Research Article

The Iterative Positive Solution for a System of Fractional $q$-Difference Equations with Four-Point Boundary Conditions

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In this work, we investigate the following system of fractional $q$-difference equations with four-point boundary problems:

\[
\begin{align*}
D^\alpha_q u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, 1 < \alpha \leq 2, \\
\eta \alpha (u(0)) &= u(1) = 0.
\end{align*}
\]  
(1)

In [8], Ferreira studied the existence of positive solutions to the nonlinear fractional $q$-difference equation:

\[
\begin{align*}
D^\alpha_q u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, 2 < \alpha \leq 3, \\
\eta \alpha (u(0)) &= u(1) = 0, \quad D^\beta_q u(1) = \beta \geq 0.
\end{align*}
\]  
(2)

1. Introduction

In [1, 2], Jackson studied the $q$-difference calculus firstly; since then, many authors have investigated this subject due to applications of the $q$-difference calculus in quantum mechanics, particle physics, hypergeometric series, and complex analysis [3, 4]. The extension of $q$-difference calculus is the fractional $q$-difference calculus, which was originally investigated by Al-Salam [5] and Agarwal [6]. In the past decade, in many works concerning nonlinear fractional $q$-difference boundary value problem, the results of the existence and the uniqueness of solutions have been given. In [7], Ferreira considered the existence of positive solutions to the nonlinear fractional $q$-difference equation:

By using a fixed-point theorem in partially ordered sets, Garzi and Agheli [9] studied the existence and uniqueness of a positive and nondecreasing solution to the fractional $q$-difference equation:

\[
\begin{align*}
D^\alpha_q u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, 3 < \alpha \leq 4, \\
u(0) = D^\alpha_q u(0) = D^\beta_q u(0) = 0, \quad D^\beta_q u(1) = \beta D^\alpha_q u(\eta),
\end{align*}
\]  
(3)

where $0 < \eta < 1$ and $1 - \beta \eta^{\alpha - 3} > 0$.

In [10], Guo and Kang obtained the existence and uniqueness of a positive solution for the fractional $q$-difference equation of the form

\[
\begin{align*}
D^\alpha_q u(t) + f(t, u(t), u(t)) + g(t, u(t)) &= 0, \quad 0 < t < 1, 1 < \alpha \leq 2, \\
u(0) = 0, \quad u(1) = \beta u(\eta),
\end{align*}
\]  
(4)

by virtue of fixed-point theorems for the mixed monotone operator. Here, $1 < \alpha \leq 2$ and $0 < \beta \eta^{\alpha - 1} < 1$.

Recently, by using the monotone iterative approach, in [11], Wang investigated the iterative positive solutions of the following fractional $q$-difference equations with three-point boundary conditions:
\[
\begin{align*}
D_q^\alpha u(x) + \lambda h(x) f(u(x)) &= 0, \quad 0 < x < 1, 2 < \alpha \leq 3, \\
u(0) &= D_q u(0) = D^\alpha_q u(1) = 0.
\end{align*}
\]

(5)

It should be noted that the existence of positive solutions of problem (5) had been studied by Li et al. [12] by means of a fixed-point theorem in cones. The novel idea of [11] is to find the positive solution.

Motivated by the above mentioned works, in this paper, we consider the following system of fractional q-difference equations with four-point boundary conditions:

\[
\begin{align*}
D_q^\alpha u(t) + f(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
D_q^\beta v(t) + g(t, u(t)) &= 0, \quad 0 < t < 1, \\
u(0) &= 0, \\
u(1) &= \gamma_1 u(\eta_1), \\
v(0) &= 0, \\
v(1) &= \gamma_2 v(\eta_2),
\end{align*}
\]

(6)

where \(D^\alpha_q\) and \(D^\beta_q\) are the fractional Riemann–Liouville q-derivative of order \(\alpha\) and \(\beta\), respectively, \(0 < q < 1\), \(1 < \beta \leq \alpha \leq 2\), \(0 < \eta_1, \eta_2 < 1\), \(0 < \gamma_1 \eta_1^{\alpha-1} < 1\), and \(0 < \gamma_2 \eta_2^{\beta-1} < 1\).

By using the monotone iterative approach, in this paper, we will construct two convergent monotone iterative schemes for seeking one coupled positive solution and obtain the coupled positive solution of problem (6). To the best of our knowledge, there is no paper to study the iterative coupled positive solutions for the coupled system of fractional q-difference boundary value problems. It is noted that we may investigate the approximate solutions of problem (6) by numerical approximation algorithms, which will be presented as another paper. For the latest development of numerical approximation algorithms of some boundary value problems, see [13–17] and the references therein.

2. Preliminaries

Let \(q \in (0, 1)\), the q-derivative of a function \(f\) is defined by

\[
(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x},
\]

\[
(D_q^\alpha f)(0) = \lim_{x \to 0} (D_q f)(x),
\]

and q-derivatives of higher order by

\[
(D_q^\alpha f)(x) = f(x), \\
(D_q^\alpha f)(x) = D_q^\alpha (D_q^{\alpha-1} f)(x), \quad n \in N.
\]

The q-integral of a function \(f\) defined in the interval \([0, b]\) is given by

\[
(I_q f)(x) = \int_0^x f(s) qds = x(1-q) \sum_{k=0}^{\infty} f(xq^k)q^k, \quad x \in [0, b].
\]

(9)

Similar to the derivatives, the operator \(I_q^\alpha f\) is given by

\[
(I_q^\alpha f)(x) = f(x), \\
(I_q^\alpha f)(x) = I_q^\alpha (I_q^{\alpha-1} f)(x), \quad n \in N.
\]

(10)

Define

\[
[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.
\]

(11)

The q-analogue of the power function \((a - b)^n\) with \(n \in \mathbb{N}_0\) is

\[
(a - b)^n = \sum_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}_0, a, b \in \mathbb{R}.
\]

(12)

Moreover, if \(a \in \mathbb{R}\), then

\[
(a - b)^{(a)} = \sum_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{an}}.
\]

(13)

Remark 1. If \(b = 0\), then \(a^{(a)} = a^a\). If \(a > 0\) and \(a - b \leq t\), then \((t - a)^{(a)} \geq (t - b)^{(a)}\).

The q-gamma function [18] is defined by

\[
\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}[0, -1, -2, \ldots].
\]

(14)

Definition 1. We say \((u_*, v_*)\) is a solution of system (6), if \((u_*, v_*)\) satisfies the first and second equations of (6) and boundary conditions of (6).

Definition 2 (see [19]). Let \(a > 0\) and \(f\) be a function defined on \([0, 1]\). The fractional q-integral of the Riemann–Liouville type is

\[
(I_q^{a} f)(x) = \frac{1}{\Gamma_q(a)} \int_0^x (x - qt)^{(a-1)} f(t) dt, \quad x \in [0, 1].
\]

(15)

Definition 3 (see [19]). The fractional q-derivative of the Riemann–Liouville type is defined by

\[
(D_q^{a} f)(x) = (D_q^{a} q^{n-a} f)(x), \quad a > 0,
\]

(16)

where \(n\) is the smallest integer greater than or equal to \(a\).

Lemma 1 (see [19]). Let \(a, \beta \geq 0\) and \(f\) be a function defined on \([0, 1]\). Then, the following formulas hold:

1. \((I_q^\alpha I_q^\beta f)(x) = (I_q^{\alpha+\beta} f)(x),\)
2. \((D_q^\alpha f)(x) = f(x).\)

Lemma 2 (see [13]). Let \(a > 0\) and \(n\) be a positive integer. Then, the following equality holds:

\[
(I_q^{a} D_q^{a} f)(x) = (D_q^{a} I_q^{a} f)(x) = \sum_{n=0}^{\infty} \frac{x^{a-nk}}{\Gamma_q(a + k - n + 1)} (D_q^{a} f)(0).
\]

(17)
By Lemmas 1 and 2, Guo and Kang in [10] obtained the following lemma.

**Lemma 3.** For any $g \in C[0, 1]$, the boundary value problem

\[
\begin{cases}
    D_q^a u(t) + g(t) = 0, & 0 < t < 1, \\
    u(0) = 0, & u(1) = \gamma_1 u(\eta_1),
\end{cases}
\]

has a unique solution:

\[
u(t) = \int_0^1 G_1(t, \eta) g(s) d\eta,
\]

where

\[
G_1(t, \eta) = \begin{cases}
    t^{a-1}(1 - \eta) - t^{a-1} \eta \left( \gamma_1 - \eta \right) - (t - q\eta)\eta^{(a-1)}(1 - \gamma_1 \eta^{a-1}), & 0 \leq \eta \leq t \leq 1, \eta \leq \eta_1, \\
    \frac{t^{a-1}(1 - \eta) - t^{a-1} \eta \left( \gamma_1 - \eta \right)}{\Gamma_q(a)(1 - \gamma_1 \eta^{a-1})}, & 0 \leq \eta_1 \leq q \eta \leq 1, \\
    \frac{t^{a-1}(1 - \eta) - t^{a-1} \eta \left( \gamma_1 - \eta \right) - (t - q\eta)\eta^{(a-1)}(1 - \gamma_1 \eta^{a-1})}{\Gamma_q(a)(1 - \gamma_1 \eta^{a-1})}, & 0 \leq t \leq \eta \leq 1, \\
    \frac{t^{a-1}(1 - q\eta)\eta^{(a-1)}}{\Gamma_q(a)(1 - \gamma_1 \eta^{a-1})}, & 0 \leq t \leq \eta \leq 1, \eta_1 \leq \eta,
\end{cases}
\]

is the Green function of BVP (18).

Similarly, we have the following.

**Lemma 4.** For any $h \in C[0, 1]$, the boundary value problem

\[
\begin{cases}
    D_q^b v(t) + h(t) = 0, & 0 < t < 1, \\
    v(0) = 0, & v(1) = \gamma_2 v(\eta_2),
\end{cases}
\]

has a unique solution:

\[
u(t) = \int_0^1 G_2(t, \eta) h(s) d\eta,
\]

where

\[
G_2(t, \eta) = \begin{cases}
    t^{b-1}(1 - \eta) - t^{b-1} \eta \left( \gamma_2 - \eta \right) - (t - q\eta)\eta^{(b-1)}(1 - \gamma_2 \eta^{b-1}), & 0 \leq \eta \leq t \leq 1, \eta \leq \eta_2, \\
    \frac{t^{b-1}(1 - \eta) - t^{b-1} \eta \left( \gamma_2 - \eta \right)}{\Gamma_q(\beta)(1 - \gamma_2 \eta^{b-1})}, & 0 \leq \eta_2 \leq q \eta \leq 1, \\
    \frac{t^{b-1}(1 - \eta) - t^{b-1} \eta \left( \gamma_2 - \eta \right) - (t - q\eta)\eta^{(b-1)}(1 - \gamma_2 \eta^{b-1})}{\Gamma_q(\beta)(1 - \gamma_2 \eta^{b-1})}, & 0 \leq t \leq \eta \leq 1, \\
    \frac{t^{b-1}(1 - \eta) - t^{b-1} \eta \left( \gamma_2 - \eta \right) - (t - q\eta)\eta^{(b-1)}(1 - \gamma_2 \eta^{b-1})}{\Gamma_q(\beta)(1 - \gamma_2 \eta^{b-1})}, & 0 \leq t \leq \eta \leq 1, \eta_2 \leq \eta,
\end{cases}
\]

is the Green function of BVP (21).

**Lemma 5.** (see [10]). For $G_1(t, \eta)$ and $G_2(t, \eta)$ defined as in Lemmas 3 and 4, respectively, we have

(i) $G_1(t, \eta)$ and $G_2(t, \eta)$ are two continuous functions

(ii) $(M_1 q \eta^{(a-1)} / \Gamma_q(a)(1 - \gamma_1 \eta^{a-1})) t^{a-1} \leq G_1(t, \eta) \leq (1 - q \eta^{(a-1)} / \Gamma_q(a)(1 - \gamma_1 \eta^{a-1})) t^{a-1}$, \(\forall 0 \leq t, \eta \leq 1\), where $0 < M_1 = \max \{1 - \gamma_1 \eta^{a-1}, \gamma_1 \eta^{a-1}(1 - \eta_1), \gamma_1 \eta^{a-1} \} < 1$
(iii) \( (M_q g s (1 - q s)^{(\beta-1)/\Gamma_q(\beta)}(1 - \gamma_2\eta_2^{\beta-1}))t^{\beta-1} \leq G_2(t, q s) \leq (1 - q s)^{(\beta-1)/\Gamma_q(\beta)}(1 - \gamma_2\eta_2^{\beta-1})t^{\beta-1}, \forall 0 \leq t, s \leq 1, \) where \( 0 < M_2 = \min\{1 - \gamma_2\eta_2^{\beta-1}, \gamma_2\eta_2^{\beta-2}(1 - \eta_2), \gamma_2\eta_2^{\beta-1}\} < 1 \)

\[ \begin{align*}
P_1 & = \left\{ u \in C[0, 1] \mid \text{there exist two positive numbers } 0 < a_i < 1 < b_i, \text{ such that } a_i t^{\alpha_i-1} \leq u(t) \leq b_i t^{\alpha_i-1}, t \in [0, 1] \right\}, \\
P_2 & = \left\{ v \in C[0, 1] \mid \text{there exist two positive numbers } 0 < a_2 < 1 < b_2, \text{ such that } a_2 t^{\beta_1-1} \leq v(t) \leq b_2 t^{\beta_1-1}, t \in [0, 1] \right\}.
\end{align*} \]

3. Main Result

In this paper, we will employ the Banach space \( C[0, 1] \), equipped with norm \( \|u\| = \sup_{t \in [0, 1]} |u(t)| \) for each \( u \in C[0, 1] \). Define two cones \( P_1 \) and \( P_2 \) in \( C[0, 1] \) as follows:

\[ \begin{align*}
P_1 & = \left\{ u \in C[0, 1] \mid \text{there exist two positive numbers } 0 < a_i < 1 < b_i, \text{ such that } a_i t^{\alpha_i-1} \leq u(t) \leq b_i t^{\alpha_i-1}, t \in [0, 1] \right\}, \\
P_2 & = \left\{ v \in C[0, 1] \mid \text{there exist two positive numbers } 0 < a_2 < 1 < b_2, \text{ such that } a_2 t^{\beta_1-1} \leq v(t) \leq b_2 t^{\beta_1-1}, t \in [0, 1] \right\}.
\end{align*} \]

Remark 2. The conditions (H1) and (H2) imply that, for all \( r > 1 \), we have \( f(t, rv) \leq r^{\alpha_i} f(t, v) \) and \( g(t, ru) \leq r^{\beta_1} g(t, u) \).

Theorem 1. Assume that conditions (H1)–(H4) hold and there exist two positive constants \( R_1 \) and \( R_2 \) such that

\[ \begin{align*}
\Gamma_q(a) & \left(1 - \gamma_1\eta_1^{\alpha_i-1}\right) \int_0^1 (1 - q s)^{(\alpha_i-1)} s, d_s \leq R_1^{1-\sigma_1}, \\
\Gamma_q(\beta) & \left(1 - \gamma_2\eta_2^{\beta_1-1}\right) \int_0^1 (1 - q s)^{(\beta_1-1)} s, d_s \leq R_2^{1-\sigma_2},
\end{align*} \]

then the fractional q-difference system (6) has one positive solution \( (u^*, v^*) \), where \( u^* \in P_1 \) and \( v^* \in P_2 \). Moreover, for each \( t \in [0, 1] \), there exist constants \( 0 < m_i < 1 < n_i (i = 1, 2) \), such that

\[ \begin{align*}
u^*(t) & \in \left[ m_1 t^{\alpha_i-1}, n_1 t^{\alpha_i-1} \right], \\
v^*(t) & \in \left[ m_2 t^{\beta_1-1}, n_2 t^{\beta_1-1} \right],
\end{align*} \]

which can be obtained by monotone iterative schemes \( \{u_n\} \) and \( \{v_n\} \) generated by

\[ \begin{align*}
u_n(t) & = \int_0^1 G_1 (t, q s) f(s, v_{n-1}(s)) d_s, \\
v_n(t) & = \int_0^1 G_2 (t, q s) g(s, u_{n-1}(s)) d_s.
\end{align*} \]

i.e., \( \|u_n - u^*\| \longrightarrow 0 \) and \( \|v_n - v^*\| \longrightarrow 0 \) as \( n \longrightarrow \infty \).

Proof. For any \( v \in P_2 \), we know that there exist two constants \( a_i \) and \( b_i \) with \( 0 < a_i < 1 < b_i \) such that

\[ a_i t^{\alpha_i-1} \leq v(t) \leq b_i t^{\alpha_i-1}, \quad t \in [0, 1]. \]

From Lemma 5 and condition (H1), we obtain
\[ T_1 v(t) = \int_0^1 G_1(t,qs)f(s,v(s))ds \]

\[ \geq \frac{qM_t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)}f(s,v(s))ds \]

\[ \geq \frac{qM_t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)}f(s,a_1s^{\beta-1})ds \]

\[ \geq \frac{qM_t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)}f(s,a_1s^{\beta-1})ds \]

\[ \geq c_1 t^{\alpha-1}, \quad (34) \]

\[ T_1 v(t) \leq \frac{b_1^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)}f(s,v(s))ds \]

\[ \leq \frac{b_1^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)}f(s,b_1s^{\beta-1})ds \]

\[ \leq \frac{b_1^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)}f(s,s^{\beta-1})ds \]

\[ \leq d_1 t^{\alpha-1}, \]

where \( d_1 \) and \( c_1 \) are two positive constants satisfying

\[ d_1 > \max \left\{ 1, \frac{b_1^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)}f(s,a_1s^{\beta-1})ds \right\}, \]

\[ 0 < c_1 < \min \left\{ 1, \frac{qM_t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)}f(s,a_1s^{\beta-1})ds \right\}, \quad (35) \]

Thus, \( T_1 \) maps \( P_2 \) into \( P_1 \). For each \( u \in P_1 \), there exist two constants \( a_2 \) and \( b_2 \) with \( 0 < a_2 < 1 < b_2 \) such that

\[ a_2 t^{\alpha-1} \leq u(t) \leq b_2 t^{\alpha-1}, \quad t \in [0,1]. \quad (36) \]

Similarly, by Lemma 5 and condition (H2), we can get that

\[ d_2 > \max \left\{ 1, \frac{b_2^{\beta-1}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})} \int_0^1 (1-qs)^{(\beta-1)}g(s,s^{\alpha-1})ds \right\}, \]

\[ 0 < c_2 < \min \left\{ 1, \frac{qM_t^{\beta-1}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})} \int_0^1 s(1-qs)^{(\beta-1)}g(s,s^{\alpha-1})ds \right\}, \quad (38) \]

\[ c_2^{\beta-1} \leq T_2 u(t) \leq d_2 t^{\beta-1}, \]

where \( d_2 \) and \( c_2 \) are two positive constants satisfying
which implies that $T_2$ maps $P_1$ into $P_2$. On the other hand, the proof of completely continuous $T_1$ and $T_2$ are as the same as in [12], and we omit it here.

Let $P_i(R) = \{u \mid u \in P_i, \|u\| \leq R\}$ ($i = 1, 2$). In the following, we will prove $T_1(P_i(R_1)) \subset P_i(R_1)$ and $T_2(P_i(R_2)) \subset P_i(R_2)$. In fact, for any $v \in P_i(R_1)$ and $u \in P_i(R_2)$, by conditions (29) and (30), we obtain

\[
T_1v(t) \leq \frac{e^{\alpha - 1}}{\Gamma_q(\alpha)(1 - \gamma_1\eta^{-1}_1)} \int_0^1 (1 - qs)^{(\alpha - 1)} f(s, v(s))d_qs
\]

\[
\leq \frac{e^{\alpha - 1}}{\Gamma_q(\alpha)(1 - \gamma_1\eta^{-1}_1)} \int_0^1 (1 - qs)^{(\alpha - 1)} f(s, s) d_qs \leq R_1,
\]

\[
T_2u(t) \leq \frac{e^{\beta - 1}}{\Gamma_q(\beta)(1 - \gamma_2\eta^{-1}_2)} \int_0^1 (1 - qs)^{(\beta - 1)} g(s, u(s))d_qs
\]

\[
\leq \frac{e^{\beta - 1}}{\Gamma_q(\beta)(1 - \gamma_2\eta^{-1}_2)} \int_0^1 (1 - qs)^{(\beta - 1)} g(s, s) d_qs \leq R_2,
\]

which implies that $\|T_1v\| \leq R_1$ and $\|T_2u\| \leq R_2$. So, $T_1(P_2(R_1)) \subset P_1(R_1)$ and $T_2(P_1(R_2)) \subset P_2(R_2)$.

Taking $e_1(t) = t^{\alpha - 1}$ and $e_2(t) = t^{\beta - 1}$, then $e_1 \in P_1$, $e_2 \in P_2$, $T_1(e_1) \in P_1$, and $T_2(e_2) \in P_2$. Thus, there exist constants $0 < m_1 < 1 < n_1$ ($i = 1, 2$) such that

\[
m_1t^{\alpha - 1} \leq T_1e_2(t) \leq n_1t^{\alpha - 1},
\]

\[
m_2t^{\beta - 1} \leq T_2e_1(t) \leq n_2t^{\beta - 1}.
\]

Let $l_1$ and $l_2$ be two positive numbers satisfying $0 < l_1 < l_2 < 1$, and

\[
l_1^{\alpha - 1} \leq m_1,
\]

\[
l_2^{\beta - 1} \leq m_2.
\]

Set

\[
u_0(t) = l_1e_1(t),
\]

\[
u_0(t) = l_2e_2(t),
\]

\[
u_0 = T_1\nu_{n-1},
\]

\[
u_0 = T_2\nu_{n-1},
\]

\[
n = 1, 2, \ldots
\]

Obviously, $u_0(t) \leq v_0(t)$ by $\beta \leq \alpha$ and $0 < l_1 < l_2 < 1$, $u_0 \in P_1(R_1)$ and $v_0 \in P_2(R_2)$. By (H1) and (H2), we have

\[
u_1(t) = T_2u_0(t) = \int_0^1 G_2(t, qs)g(s, u_0(s))d_qs
\]

\[
u_1(t) = T_2u_0(t) = \int_0^1 G_2(t, qs)g(s, u_0(s))d_qs
\]

\[
\geq l_2^{\beta - 1} \int_0^1 G_2(t, qs)g(s, e_2(s))d_qs
\]

\[
= l_2^{\beta - 1} l_1e_1(t) \geq l_1 e_1(t) = u_0(t),
\]

\[
\geq l_2^{\beta - 1} \int_0^1 G_2(t, qs)g(s, e_2(s))d_qs
\]

\[
= l_2^{\beta - 1} l_2e_2(t) \geq l_2 e_2(t) = v_0(t).
\]

From conditions (H1) and (H2), we know that $T_1$ and $T_2$ are two nondecreasing operators. Thus, by induction, we can obtain

\[
u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots,
\]

\[
u_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots,
\]

\[
u_n \in P_1(R_1),
\]

\[
u_n \in P_2(R_2),
\]

\[
n = 1, 2, \ldots
\]

By the compactness of the operators $T_1$ and $T_2$, we have that $\{u_n\}$ and $\{v_n\}$ are two sequentially compact sets. Therefore, there exist $u_s \in P_1(R_1)$ and $v_s \in P_2(R_2)$, such that $u_n$ converges to $u_s$ and $v_n$ converges to $v_s$ as $n \to \infty$, respectively. Since the operators $T_1$ and $T_2$ are continuous, $u_n = T_1v_{n-1}$ and $v_n = T_2u_{n-1}$, and we obtain $u^* = T_1v^*$ and $v^* = T_2u^*$ as $n \to \infty$, which implies that system (6) has a positive solution $(u^*, v^*)$, and $u^* = [m_1t^{\alpha - 1}, n_1t^{\alpha - 1}]$, $v^* = [m_2t^{\beta - 1}, n_2t^{\beta - 1}]$, and $\forall t \in [0, 1]$, where $m_1$ and $n_1$ are constants and $0 < m_1 < 1 < l_1$ ($i = 1, 2$), which can be achieved by the monotone scheme:

\[
u_n(t) = \int_0^1 G_2(t, qs)g(s, u_{n-1}(s))d_qs,
\]

\[
u_n(t) = \int_0^1 G_2(t, qs)g(s, u_{n-1}(s))d_qs,
\]

with initial values $u_0(t)$ and $v_0(t)$ defined as in (42).

In the following, we give an example to illustrate the existence of positive solutions of BVP (6).

**Example 1.** Consider the following system of fractional $q$-difference with boundary conditions:

...
where $q = (1/3)$, $a = (5/3)$, $\beta = (3/2)$, $\eta_1 = (3/4)$, $\eta_2 = (1/2)$, $\gamma_1 = 1$, $\gamma_2 = (5/4)$, and

$$f(t, v) = \frac{1}{8} tv^{(3/2)},$$

$$g(t, u) = \sqrt{t} \left( u^{(1/4)} + \frac{u^{(1/3)}}{1 + u^{(1/4)}} \right),$$

(48)

Obviously, $f(t, v)$ and $g(t, u)$ are nondecreasing with respect to $v$ and $u$, respectively, and

$$0 < \gamma_1 \eta_1^{-\alpha - 1} = \left( \frac{3}{4} \right)^{2/3} < 1,$$

$$0 < \gamma_2 \eta_2^{-\beta - 1} = \frac{5}{4} \left( \frac{1}{2} \right)^{1/2} < 1.$$  

Choosing $\alpha_1 = 2 > 1$ and $\alpha_2 = (1/3) < 1$, we have

$$f(t, rv) = \frac{1}{8} tr^{(3/2)} v^{(3/2)} \geq r^2 f(t, v), \quad \forall v \in [0, +\infty), r \in (0, 1],$$

$$g(t, ru) = \sqrt{r} \left( u^{(1/4)} + \frac{r^{(1/3)} u^{(1/3)}}{1 + r^{(1/4)} u^{(1/4)}} \right),$$

$$\geq \sqrt{r} \left( u^{(1/3)} + \frac{r^{(1/3)} u^{(1/3)}}{1 + u^{(1/4)}} \right) = r^{(1/3)} g(t, u),$$

$$\forall u \in [0, +\infty), r \in (0, 1].$$

(50)

So, conditions (H1) and (H2) hold. Moreover, we can show that

$$0 < \int_0^1 \left( 1 - \frac{1}{3} s \right)^{(2/3)} f(s, 1) ds = \frac{1}{8} \int_0^1 \left( 1 - \frac{1}{3} s \right)^{(2/3)} s ds < \infty,$$

$$0 < \int_0^1 \left( 1 - \frac{1}{3} s \right)^{(4/3)} g(s, 1) ds = \frac{3}{2} \int_0^1 \left( 1 - \frac{1}{3} s \right)^{(1/2)} \sqrt{s} ds < \infty,$$

(51)

which implies that (H3) and (H4) hold. Moreover, we know that there exist two positive constants $R_1$ and $R_2$ such that (29) and (30) hold, respectively. Thus, it follows from Theorem 1 that boundary value problem of fractional $q$-difference system (47) has one iterative positive solution ($u^*, v^*$) which can be obtained with the aid of monotone iterative sequences.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the manuscript.

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**References**


