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Research Article

Pullback Attractors for a Nonautonomous Retarded Degenerate Parabolic Equation

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This paper is devoted to a nonautonomous retarded degenerate parabolic equation. We first show the existence and uniqueness of a weak solution for the equation by using the standard Galerkin method. Then we establish the existence of pullback attractors for the equation by proving the existence of compact pullback absorbing sets and the pullback asymptotic compactness.

1. Introduction

In this paper, we study the following nonautonomous retarded degenerate parabolic equation defined on an arbitrary domain (bounded or unbounded) $D_N \subset \mathbb{R}^N$, $N \geq 2$:

$$\frac{\partial u}{\partial t} + \lambda u - \operatorname{div}(\sigma(x)\nabla u) = f(u^t) + g(x,t), \quad x \in D_N, \quad t < 0, \tag{1}$$

with the initial value and boundary condition

$$u(x,t) = u^0(x,t), \quad u(x,t)\big|_{\partial D_N} = 0, \quad x \in D_N, \quad t \in [-\nu, 0], \tag{2}$$

where λ is a fixed positive constant, and we suppose that $\sigma : D_N \to \mathbb{R}$ satisfies the following assumptions:

$\mathscr{H}_\alpha$: when D_N is bounded, we suppose that $\sigma \in L^1_{loc}(D_N)$ and $\lim\inf_{x\to z}|x-z|^{-\alpha}\sigma(x) > 0$ for some $\alpha \in (0,2)$ and each $z \in \overline{D_N}$;

$\mathscr{H}_\alpha^\beta$: when D_N is unbounded, we suppose that σ satisfies condition $\mathscr{H}_\alpha$ and $\lim\inf_{|x|\to\infty}|x|^{-\beta}\sigma(x) < 0$ for some $\beta > 2$.

For $\nu > 0$, we denote by $\mathscr{F}$ the Banach space which is consisted of all continuous functions $\xi : [-\nu, 0] \to L^2(D_N)$ endowed with norm $\|\xi\|_{\mathscr{F}} = \sup_{s\in[-\nu,0]}\|\xi(s)\|$. For all real number $a \leq b$, $t \in [a,b]$, and any continuous function $u : [a-\nu, b] \to L^2(D_N)$, we denote by u^t the element of $\mathscr{F}$, and $u^t(s) = u(t+s)$ for $s \in [-\nu, 0]$.

f is a nonlinear function satisfying the following conditions:

(i) $f(0) = 0$;

(ii) there exists a positive continuous function $l_f(r)$ with

$$\lim_{r\to\infty}\frac{l_f(r)}{r^{k_0}} = 0, \tag{3}$$

for some positive integers k_0 such that if $\xi, \eta \in \mathscr{F}$, $\|\xi\| \leq r$ and $\|\eta\| \leq r$, then we have

$$\|f(\xi) - f(\eta)\| \leq l_f(r)\|\xi - \eta\|_{\mathscr{F}}; \tag{4}$$

(iii) there exists a positive constant c_f such that, for any $t > 0$, $u \in C([-\nu, t]; L^2(D_N))$, and $x \in D_N$,

$$\int_0^t |f(u^s)(x)|^2 ds \leq c_f^2 \int_{-\nu}^t |u(s)(x)|^2 ds; \tag{5}$$

(iv) $\lambda > c_f$.

g is a given function satisfying

$$g \in L^2_{loc}(\mathbb{R}; L^2(D_N)). \tag{6}$$

It is well known that problem (1) can be seen as a simple model for neutron diffusion (see [1]). For this case, u and σ represent the neutron flux and neutron diffusion, respectively.

As we know, the most interesting problem about the partial differential equations is to investigate the asymptotic behavior of the solutions when time tends to infinity. In addition, we can get the useful information about the future of the model from it. Besides, when we want to model some phenomena arising in physics, chemistry, biology and other fields, some hereditary variables such as aftereffect, time-lag, and time delay can be added in the variables. For example, the stochastic retarded reaction-diffusion equation on unbounded domains was considered in [2]. For this kind of equation with the delay term, the reader is referred to [2–4] and the references therein.

The theory of pullback attractors is a good choice to investigate the long time behavior of evolution systems. Anh and Bao [5] proved the existence of a pullback attractor for a nonautonomous semi-linear degenerate parabolic equation. Cui and Li [6] investigated the existence and upper semicontinuity of random attractors for stochastic degenerate parabolic equations with multiplicative noises. For nonautonomous stochastic PDEs, the existence of attractors and well-posedness of the problem have been studied in [7–15]. In addition, the existence of a random attractor is generally ensured by the well-posedness (i.e., existence and uniqueness) of the problem combined with a compact absorbing set. When we want to get the compactness of the absorbing sets and get the compactness of the system, Sobolev compact embedding is a good choice, see, e.g., [5, 16].

Motivated by [5] and [2], in this article, we devote to study a nonautonomous retarded degenerate parabolic equation. In fact, as we can see, the term of delay makes the phase space not reflexive and the uniform estimates of solutions will be difficult. The main result of the paper is Theorem 2.12.

The rest of this paper consists of the following contents. For convenience, in Section 2, we first introduce some preliminaries about function spaces and operators as well as dynamical systems and pullback attractors. Then, in Section 3, we get the existence and uniqueness of a weak solution for the problem (1) by using the standard Galerkin method. Section 4 is devoted to prove the existence of the pullback attractors.

2. Preliminaries

2.1. Function Spaces and Operators. We recall some basic concepts related to the function spaces and operators which we will use, the reader is referred to [5, 7, 16] for more details.

In order to study problem (1), we define the space $D_0^{1,2}(D_N,\sigma)$ as the closure of $C_0^\infty(D_N)$ with respect to the norm

$$\|u\|_{D_0^{1,2}(D_N,\sigma)} := \left(\int_{D_N}\sigma(x)|\nabla u|^2dx\right)^{1/2} < \infty. \quad (7)$$

This space is a Hilbert space with respect to the inner product

$$(u,v)_\sigma := \int_{D_N}\sigma(x)\nabla u\nabla v dx. \quad (8)$$

We denote by $\|\cdot\|$, $(\cdot,\cdot)$, $\|\cdot\|_\sigma$, $(\cdot,\cdot)_\sigma$ the norms and inner products in $L^2(D_N)$ and $D_0^{1,2}(D_N,\sigma)$, respectively, and $\|\cdot\|_{\mathscr{F}}$ the norm in $\mathscr{F}$. And also, define

$$\|\xi\|_{\sigma,\mathscr{F}} = \sup_{s\in[-\nu,0]}\|\xi(s)\|_\sigma. \quad (9)$$

Let c be a normal positive number which might vary from line to line. We have the following lemma which is a generalized version of the Poincaré inequality in [5].

Lemma 2.1. *Suppose that D_N is a bounded (unbounded) domain of $\mathbb{R}^N, N\ge 2$ and suppose that the condition $\mathscr{H}_\alpha$ ($\mathscr{H}_\alpha^\beta$) is satisfied. Then there is a positive constant c such that*

$$\int_{D_N}|u|^2dx \le c\int_{D_N}\sigma(x)|\nabla u|^2dx, \text{ for every } u\in C_0^\infty(D_N). \quad (10)$$

The following results can be found in [16].

Remark 2.2. Let $N\ge 2, \alpha\in(0,2)$, and

$$2_\alpha^* = \begin{cases} \frac{4}{\alpha}\in(2,\infty) & \text{if } N=2,\\ \frac{2N}{N-2+\alpha}\in\left(2,\frac{2N}{N-2}\right) & \text{if } N\ge 3.\end{cases} \quad (11)$$

The given exponent 2_α^* has the role of the critical exponent in the classical Sobolev embedding.

Lemma 2.3. *Suppose that D_N is a bounded domain in $\mathbb{R}^N, N\ge 2$ and σ satisfies ($\mathscr{H}_\alpha$). We get the following embedding:*

(1) *$D_0^{1,2}(D_N,\sigma)\hookrightarrow L^{2_\alpha^*}(D_N)$ continuously,*
(2) *$D_0^{1,2}(D_N,\sigma)\hookrightarrow L^p(D_N)$ compactly for every $p\in[1,2_\alpha^*)$.*

Lemma 2.4. *Suppose that D_N is an unbounded domain in $\mathbb{R}^N, N\ge 2$ and σ satisfies ($\mathscr{H}_\alpha^\beta$). We get the following embedding:*

(1) *$D_0^{1,2}(D_N,\sigma)\hookrightarrow L^p(D_N)$ continuously, if $p\in[2_\beta^*,2_\alpha^*]$,*
(2) *$D_0^{1,2}(D_N,\sigma)\hookrightarrow L^p(D_N)$ compactly, if $p\in(2_\beta^*,2_\alpha^*)$.*

We now investigate the case where D_N is a bounded domain (the unbounded case is considered similarly).

We define a linear operator determined by the leading term in (1)

$$Au = -\text{div}(\sigma(x)\nabla u). \quad (12)$$

Under the conditions $\mathscr{H}_\alpha$ or $\mathscr{H}_\alpha^\beta$, the operator A is a positive and self-adjoint linear operator with the domain of definition

$$D(A) = \left\{u\in D_0^{1,2}(D_N,\sigma) : Au\in L^2(D_N)\right\}. \quad (13)$$

The eigenvectors of the operator A construct the complete orthonormal family $\{e_j\}_{j=1}^\infty$ in $L^2(D_N)$ and the relevant spectrum is discrete and represented by $\{\lambda_j\}_{j=1}^\infty$ such that

$$(e_i,e_j)_{L^2} = \delta_{ij}, \text{ and } Ae_j = \lambda_j e_j,\ i,j=1,2,\ldots, \quad (14)$$

and

$$0 < \lambda_1 < \lambda_2 < \cdots, \ \lambda_j \to +\infty \text{ as } j \to +\infty. \quad (15)$$

Furthermore

$$\lambda_1 = \inf\left\{ \frac{\|u\|_\sigma^2}{\|u\|^2} : u \in D_0^{1,2}(D_N, \sigma), u \neq 0 \right\}. \quad (16)$$

Noting that

$$\|u\|_\sigma^2 \geq \lambda_1 \|u\|^2, \text{ for all } u \in D_0^{1,2}(D_N, \sigma). \quad (17)$$

Define $D^{-1}(D_N, \sigma)$ as the conjugate space of the space $D_0^{1,2}(D_N, \sigma)$. Since $2_\alpha^* > 2$, we get an evolution triple

$$D_0^{1,2}(D_N, \sigma) \hookrightarrow L^2(D_N) \hookrightarrow D^{-1}(D_N, \sigma). \quad (18)$$

2.2. Dynamical Systems and Pullback Attractors. Next, we will show some theories about the dynamical systems and pullback attractors, the following contents can be found in [2, 5, 7].

Let Ω be a nonempty set and X be a metric space endowed with metric d. For any $A, B \subset X$, define the Hausdorff semi-distance between A and B

$$\operatorname{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y). \quad (19)$$

Definition 2.5. A family of mapping $\{\theta_t\}_{t\in\mathbb{R}}$ from Ω to itself is said to be a family of shift operates on Ω if for $\{\theta_t\}_{t\in\mathbb{R}}$, the following two group properties are satisfied:

(i) $\theta_0\omega = \omega$, for any $\omega \in \Omega$,
(ii) $\theta_t(\theta_\tau\omega) = \theta_{t+\tau}\omega$, for any $\omega \in \Omega$ and $t, \tau \in \mathbb{R}$.

Definition 2.6. Suppose that $\{\theta_t\}_{t\in\mathbb{R}}$ is a family of shift operators on Ω. Then a mapping on X

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \to \phi(t, \omega, x), \quad (20)$$

is called a continuous θ-cocycle if for any $\omega \in \Omega$ and $t, \tau \in \mathbb{R}$, it satisfies:

(i) $\phi(0, \omega, \cdot)$ is the identity on X,
(ii) $\phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau\omega, \cdot) \circ \phi(\tau, \omega, \cdot)$,
(iii) $\phi(t, \omega, \cdot) : X \to X$ is continuous.

In what follows, we will always suppose that ϕ is a continuous θ-cocycle on X and $\mathscr{D}$ is a collection of families of subsets of X:

$$\mathscr{D} = \{D = \{D(\omega)\}_{\omega\in\Omega} : D(\omega) \subset X \text{ for every } \omega \in \Omega\}. \quad (21)$$

Definition 2.7. Suppose that $\mathscr{D}$ is a collection of families of subsets of X. Then $\mathscr{D}$ is said to be inclusion closed if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subset X : \omega \in \Omega\}$ with $\tilde{D}(\omega) \subset D(\omega)$ for any $\omega \in \Omega$ imply that $\tilde{D} \in \mathscr{D}$.

Definition 2.8. Suppose that $\mathscr{D}$ is a collection of families of subsets of X. Then $\{K(\omega)\}$ is said to be a $\mathscr{D}$-pullback absorbing set for ϕ if for any $B \in \mathscr{D}$ and $\omega \in \Omega$, there is $t(\omega, B) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \forall t \geq t(\omega, B). \quad (22)$$

Definition 2.9. Suppose that $\mathscr{D}$ is a collection of families of subsets of X. Then ϕ is called $\mathscr{D}$-pullback asymptotically compact in X if for all $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}$ possesses a convergent sub-sequence in X whenever $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$.

Definition 2.10. Suppose that $\mathscr{D}$ is a collection of families of subsets of X and $\mathscr{A}(\omega)_{\omega\in\Omega} \in \mathscr{D}$. Then $\mathscr{A}(\omega)_{\omega\in\Omega} \in \mathscr{D}$ is said to be a $\mathscr{D}$-pullback attractor for ϕ if for any $\omega \in \Omega$:

(i) $\mathscr{A}(\omega)$ is compact,
(ii) $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\phi(t, \omega, \mathscr{A}(\omega)) = \mathscr{A}(\theta_t\omega), \quad \forall t \geq 0, \quad (23)$$

(iii) $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathscr{D}$, that is, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$,

$$\lim_{t\to\infty} d(\phi(t, \theta_{-t}, B(\theta_{-t}\omega)), \mathscr{A}(\omega)) = 0. \quad (24)$$

Proposition 2.11. *Suppose that $\mathscr{D}$ is an inclusion-closed collection of families of subsets of X and ϕ is a continuous θ-cocycle in X. Assume that $\{K(\omega)\}_{\omega\Omega} \in \mathscr{D}$ is a closed absorbing set for ϕ in $\mathscr{D}$ and ϕ is $\mathscr{D}$-pullback asymptotically compact in X. Then ϕ possesses a unique $\mathscr{D}$-pullback attractor $\mathscr{A}(\omega)_{\omega\in\Omega} \in \mathscr{D}$ such that*

$$\mathscr{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \quad (25)$$

In this article, let $\mathscr{D}$ be the collection of families of subsets of $\mathscr{F}$. Then, we will give the main theorem of this article.

Theorem 2.12. *(Main Theorem) Assume that (i)–(iv) hold and (6) satisfies. Then the problem (1)–(2) has a $\mathscr{D}$-pullback attractor $\mathscr{A}(\omega)_{\omega\in\Omega}$.*

3. Existence and Uniqueness of the Solution

In this section, we will prove the existence and uniqueness of the solution for problem (1) and (2).

Theorem 3.1. *Suppose that f satisfies (i)–(iv) and $g \in L^2_{loc}(\mathbb{R}; L^2(D_N))$, then for any given $T > 0$, $u^0 \in \mathscr{F}$, the problem (1) and (2) has a unique weak solution*

$$u \in L^\infty(0, T; L^2(D_N)) \cap L^2(-\nu, T; L^2(D_N)) \cap L^2(0, T; D_0^{1,2}(D_N, \sigma)), \quad (26)$$

and the equation is satisfied in the sense of distribution.

Proof. Consider the approximating solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^{n} u_{nk}(t)e_k, \quad (27)$$

where $\{e_j\}_{j=1}^{\infty}$ are eigenvectors of the linear operator A. We have the following equations for u_n

$$\begin{aligned}\left\langle \frac{\partial u_n(t)}{\partial t}, e_k \right\rangle &+ \langle \lambda u_n(t), e_k\rangle + \langle A u_n(t), e_k\rangle \\ &= \langle f(u_n^t), e_k\rangle + \langle g(x,t), e_k\rangle,\end{aligned} \quad (28)$$

$$(u_n(t), e_k) = (u_n^0(t), e_k), \quad t \in [-\nu, 0], \quad k = 1, \ldots, n. \quad (29)$$

Hence, we get the local existence of u_n. We now give some priori estimates for u_n. We have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \lambda\|u_n\|^2 + \|u_n\|_\sigma^2 = \int_{D_N} f(u_n^t)u_n dx + \int_{D_N} g(x,t)u_n dx. \quad (30)$$

By the Young inequality, we get

$$\begin{aligned}\int_{D_N} f(u_n^t)u_n dx &\le \frac{c_f}{2}\|u_n\|^2 + \frac{1}{2c_f}\|f(u_n^t)\|^2, \\ \int_{D_N} g(x,t)u_n dx &\le \frac{1}{2\alpha}\|g(x,t)\|^2 + \frac{\alpha}{2}\|u_n^t\|^2,\end{aligned} \quad (31)$$

where c_f is given by (5), α is a positive constant. By the condition (iv) of f, we can choose α small enough such that $2\lambda > 2c_f + \alpha$, then

$$\frac{d}{dt}\|u_n\|^2 + (2\lambda - c_f - \alpha)\|u_n\|^2 + 2\|u_n\|_\sigma^2 \le \frac{1}{\alpha}\|g(x,t)\|^2 + \frac{1}{c_f}\|f(u_n^t)\|^2. \quad (32)$$

Integrating (32) on $[0,t]$, $0 \le t \le T$, then from the conditions (iii) and (iv) of f, we can get

$$\begin{aligned}&\|u_n(t)\|^2 + (2\lambda - c_f - \alpha)\int_0^t \|u_n(s)\|^2 ds + 2\int_0^t \|u_n(s)\|_\sigma^2 ds \\ &\quad \le \|u_n(0)\|^2 + \frac{1}{\alpha}\int_0^t \|g(x,s)\|^2 ds + c_f\int_{-\nu}^t \|u_n(s)\|^2 ds \\ &\quad = \|u_n(0)\|^2 + \frac{1}{\alpha}\int_0^t \|g(x,s)\|^2 ds + c_f\int_{-\nu}^0 \|u_n(s)\|^2 ds + c_f\int_0^t \|u_n(s)\|^2 ds \\ &\quad \le \|u_n(0)\|^2 + c_f\nu\|u_n^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^t \|g(x,s)\|^2 ds + c_f\int_0^t \|u_n(s)\|^2 ds.\end{aligned} \quad (33)$$

Noting that $2\lambda > 2c_f + \alpha$, we obtain

$$\begin{aligned}\|u_n(t)\|^2 + 2\int_0^t \|u_n(s)\|_\sigma^2 ds &\le \|u_n(0)\|^2 + c_f\nu\|u_n^0\|_{\mathscr{F}}^2 \\ &\quad + \frac{1}{\alpha}\int_0^t \|g(x,s)\|^2 ds.\end{aligned} \quad (34)$$

By (34), we also have

$$\|u_n(t)\|^2 \le (1 + c_f\nu)\|u_n^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^t \|g(x,s)\|^2 ds, \quad (35)$$

for fixed $\rho \in [-\nu, 0]$, we obtain that, for $t \in (-\rho, T]$,

$$\|u_n(t+\rho)\|^2 \le (1 + c_f\nu)\|u_n^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^{t+\rho} \|g(x,s)\|^2 ds, \quad (36)$$

and for $t \in [0, -\rho]$,

$$\begin{aligned}\|u_n(t+\rho)\|^2 &\le \|u_n^0\|_{\mathscr{F}}^2 \\ &\le (1 + c_f\nu)\|u_n^0\|_{\mathscr{F}}^2.\end{aligned} \quad (37)$$

Hence, from (36) and (38), we get for all $t \in [0,T]$

$$\|u_n^t\|_{\mathscr{F}}^2 \le (1 + c_f\nu)\|u_n^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^T \|g(x,s)\|^2 ds, \quad (38)$$

The (34) and (38) imply that for each $u_n^0 \in \mathscr{F}$,

$$\{u_n\} \text{ is bounded in } L^\infty(-\nu, T; L^2(D_N)), \quad (39)$$

$$\{u_n\} \text{ is bounded in } L^2(-\nu, T; L^2(D_N)), \quad (40)$$

$$\{u_n\} \text{ is bounded in } L^2(0, T; D_0^{1,2}(D_N, \sigma)). \quad (41)$$

Noting that from the condition of f, we conclude $\{f(u_n^t)\}$ is bounded in $L^2(0,T;L^2(D_N))$, then we get

$$f(u_n^t) \rightharpoonup \xi \text{ in } L^2(0,T;L^2(D_N)). \quad (42)$$

Hence we get

$$u_n \rightharpoonup u \text{ in } L^2(0,T;D_0^{1,2}(D_N,\sigma)), \quad (43)$$

$$f(u_n^t) \rightharpoonup \xi \text{ in } L^2(0,T;L^2(D_N)), \quad (44)$$

$$Au_n \rightharpoonup Au \text{ in } L^2(0,T;D^{-1}(D_N,\sigma). \quad (45)$$

We also have the following equation which has another form

$$\frac{\partial u_n}{\partial t} = -\lambda u_n - Au_n + f(u_n^t) + g, \quad (46)$$

we obtain that $\{\partial u_n/\partial t\}$ is bounded in $L^2(0,T;D^{-1}(D_N,\sigma) \bigcup L^2(D_N))$. We also see the following triple

$$D_0^{1,2}(D_N,\sigma)) \subset\subset L^2(D_N) \subset D^{-1}(D_N,\sigma) \cup L^2(D_N), \quad (47)$$

applying the compactness lemma we can suppose that $u_n \to u$ strongly in $L^2(-\nu,T;L^2(D_N))$. Thus $u_n \to u$ a.e. in $D_N \times [-\nu,T]$ and $u_n^t \to u^t$ a.e. in $D_N \times [0,T]$. Since f is continuous, $f(u_n^t) \to f(u^t)$ a.e. in $D_N \times [0,T]$, we get

$$f(u_n^t) \to f(u^t) \text{ in } L^2(0,T;L^2(D_N)). \quad (48)$$

Therefore by (46) we get

$$\frac{\partial u}{\partial t} = -\lambda u - Au + f(u^t) + g \text{ in } L^2(0,T;D^{-1}(D_N,\sigma) \cup L^2(D_N)). \quad (49)$$

By a standard argument, and using the strong convergence $u_n \to u$ in $C([0,T]; L^2(D_N))$ and the Doubinskii's theorem [4], we get that any weak-*limit is a solution of problem (1) subject to the initial conditions. Hence we obtain the existence of the solution.

Next we investigate the uniqueness of the solution. Let u_1, u_2 be the two solutions of problem (1) with the initial value u_1^0, u_2^0, respectively. Thus from (1) we get

$$\frac{\partial}{\partial t}(u_1 - u_2) + \lambda(u_1 - u_2) + (Au_1 - Au_2) = (f(u_1^t) - f(u_2^t)). \tag{50}$$

By the condition (ii) of f we conclude that

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|u_1 - u_2\|^2 &\leq \|f(u_1^t) - f(u_2^t)\|\|u_1 - u_2\| \\ &\leq l_f(r(T))\|u_1^t - u_2^t\|_{\mathscr{F}}\|u_1 - u_2\| \\ &\leq l_f(r(T))\|u_1^t - u_2^t\|_{\mathscr{F}}^2. \end{aligned} \tag{51}$$

Integrating on $[0,t]$, $0 < t \leq T$,

$$\|u_1(t) - u_2(t)\|^2 \leq \|u_1(0) - u_2(0)\|^2 + l_f(r(T))\int_0^t \|u_1^s - u_2^s\|_{\mathscr{F}}^2 ds. \tag{52}$$

Therefore, for fixed $\rho \in [-\nu, 0]$, we have that, for $t \in (-\rho, T]$,

$$\begin{aligned} \|u_1(t+\rho) - u_2(t+\rho)\|^2 &\leq \|u_1(0) - u_2(0)\|^2 \\ &+ l_f(r(T))\int_0^{t+\rho} \|u_1^s - u_2^s\|_{\mathscr{F}}^2 ds \leq \|u_1^0 - u_2^0\|_{\mathscr{F}}^2 \\ &+ l_f(r(T))\int_0^{t} \|u_1^s - u_2^s\|_{\mathscr{F}}^2 ds, \end{aligned} \tag{53}$$

for $t \in [0, -\rho]$,

$$\|u_1(t+\rho) - u_2(t+\rho)\|^2 \leq \|u_1^0 - u_2^0\|_{\mathscr{F}}^2. \tag{54}$$

Thus, from (53) and (54) we get that, for all $t \in [0,T]$,

$$\|u_1^t - u_2^t\|_{\mathscr{F}}^2 \leq \|u_1^0 - u_2^0\|_{\mathscr{F}}^2 + l_f(r(T))\int_0^t \|u_1^s - u_2^s\|_{\mathscr{F}}^2 ds. \tag{55}$$

By using the Gronwall inequality we obtain that, for all $t \in [0,T]$,

$$\|u_1^t - u_2^t\|_{\mathscr{F}}^2 \leq \|u_1^0 - u_2^0\|_{\mathscr{F}}^2 \left(1 + l_f(r(T))te^{l_f(r(T))t}\right). \tag{56}$$

Hence we obtain the uniqueness (if $u_1^0 = u_2^0$) of the solution. □

4. Existence of Pullback Attractors

In this section, we will prove the existence of $\mathscr{D}$-pullback attractors of this nonautonomous retarded degenerate parabolic equation by proving the existence of the $\mathscr{D}$-pullback absorbing sets and the $\mathscr{D}$-pullback asymptotic compactness.

We will first construct a θ-cocycle for the nonautonomous retarded degenerate parabolic equation defined on D_N. As in the case of bounded domains, the Theorem 3.1 implies that the problem (1) and (2) is well defined. In order to construct a cocycle ϕ for the problem (1) and (2), let $\Omega = D_N$ and define a shift operator θ_t on Ω, for any $t \in \mathbb{R}$,

$$\theta_t(\tau) = t + \tau, \quad \forall \tau \in \mathbb{R}. \tag{57}$$

Then we can define ϕ as a mapping from $\mathbb{R}^+ \times \Omega \times \mathscr{F}$ to $\mathscr{F}$ such that

$$\phi(t, \tau, u_\tau) = u(t+\tau, \tau, u_\tau), \tag{58}$$

where $t \geq 0$, $\tau \in \mathbb{R}$, $u_\tau \in \mathscr{F}$, and u is the solution of the problem (1) and (2). It is not difficult to see that ϕ is a continuous θ-cocycle on $\mathscr{F}$.

Then, we will derive uniform estimates on the weak solutions of problem (1) and (2) when $t \to \infty$, which is essential to prove the existence of a bounded pullback absorbing set and the pullback asymptotic compactness for ϕ.

Lemma 4.1. *Suppose that (i)–(iv) hold and (6) satisfies. Then there exists $T > 0$ such that for any $t \geq T$,*

$$\|u^t\|_{\mathscr{F}}^2 \leq K_1, \tag{59}$$

which implies the existence of $\mathscr{D}$-pullback absorbing set in $\mathscr{F}$.

Proof. Taking the inner product of (1) with u over D_N, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2 + \lambda\|u\|^2 + \|u\|_\sigma^2 + \int_{D_N} f(u^t)u dx = \int_{D_N} g(x,t)u dx. \tag{60}$$

By using the Young inequality and replacing u_n with u in (35) and (38), we can get for all $t \in [0,T]$,

$$\|u(t)\|^2 \leq (1 + c_f\nu)\|u^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^t \|g(x,s)\|^2 ds, \tag{61}$$

$$\|u^t\|_{\mathscr{F}}^2 \leq (1 + c_f\nu)\|u^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^T \|g(x,s)\|^2 ds. \tag{62}$$

Since $g \in L_{loc}^2(\mathbb{R}; L^2(D_N))$, let K_1 be

$$K_1 = (1 + c_f\nu)\|u^0\|_{\mathscr{F}}^2 + \frac{1}{\alpha}\int_0^T \|g(x,s)\|^2 ds, \tag{63}$$

then for any $t \geq T$, K_1 is bounded. Hence we get the lemma immediately. □

Lemma 4.2. *Suppose that (i)–(iv) hold and (6) satisfies. Then for any $t \geq T$,*

$$\int_0^{t+1} \|u(s)\|_\sigma^2 ds \leq K_2. \tag{64}$$

Proof. Noting that from (32) we obtain that

$$\frac{d}{dt}\|u\|^2 + (2\lambda - c_f - \alpha)\|u\|^2 + 2\|u\|_\sigma^2 \leq \frac{1}{\alpha}\|g(x,t)\|^2 + \frac{1}{c_f}\|f(u^t)\|^2. \tag{65}$$

Integrating (65) over the interval $[t, t+1]$, we get that

$$\|u(t+1)\|^2 - \|u(t)\|^2 + \left(2\lambda - c_f - \alpha\right)\int_t^{t+1} \|u(s)\|^2 ds + 2\int_t^{t+1} \|u(s)\|_\sigma^2 ds \leq \frac{1}{\alpha}\int_t^{t+1} \|g(x,s)\|^2 + \frac{1}{c_f}\int_t^{t+1} \|f(u^s)\|^2 ds. \tag{66}$$

It follows from the Young inequality and (iii) of f that

$$\begin{aligned}\frac{1}{c_f}\int_t^{t+1} \left\|f(u^s)\right\|^2 ds &\leq c_f \int_{t-\nu}^{t+1} \|u(s)\|^2 ds \\ &\leq c_f \int_{t-\nu}^{t} \|u(s)\|^2 ds + c_f \int_t^{t+1} \|u(s)\|^2 ds.\end{aligned} \tag{67}$$

By (66) and (67), we find that

$$\|u(t+1)\|^2 + \left(2\lambda - 2c_f - \alpha\right)\int_t^{t+1} \|u(s)\|^2 ds + 2\int_t^{t+1} \|u(s)\|_\sigma^2 ds \leq \|u(t)\|^2 + \frac{1}{\alpha}\int_t^{t+1} \left\|g(x,s)\right\|^2 + c_f \int_{t-\nu}^{t} \|u(s)\|^2 ds. \tag{68}$$

Thus, thanks to $2\lambda > 2c_f + \alpha$, we have that

$$\begin{aligned}2\int_t^{t+1} \|u(s)\|_\sigma^2 ds &\leq \|u(t)\|^2 + \frac{1}{\alpha}\int_t^{t+1} \left\|g(x,s)\right\|^2 + c_f \int_{t-\nu}^{t} \|u(s)\|^2 ds \\ &\leq \left(1 + c_f\nu\right)\|u(t)\|_{\mathcal{F}}^2 + \frac{1}{\alpha}\int_t^{t+1} \left\|g(x,s)\right\|^2 \\ &\leq \left(1 + c_f\nu\right)K_1 + \frac{1}{\alpha}\int_t^{t+1} \left\|g(x,s)\right\|^2.\end{aligned} \tag{69}$$

Therefore,

$$\int_t^{t+1} \|u(s)\|_\sigma^2 ds \leq \frac{1 + c_f\nu}{2}K_1 + \frac{1}{2\alpha}\int_t^{t+1} \left\|g(x,s)\right\|^2 := K_2, \tag{70}$$

where K_1 is given by (59). It follows from (6) and Lemma 4.1 we find that K_2 is bounded. Thus we finish the proof of the Lemma 4.2.

Lemma 4.3. *Suppose that (i)–(iv) hold and (6) satisfies. Then for any $t \geq T + \nu + 1$ and $\rho_1, \rho_2 \in [-\nu, 0]$,*

$$\|u(t)\|_\sigma^2 \leq K_3, \tag{71}$$

$$\int_{t+\rho_1}^{t+\rho_2} \|Au(s)\|^2 ds \leq K_4, \tag{72}$$

where K_3, K_4 are two positive constants.

Proof. Taking the inner product of (1) with Au we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_\sigma^2 + \lambda\|u\|_\sigma^2 + \|Au\|^2 = \left(f(u^t), Au\right) + \left(g(x,t), Au\right). \tag{73}$$

Applying the Young inequality, we find that

$$\left(f(u^t), Au\right) \leq \left\|\left(f(u^t)\right\|\|Au\| \leq \left\|f(u^t)\right\|^2 + \frac{1}{4}\|Au\|^2, \tag{74}$$

$$\left(g(x,t), Au\right) \leq \left\|g(x,t)\right\|\|Au\| \leq \left\|g(x,t)\right\|^2 + \frac{1}{4}\|Au\|^2. \tag{75}$$

It follows from (73)–(75) that

$$\frac{1}{2}\frac{d}{dt}\|u\|_\sigma^2 + \lambda\|u\|_\sigma^2 + \frac{1}{2}\|Au\|^2 \leq \left\|f(u^t)\right\|^2 + \left\|g(x,t)\right\|^2. \tag{76}$$

Thus,

$$\frac{d}{dt}\|u\|_\sigma^2 + \|Au\|^2 \leq 2\left\|f(u^t)\right\|^2 + 2\left\|g(x,t)\right\|^2. \tag{77}$$

Let $T > 0$ be the constant in Lemma 4.1, then take $t \geq T$ and $s \in (t, t+1)$. Integrating (77) over the interval $[s, t+1]$ leads to

$$\begin{aligned}\|u(t+1)\|_\sigma^2 &\leq \|u(s)\|_\sigma^2 + 2\int_s^{t+1} \left\|f(u^\tau)\right\|^2 d\tau + 2\int_s^{t+1} \left\|g(x,\tau)\right\|^2 d\tau \\ &\leq \|u(s)\|_\sigma^2 + 2\int_t^{t+1} \left\|f(u^\tau)\right\|^2 d\tau + 2\int_t^{t+1} \left\|g(x,\tau)\right\|^2 d\tau.\end{aligned} \tag{78}$$

Integrating the above inequality with respect to s over the integral $[t, t+1]$, we obtain that

$$\begin{aligned}\|u(t+1)\|_\sigma^2 &\leq \int_t^{t+1} \|u(s)\|_\sigma^2 ds + 2\int_t^{t+1} \left\|f(u^\tau)\right\|^2 d\tau + 2\int_t^{t+1} \left\|g(x,\tau)\right\|^2 d\tau \\ &\leq K_2 + 2l_f^2 \int_t^{t+1} \|u^s\|_{\mathcal{F}}^2 ds + 2\int_t^{t+1} \left\|g(x,\tau)\right\|^2 d\tau \\ &\leq K_2 + 2l_f^2 K_1 + 2\int_t^{t+1} \left\|g(x,\tau)\right\|^2 d\tau.\end{aligned} \tag{79}$$

Thus we get, for all $t \geq T + 1$,

$$\|u(t)\|_\sigma^2 \leq K_2 + 2l_f^2 K_1 + 2\int_t^{t+1} \left\|g(x,\tau)\right\|^2 d\tau := K_3. \tag{80}$$

From Lemma 4.1, Lemma 4.2, and (6) we know K_3 is bounded. Let $t \geq \nu$, $-\nu \leq \rho_1 \leq \rho_2 \leq 0$. Integrating (77) over the interval $[t + \rho_1, t + \rho_2]$ we get

$$\begin{aligned}&\left\|u(t+\rho_2)\right\|_\sigma^2 + \int_{t+\rho_1}^{t+\rho_2} \|Au(s)\|^2 ds \\ &\quad\leq \left\|u(t+\rho_1)\right\|_\sigma^2 + 2\int_{t+\rho_1}^{t+\rho_2} \left\|f(u^s)\right\|^2 ds \\ &\qquad + 2\int_{t+\rho_1}^{t+\rho_2} \left\|g(x,s)\right\|^2 ds.\end{aligned} \tag{81}$$

Thus, for all $t \geq T + \nu + 1$ and $\rho_1, \rho_2 \in [-\nu, 0]$,

$$\begin{aligned}\int_{t+\rho_1}^{t+\rho_2} \|Au(s)\|^2 ds &\leq \left\|u(t+\rho_1)\right\|_\sigma^2 + 2l_f^2 \int_{t-\nu}^{t} \|u^s\|_{\mathcal{F}}^2 ds + 2\int_{t-\nu}^{t} \left\|g(x,s)\right\|^2 ds \\ &\leq \left\|u(t+\rho_1)\right\|_\sigma^2 + 2l_f^2 \nu K_1 + 2\int_{t-\nu}^{t} \left\|g(x,s)\right\|^2 ds \\ &\leq K_3 + 2l_f^2 \nu K_1 + 2\int_{t-\nu}^{t} \left\|g(x,s)\right\|^2 ds,\end{aligned} \tag{82}$$

where K_1 is given by (59), K_3 is given by (71). Then we let K_4 be

$$K_4 = K_3 + 2l_f^2 \nu K_1 + 2\int_{t-\nu}^{t} \left\|g(x,s)\right\|^2 ds. \tag{83}$$

It is not difficult to find that K_4 is bounded. Thus we finish the proof. ☐

Lemma 4.4. *ϕ is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{F}$; that is, for any $\omega \in \Omega$, the sequence $\left\{\phi\left(t_n, \theta_{-t_n}\omega, u_n^0\left(\theta_{-t_n}\omega\right)\right)\right\}_{n=1}^{\infty}$*

possesses a convergence subsequence in $\mathscr{F}$ *provided* $t_n \to \infty$, $B \in \mathscr{D}$, *and* $u_n^0(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.

Proof. The proof is quite similar to the Lemma 18 in [2], more details can be found in [7, 9, 17, 18] and thus we omit the proof here. □

Proof of Theorem 2.12. Noting that from Lemma 4.1, we get that ϕ has a $\mathscr{D}$-pullback absorbing set. In addition, ϕ is $\mathscr{D}$-pullback asymptotically compact in $\mathscr{F}$ by Lemma 4.4. Therefore, the existence of a unique $\mathscr{D}$-pullback attractor $\mathscr{A}(\omega)_{\omega\in\Omega}$ for ϕ follows from Proposition 2.11.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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