (3), and (D₂), we obtain

$$\begin{aligned} V^{(\alpha)}(t) &= r^{(\alpha)}(t) \frac{p(t)y^{(\alpha+1)}(t)}{y(t)} + r(t) \left[\frac{p(t)y^{(\alpha+1)}(t)}{y(t)} \right]^{(\alpha)} \\ &= \frac{r^{(\alpha)}(t)}{r(t)} V(t) + r(t) \frac{[p(t)y^{(\alpha+1)}(t)]^{(\alpha)} y(t) - y^{(\alpha)}(t)p(t)y^{(\alpha+1)}(t)}{y^2(t)} \\ &= \frac{r^{(\alpha)}(t)}{r(t)} V(t) - r(t)q(t) - r(t) \frac{y^{(\alpha+1)}(t)}{y(t)} - r(t) \frac{p(t)y^{(\alpha)}(t)y^{(\alpha+1)}(t)}{y^2(t)} \\ &= \frac{r^{(\alpha)}(t)}{r(t)} V(t) - r(t)q(t) - \frac{V(t)}{p(t)} - \frac{V^2(t)}{r(t)p(t)} \cdot \frac{y^{(\alpha)}(t)}{y^{(\alpha+1)}(t)} \\ &\leq \frac{r^{(\alpha)}(t)p(t) - r(t)}{r(t)p(t)} V(t) - \frac{k}{r(t)p(t)} V^2(t) - r(t)q(t) \\ &\leq \frac{M_+(t)}{r(t)p(t)} V(t) - \frac{k}{r(t)p(t)} V^2(t) - r(t)q(t), \end{aligned} \quad (12)$$

that is,

$$V^{(\alpha)}(t) \leq \frac{M_+(t)}{r(t)p(t)} V(t) - \frac{k}{r(t)p(t)} V^2(t) - r(t)q(t). \quad (13)$$

Multiply both sides of (13) by $H(t, s)$, and integrate from T to t_0 ($T \geq t_0 \geq t_0$). In accordance with (D₁), (C₂), (13), and the integration by parts, we get

$$\begin{aligned} \int_T^t H(t, s)r(s)q(s)d_{t_0}^\alpha s &\leq - \int_T^t H(t, s)V^{(\alpha)}(s)d_{t_0}^\alpha s + \int_T^t H(t, s)\frac{M_+(s)}{r(s)p(s)}V(s)d_{t_0}^\alpha s \\ &\quad - \int_T^t H(t, s)\frac{k}{r(s)p(s)}V^2(s)d_{t_0}^\alpha s = H(t, s)v(T) - \int_T^t v(s)H^{(\alpha)}(t, s)d_{t_0}^\alpha s \\ &\quad + \int_T^t H(t, s)\frac{M_+(s)}{r(s)p(s)}V(s)d_{t_0}^\alpha s - \int_T^t H(t, s)\frac{k}{r(s)p(s)}V^2(s)d_{t_0}^\alpha s \\ &= H(t, s)V(T) - \int_T^t (s-t_0)^{1-\alpha}h(t, s)\sqrt{H(t, s)}V(s)d_{t_0}^\alpha s + \int_T^t H(t, s)\frac{M_+(s)}{r(s)p(s)}V(s)d_{t_0}^\alpha s \\ &\quad - \int_T^t H(t, s)\frac{k}{r(s)p(s)}V^2(s)d_{t_0}^\alpha s \\ &= H(t, T)V(T) \\ &\quad - \int_T^t \left\{ H(t, s)\frac{k}{r(s)p(s)}V^2(s) + \left[(s-t_0)^{1-\alpha}h(t, s)\sqrt{H(t, s)} - H(t, s)\frac{M_+(s)}{r(s)p(s)} \right] V(s) \right\} d_{t_0}^\alpha s. \end{aligned} \quad (14)$$

Let $\lambda = 2$, $A = \sqrt{(s-t_0)^{1-\alpha}h(t, s)} \cdot [H(t, s)]^{1/4}$ and $B = (1/2)(M_+(s)/r(s)p(s)) \cdot [H(t, s)]^{3/4}/\sqrt{(s-t_0)^{1-\alpha}h(t, s)}$.
From

$$A^\lambda - \lambda AB^{\lambda-1} \geq (\lambda-1)B^\lambda, \quad \lambda > 1, \quad (15)$$

we can have

$$\begin{aligned} \int_T^t H(t, s)r(s)q(s)d_{t_0}^\alpha s &\leq H(t, T)V(T) \\ &\quad - \int_T^t \left\{ H(t, s)\frac{k}{r(s)p(s)}V^2(s) + \frac{1}{4} \frac{M_+^2(s)}{[r(s)p(s)]^2} H(t, s) \cdot \frac{\sqrt{H(t, s)}}{(s-t_0)^{1-\alpha}h(t, s)} V(s) \right\} d_{t_0}^\alpha s \\ &= H(t, T)V(T) \\ &\quad - \int_T^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s + \int_T^t \frac{1}{16k} Z^2(s) d_{t_0}^\alpha s \\ &\leq H(t, T)V(T) + \int_T^t \frac{1}{64k} Z^2(s) d_{t_0}^\alpha s, \end{aligned} \quad (16)$$

where $Z(s) = (r(s)p(s))^{-(3/2)} (M_+(s)H(t, s)/(s-t_0)^{1-\alpha}h(t, s))$.

For $t \geq T \geq t_0$, we have

$$\begin{aligned} \int_T^t \left[H(t, s)r(s)q(s) - \frac{1}{64k} Z^2(s) \right] d_{t_0}^\alpha s &\leq H(t, T)V(T) \\ &\leq H(t, T)|V(T)| \leq H(t, t_0)|V(T)|. \end{aligned} \quad (17)$$

Letting $T = t_0 + 1$, we get

$$\int_{t_0+1}^t \left[H(t, s)r(s)q(s) - \frac{1}{64k} Z^2(s) \right] d_{t_0}^\alpha s \leq H(t, t_0)|V(t_0+1)|. \quad (18)$$

Dividing both sides of the above inequality by $H(t, t_0)$ and taking the upper limit in both sides as $t \rightarrow \infty$, we have the contradict with (9). The proof is complete.

Letting m be an integer with $m > 1$, then $H(t, t) = 0$, $H(t, s) > 0$. We have $h(t, s) = m(t - s)^{(m-2)/2}$, so equation (9) is transformed to the following. $\square$

Corollary 1. *If there exists a function $r(t) \in C([t_0, \infty), R^+)$ and*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0+1}^t (t-s)^m \left[r(s)q(s) - \frac{1}{64k} \frac{J^2(s)}{H(t, s)} \right] d_{t_0}^\alpha s = \infty, \quad (19)$$

then equation (3) is oscillatory, where $J(s) = (r(s)p(s))^{-(3/2)}$ ($M_+(s)(t-s)^{(m+2)/2}/(s-t_0)^{1-\alpha}m$).

Before we introduce Theorem 2, we present the properties of the conformable fractional integral.

(D₄): the left conformable fractional integral of order $0 < \alpha \leq 1$ starting at t_0 is defined by

$$(I_{t_0}^\alpha f)(t) = \int_{t_0}^t (s-t_0)^{\alpha-1} f(s) ds = \int_{t_0}^t f(s) d_{t_0}^\alpha s. \quad (20)$$

We call the conformable fractional integral of a given function f is α -integrable if it exists.

(D₅): assume that $f: [\alpha, \infty) \rightarrow R$ is continuous and $0 < \alpha \leq 1$. For all $t > t_0$, we can get

$$\begin{aligned} I_{t_0}^\alpha T_{t_0}^\alpha(f)(t) &= \int_{t_0}^t (s-t_0)^{\alpha-1} (s-t_0)^{1-\alpha} f(s) ds \\ &= f(t) - f(t_0), \\ T_{t_0}^\alpha I_{t_0}^\alpha(f)(t) &= f(t). \end{aligned} \quad (21)$$

(D₆) (Cauchy-Schwarz inequality with conformable fractional derivative). Let $f, g: [t_0, \infty) \rightarrow R$ be two functions such that f^2 and g^2 are α -integrable. Then,

$$\left| \int_{t_0}^t f(s)g^{(\alpha)}(s) d_{t_0}^\alpha s \right|^2 \leq \int_{t_0}^t |f(s)|^2 d_{t_0}^\alpha s \cdot \int_{t_0}^t |g(s)|^2 d_{t_0}^\alpha s. \quad (22)$$

Theorem 2. *Let $H(t, s)$ be defined as in (C₂) and (C₁) also hold. Suppose that*

$$0 < \inf_{s \geq t_0} \left[\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] \leq \infty, \quad (23)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{Z^2(s)}{16k} d_{t_0}^\alpha s < \infty, \quad (24)$$

hold. For every $T \geq t_0$, if there exists functions $S \in C([t_0, \infty), R)$, $r(t) \in C([t_0, \infty), R^+)$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, s)} \int_T^t \left[H(t, s)r(s)q(s) - \frac{1}{64k} Z(s) \right] d_{t_0}^\alpha s \geq S(T), \quad (25)$$

then equation (3) is oscillatory provided

$$\int_{t_0+1}^\infty k \frac{S_+^2(s)}{r(s)p(s)} d_{t_0}^\alpha s = \infty, \quad (26)$$

where $r(t) \in C[t_0, \infty)$, $Z(s) = (r(s)p(s))^{-(3/2)}$ ($M_+(s)H(t, s)/(s-t_0)^{1-\alpha}h(t, s)$), and $S_+(t) = \max\{S(t), 0\}$.

Proof. On the contrary, for all $t \geq t_0$, we suppose that equation (3) has a nonoscillation solution $y(t) \neq 0$. Define $V(t)$ as a generalized Riccati substitution:

$$V(t) = r(t) \frac{p(t)y^{(\alpha+1)}(t)}{y(t)}. \quad (27)$$

Proceeding as the proof of Theorem 1, for $t \geq T \geq t_0$, from (16), we can get

$$\begin{aligned} &\int_T^t \left[H(t, s)r(s)q(s) - \frac{1}{64k} Z(s) \right] d_{t_0}^\alpha s \\ &\leq H(t, T)V(T) - \int_T^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s. \end{aligned} \quad (28)$$

Dividing both sides of the above inequality by $H(t, T)$, we obtain

$$\begin{aligned} &\frac{1}{H(t, T)} \int_T^t \left[H(t, s)r(s)q(s) - \frac{1}{64k} Z(s) \right] d_{t_0}^\alpha s \\ &\leq V(T) - \frac{1}{H(t, T)} \int_T^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s. \end{aligned} \quad (29)$$

Immediately, it shows the following:

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{1}{H(t, s)} \int_T^t \left[H(t, s)r(s)q(s) - \frac{1}{64k} Z(s) \right] d_{t_0}^\alpha s \\ &\leq V(T) - \liminf_{t \rightarrow \infty} \frac{1}{H(t, s)} \int_T^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s. \end{aligned} \quad (30)$$

So, for all $T \geq t_0$, from (25), we have

$$V(T) \geq S(T) + \liminf_{t \rightarrow \infty} \frac{1}{H(t, s)} \int_T^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s. \quad (31)$$

It means that, for every $T \geq t_0$,

$$\begin{aligned} &V(T) \geq S(T), \\ &\liminf_{t \rightarrow \infty} \frac{1}{H(t, s)} \int_T^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s < \infty. \end{aligned} \quad (32)$$

Then, for every $T_0 > t_0$, we obtain

$$\begin{aligned}
\infty &> \liminf_{t \rightarrow \infty} \frac{1}{H(t, T_0)} \int_{T_0}^t \left[\sqrt{\frac{kH(t, s)}{r(s)p(s)}} V(s) + \frac{1}{8\sqrt{k}} Z(s) \right]^2 d_{t_0}^\alpha s \\
&\geq \liminf_{t \rightarrow \infty} \left[\frac{1}{H(t, T_0)} \int_{T_0}^t \frac{kH(t, s)}{r(s)p(s)} V^2(s) d_{t_0}^\alpha s \right. \\
&\quad \left. + \frac{1}{H(t, T_0)} \int_{T_0}^t \frac{1}{4} \sqrt{\frac{H(t, s)}{r(s)p(s)}} Z(s) V(s) d_{t_0}^\alpha s \right].
\end{aligned} \tag{33}$$

For $t \geq T_0 > t_0$, we let

$$\begin{aligned}
E(t) &= \frac{1}{H(t, T_0)} \int_{T_0}^t \frac{kH(t, s)}{r(s)p(s)} V^2(s) d_{t_0}^\alpha s, \\
F(t) &= \frac{1}{H(t, T_0)} \int_{T_0}^t \frac{1}{4} \sqrt{\frac{H(t, s)}{r(s)p(s)}} Z(s) V(s) d_{t_0}^\alpha s.
\end{aligned} \tag{34}$$

From (3), we have

$$\liminf_{t \rightarrow \infty} [F(t) + E(t)] < \infty. \tag{35}$$

Next, we claim that

$$\int_{T_0}^\infty k \cdot \frac{V^2(s)}{r(s)p(s)} d_{t_0}^\alpha s < \infty. \tag{36}$$

From (23), we can find a constant β that will allow

$$\inf_{s \geq t_0} \left[\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] > \beta > 0. \tag{37}$$

If (36) is not true, then for every $\tau > 0$, we can find $T_1 > T_0$ such that

$$\int_{T_0}^t k \cdot \frac{V^2(s)}{r(s)p(s)} d_{t_0}^\alpha s \geq \frac{2\tau}{\beta}, \quad t > T_1. \tag{38}$$

From (D₃), we obtain the following:

$$\begin{aligned}
E(t) &= \frac{1}{H(t, T_0)} \int_{T_0}^t H(t, s) \left[\int_{T_0}^s k \cdot \frac{V^2(\xi)}{r(\xi)p(\xi)} d_{t_0}^\alpha \xi \right]^{(\alpha)} d_{t_0}^\alpha s \\
&= -\frac{1}{H(t, T_0)} \int_{T_0}^t \left[\int_{T_0}^s k \cdot \frac{V^2(\xi)}{r(\xi)p(\xi)} d_{t_0}^\alpha \xi \right] [H(t, s)]_s^{(\alpha)} d_{t_0}^\alpha s \\
&\geq -\frac{1}{H(t, T_0)} \int_{T_1}^t \left[\int_{T_0}^s k \cdot \frac{V^2(\xi)}{r(\xi)p(\xi)} d_{t_0}^\alpha \xi \right] [H(t, s)]_s^{(\alpha)} d_{t_0}^\alpha s \\
&\geq -\frac{2\tau}{\beta} \frac{1}{H(t, T_0)} \int_{T_1}^t [H(t, s)]_s^{(\alpha)} d_{t_0}^\alpha s \\
&\geq \frac{2\tau}{\beta} \frac{H(t, T_1)}{H(t, T_0)}, \quad t \geq T_1 > T_0.
\end{aligned} \tag{39}$$

Thus, (4) and (7) give $[H(t, s)]_s^{(\alpha)} \leq 0$, for $t \geq T_1 > T_0$, and we have $H(t, T_0) \geq H(t, T_1) > 0$.

However,

$$\liminf_{t \rightarrow \infty} \frac{H(t, T_1)}{H(t, T_0)} > \beta, \quad T_1 > T_0. \tag{40}$$

Noticing that $T_0 > t_0$ can be arbitrarily close to t_0 and H is continuous, we choose $T_2 \geq T_1$ such that, for $t \geq T_2$,

$$\frac{H(t, T_1)}{H(t, T_0)} \geq \frac{\beta}{2}. \tag{41}$$

So, from (39) and (41), we have

$$E(t) \geq \tau, \quad t \geq T_2, \tag{42}$$

and $\tau > 0$ is arbitrary, and we obtain

$$\lim_{t \rightarrow \infty} E(t) = \infty. \tag{43}$$

In the interval $[T_0, \infty)$, we select a sequence t_i , $i = 1, 2, \dots$, which makes $\lim_{i \rightarrow \infty} t_i = \infty$, and

$$\lim_{i \rightarrow \infty} [E(t_i) + F(t_i)] = \liminf_{t \rightarrow \infty} [E(t) + F(t)]. \tag{44}$$

From (35), we can always find a constant M which lets

$$E(t_i) + F(t_i) \leq M, \quad i = 1, 2, \dots \tag{45}$$

Through (43) and (45), we gain

$$\lim_{i \rightarrow \infty} E(t_i) = \infty, \tag{46}$$

$$\lim_{i \rightarrow \infty} F(t_i) = -\infty. \tag{47}$$

The above also show that

$$\lim_{i \rightarrow \infty} \frac{F^2(t_i)}{E(t_i)} = \infty. \tag{48}$$

Using (46), divide both sides of (45) by $E(t_i)$, and because i is sufficiently large, we receive

$$1 + \frac{F(t_i)}{E(t_i)} \leq \frac{M}{E(t_i)} < \frac{1}{2}. \tag{49}$$

So, we can get

$$\frac{F(t_i)}{E(t_i)} < -\frac{1}{2}. \tag{50}$$

By (D₆), and for every positive integer i , we gain

$$\begin{aligned}
F^2(t_i) &= \left[\frac{1}{H^2(t_i, T_0)} \int_{T_0}^{t_i} \frac{1}{4} \sqrt{\frac{H(t_i, s)}{r(s)p(s)}} Z(s) V(s) d_{t_0}^\alpha s \right]^2 \\
&\leq \left[\frac{1}{H(t_i, T_0)} \int_{T_0}^{t_i} \frac{1}{16k} Z^2(s) d_{t_0}^\alpha s \right] \\
&\quad \cdot \left[\frac{1}{H(t_i, T_0)} \int_{T_0}^{t_i} \frac{kH(t_i, s)}{r(s)p(s)} V^2(s) d_{t_0}^\alpha s \right] \\
&= \left[\frac{1}{H(t_i, T_0)} \int_{T_0}^{t_i} \frac{Z^2(s)}{16k} d_{t_0}^\alpha s \right] E(t_i).
\end{aligned} \tag{51}$$

That is,

$$\frac{F^2(t_i)}{E(t_i)} \leq \frac{1}{H(t_i, T_0)} \int_{T_0}^{t_i} \frac{Z^2(s)}{16k} d_{t_0}^\alpha s. \quad (52)$$

Using (37), we obtain

$$\liminf_{t \rightarrow \infty} \frac{H(t, T_0)}{H(t, t_0)} > \beta. \quad (53)$$

Then, there exists $T'_0 \geq T_0$, for $t \geq T'_0$ such that

$$\frac{H(t, T_0)}{H(t, t_0)} \geq \beta. \quad (54)$$

So, we have for sufficiently large i ,

$$\frac{H(t_i, T_0)}{H(t_i, t_0)} \geq \beta. \quad (55)$$

For any large i , (52) is going to be equal to

$$\begin{aligned} \frac{F^2(t_i)}{E(t_i)} &\leq \frac{1}{\beta} \frac{1}{H(t_i, t_0)} \int_{T_0}^{t_i} \frac{Z^2(s)}{16k} d_{t_0}^\alpha s \\ &\leq \frac{1}{\beta} \frac{1}{H(t_i, t_0)} \int_{t_0}^{t_i} \frac{Z^2(s)}{16k} d_{t_0}^\alpha s. \end{aligned} \quad (56)$$

Since $\lim_{i \rightarrow \infty} (F^2(t_i)/E(t_i)) = \infty$, that implies

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{Z^2(s)}{16k} d_{t_0}^\alpha s = \infty, \quad (57)$$

which contradicts with (24). This gives (36), that is,

$$\int_{t_0+1}^{\infty} k \cdot \frac{V^2(s)}{r(s)p(s)} d_{t_0}^\alpha s < \infty, \quad (58)$$

and by using $V(T) \geq S(T) > 0$, we can gain

$$\int_{t_0+1}^{\infty} k \cdot \frac{S_+^2(s)}{r(s)p(s)} d_{t_0}^\alpha s \leq \int_{t_0+1}^{\infty} k \cdot \frac{V^2(s)}{r(s)p(s)} d_{t_0}^\alpha s < \infty, \lim, \quad (59)$$

which contradicts with (26). The proof is complete. $\square$

Corollary 2. If there exists a function $r(t) \in C([t_0, \infty), R^+)$ and a function $S \in C([t_0, \infty), R)$, $H(t, s) = (t-s)^m$, $m > 1$. So, (24) and (25) are replaced by

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t \frac{J^2(s)}{16k} d_{t_0}^\alpha s < \infty, \quad (60)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_T^t \left[(t-s)^m r(s) q(s) - \frac{1}{64k} J(s) \right] d_{t_0}^\alpha s \geq S(T), \quad (61)$$

where $J(s) = (r(s)p(s))^{-(3/2)} (M_+(s)(t-s)^{(m+2)/2}/(s-t_0)^{1-\alpha m})$; then, equation (3) is oscillatory.

3. Example

Consider the oscillation for the conformable fractional differential equations

$$(y^{(\alpha+1)}(t))^{(\alpha)} + y^{(\alpha+1)}(t) + y(t) = 0, t \geq t_0, \quad 0 < \alpha \leq 1. \quad (62)$$

Here, $y^{(\alpha)}(t)$ is the conformable fractional derivative. Compared with (1.1), here $p(s) = 1$, and let $r(s) = p(s) = 1$, that is, $J(s) = 0$. Using Corollary 1, we can verify that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0+1}^t (t-s)^m \left[r(s)q(s) - \frac{1}{64k} \frac{J^2(s)}{H(t, s)} \right] d_{t_0}^\alpha s \\ = \frac{1}{t^m} \int_{t_0+1}^t (t-s)^m d_{t_0}^\alpha s \\ = \frac{1}{t^m} \int_{t_0+1}^t (t-s)^m (s-t_0)^{\alpha-1} ds \\ \geq \frac{1}{t^m} \int_{t_0+1}^t (t-s)^m (s-T_1)^{\alpha-1} ds. \end{aligned} \quad (63)$$

Letting $s - T_1 = \xi$, we have

$$\begin{aligned} \int_{t_0+1}^t (t-s)^m (s-T_1)^{\alpha-1} ds \\ = \int_0^{t-T_1} \xi^{\alpha-1} [t - (\xi + T_1)]^m d\xi \\ = \int_0^{t-T_1} \xi^{\alpha-1} [(t-T_1) - \xi]^m d\xi \\ = \int_0^{t-T_1} \xi^{\alpha-1} \left(1 - \frac{\xi}{t-T_1} \right)^m (t-T_1)^m d\xi \\ = (t-T_1)^{m+1} (t-T_1)^{\alpha-1} \\ \cdot \int_0^{t-T_1} \left(\frac{\xi}{t-T_1} \right)^{\alpha-1} \left(1 - \frac{\xi}{t-T_1} \right)^m d\left(\frac{\xi}{t-T_1} \right). \end{aligned} \quad (64)$$

Next, letting $v = (\xi/t - T_1)$, then the above one

$$\begin{aligned} = (t-T_1)^{m+1} (t-T_1)^{\alpha-1} \\ \cdot \int_0^{t-T_1} \left(\frac{\xi}{t-T_1} \right)^{\alpha-1} \left(1 - \frac{\xi}{t-T_1} \right)^m d\left(\frac{\xi}{t-T_1} \right) \\ = (t-T_1)^{m+\alpha} \int_0^1 v^{\alpha-1} (1-v)^m dv. \end{aligned} \quad (65)$$

That is,

$$\begin{aligned} \int_{t_0+1}^t (t-s)^m (s-T_1)^{\alpha-1} ds \\ = (t-T_1)^{m+\alpha} B(m+1, \alpha), \end{aligned} \quad (66)$$

where $B(\cdot, \cdot)$ is the classical beta function defined by

$$B(m, n) = \int_0^1 (1-s)^{m-1} s^{n-1} ds. \quad (67)$$

So, we have $(1/t^m) \int_{t_0+1}^t (t-s)^m d_{t_0}^\alpha s = (1/t^m)(t-T_1)^{m+\alpha} B(m+1, \alpha) = \infty$, which shows (19) holds. From Corollary 1, we have equation (3) is oscillatory.

4. Conclusion

In the paper, we concerned with the oscillation of the Kamenev type for linear conformable fractional differential equations by using generalized Riccati transformation, Cauchy-Schwarz inequality for the conformable fractional derivative, and average inequality technique. Because conformable derivative combines the best qualities of fractional derivatives, it is now well developed. We will continue to refine this part of the theory and continue to study the oscillation of fractional differential equations.

Data Availability

The authors declare that the data and material in the paper can be used publicly.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

HL carried out the main results and completed the corresponding proof. RX participated in the proof and helped to complete Section 3-Examples. All authors read and approved the final manuscript.

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