

Research Article

Dynamical Analysis for the Hybrid Network Model of Delayed Predator-Prey Gompertz Systems with Impulsive Diffusion between Two Patches

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In this paper, we consider a hybrid network model of delayed predator-prey Gompertz systems with impulsive diffusion between two patches, in which the patches represent nodes of the network such that the prey population interacts locally in each patch and diffusion occurs along the edges connecting the nodes. Using the discrete dynamical system determined by the stroboscopic map which has a globally stable positive fixed point, we obtain the global attractive condition of predator-extinction periodic solution for the network system. Furthermore, by employing the theory of delay functional and impulsive differential equation, we obtain sufficient condition with time delay for the permanence of the network.

1. Introduction

Along with the continuous development of the network science, the mathematical models organized as networks have received considerable attention [1]–[3]. Taking epidemic models for an example, locations such as cities or urban areas can be represented as nodes of a network; individuals can be divided into different states, such as infection, susceptibility, immunity, etc. These individuals interact moving between connecting nodes [2, 3]. Furthermore, in the study of population dynamical systems, due to the universality and importance of the predator-prey relationship, the dynamics of the predator-prey system has been widely concerned. In recent decades, the dynamical behaviors of the predator-prey model defined on the network have enjoyed remarkable progress [4–8]. In [6], each node of the coupled network represents a discrete predator-prey system, and the network dynamics are investigated. In [7], Chang studied instability induced by time delay for a predator-prey model on complex networks and instability conditions were obtained via linear stability analysis of network organized systems.

Since the severe competition, natural enemy, or deterioration of the patch environment, the population dispersal phenomena of biological species can often occur between patches. Therefore, the effect of spatial factors in population dynamics becomes a very hot subject [9, 10]. Concerning qualitative analysis for predator-prey models with diffusion, such as local (or global) stability of equilibria and the existence of periodic solutions, many nice results have been obtained (see also, e.g., [11–13]). Regrettably, in all of the above population dispersion systems, dispersal behavior of the populations is occurring at every time. That is, it is a continuous dispersal. In practice, it is often the case that population diffusion occurs in regular pulses. For example, when winter comes, birds will migrate between patches in search for a better environment, whereas they do not diffuse during other seasons. Thus, impulsive diffusion provides a more natural behavior phenomenon. At present, many scholars have applied the theory of impulsive differential equations to population dynamics, and many important studies have been performed [14–19]. Accordingly, it is an interesting subject to analyze the dynamic behaviors of the

system by extending the predator-prey model with impulsive diffusion to the network version. In addition, in the 1825s, Benjamin Gompertz established the Gompertz function $y = ke^{-e^{-bx}}$, which can be translated into a Gompertz differential equation $dy/dx = by \ln(k/y)$ (see [20, 21]). Compared with the logistic function, it has been proven to be a simple example to generate an asymmetric S-shaped curve [22]. Since then, many models have been established for biological growth by using the Gompertz function (e.g., [23, 24]). Furthermore, many species usually go through two distinct life stages, immature and mature. Considering that the immature becomes the mature need to spend units of time and the number of deaths in the juvenile period, it is essential to consider time-delay in stage-structured model. Many stage-structured predator-prey models with time delay and impulsive diffusive were investigated [25–28]. Liu [25] studied a delayed predator-prey model with impulsive perturbations and gave the predator-extinction periodic solution of the model, which is globally attractive and permanence. Jiao et al. [26] and Dhar and Jatav [27] investigated a delayed predator-prey model with impulsive diffusion and sufficient conditions of the global attractiveness of the predator-extinction periodic solution and the permanence were derived.

Motivated by the above discussion, in this paper, we shall organize the patches into networks to investigate a delayed stage-structured functional response predator-prey Gompertz model with impulsive diffusion between two predators territories. We also consider the harvesting effort of the two mature predators. By employing the comparison theorem of impulsive differential equations and the global attractivity of the first order time-delay system, we will obtain some sufficient conditions on the global attractiveness of predator-extinction periodic solution and permanence of our model. The results can provide a reliable strategic basis for the protection of biological resources.

The paper is organized as follows. In the next section, introduce model development. In Section 3, some useful preliminaries are given. In Section 4, we give the conditions of the global attractivity for our model. In Section 5, we give the conditions of permanence for our model. Finally, discussion is given in Section 6.

2. Hybrid Network Model-Organized Predator-Prey System

Aiello and Freedman [29] introduced the following stage-structured single species model:

$$\begin{cases} x_1'(t) = \alpha x_2(t) - \gamma_1 x_1(t) - ae^{-\gamma_1 t} x_2(t - \tau), \\ x_2'(t) = ae^{-\gamma_1 t} x_2(t - \tau) - \gamma_2 x_2^2(t), \end{cases} \quad (1)$$

where $x_1(t)$ and $x_2(t)$ denote the immature and mature population densities, respectively, $\alpha > 0$ represents the birth rate, $\gamma_1 > 0$ is the immature death rate, $\gamma_2 > 0$ is the mature death and overcrowding rate, and τ represents the mean

length of the juvenile period. The term $ae^{-\gamma_1 t} x_2(t - \tau)$ represents the immature populations who were born at time $t - \tau$ and survived at time t (with the immature death rate γ_1) and therefore represents the transformation of immature to mature.

Wang et al. [23] considered the following model:

$$\begin{cases} \left. \begin{aligned} x_1'(t) &= r_1 x_1(t) \ln \frac{k_1}{x_1}, \\ x_2'(t) &= r_2 x_2(t) \ln \frac{k_2}{x_2}, \end{aligned} \right\} t \neq n\tau, \\ \left. \begin{aligned} \Delta x_1(t) &= d_1(x_2(t) - x_1(t)), \\ \Delta x_2(t) &= d_2(x_1(t) - x_2(t)), \end{aligned} \right\} t = n\tau, \quad n \in \mathbb{N}, \end{cases} \quad (2)$$

where x_i ($i = 1, 2$) is the density of species in the i th patch, r_i ($i = 1, 2$) is the intrinsic rate of natural increase of population in the i th patch, k_i ($i = 1, 2$) denotes the carrying capacity in the i th patch, and d_i ($i = 1, 2$) is dispersal rate in the i th patch. It is assumed here that the net exchange from the j th to i th patch is proportional to the difference $x_j - x_i$ of population densities. The pulse diffusion occurs every τ period (τ is a positive constant). Here $\Delta x_i(t) = x_i(n\tau^+) - x_i(n\tau^-)$, where $x_i(n\tau^+)$ represents the density of population in the i th patch immediately after the n th diffusion pulse, and $x_i(n\tau^-)$ represents the density of population in the i th patch before the n th diffusion pulse at time $t = n\tau$. r_i , k_i and d_i ($i = 1, 2$) are positive constants.

According to the model formulation in the literature [15, 23, 25–27], in the following, we shall extend predator-prey model to the network analogue version. Firstly, we propose in this paper a predator-prey model on the network with the following assumptions:

- (A1) The patches are created by predator territories and are represented as nodes of the network.
- (A2) The prey population in different nodes has different growth rates. The prey population interacts locally in each patch and impulsively diffuses through connected nodes.
- (A3) The predator population is divided into immature and mature. Immature becoming mature requires a constant time.
- (A4) Mature predator in different nodes has a different conversion rate.
- (A5) Immature predators only feed on mature predators and can not reproduce.
- (A6) Mature predators in different nodes have different harvest efforts.

We formulate the following hybrid network model of delayed predator-prey Gompertz system with impulsive diffusion between two patches:

$$\left. \begin{aligned}
 & x_1'(t) = r_1 x_1(t) \ln \frac{k_1}{x_1(t)} - p_1(x_1(t)) z_1(t), \\
 & y_1'(t) = \alpha_1 z_1(t) - \alpha_1 e^{-\omega_1 \tau_1} z_1(t - \tau_1) - \omega_1 y_1(t), \\
 & z_1'(t) = \alpha_1 e^{-\omega_1 \tau_1} z_1(t - \tau_1) + \beta_1 p_1(x_1(t)) z_1(t) - E_1 z_1(t) - \gamma_1 z_1^2(t), \\
 & x_2'(t) = r_2 x_2(t) \ln \frac{k_2}{x_2(t)} - p_2(x_2(t)) z_2(t), \\
 & y_2'(t) = \alpha_2 z_2(t) - \alpha_2 e^{-\gamma_1 \tau_2} z_2(t - \tau_2) - \gamma_1 y_2(t), \\
 & z_2'(t) = \alpha_2 e^{-\omega_2 \tau_1} z_1(t - \tau_1) + \beta_2 p_2(x_2(t)) z_2(t) - E_2 z_2(t) - \gamma_2 z_2^2(t),
 \end{aligned} \right\} t \neq n\tau,$$

$$\left. \begin{aligned}
 & \Delta x_1(t) = d_1(x_2(t) - x_1(t)), \\
 & \Delta x_2(t) = d_2(x_1(t) - x_2(t)), \\
 & \Delta y_1(t) = 0, \\
 & \Delta y_2(t) = 0, \\
 & \Delta z_1(t) = 0, \\
 & \Delta z_2(t) = 0,
 \end{aligned} \right\} t = n\tau, n \in \mathbb{N},$$
(3)

where x_i ($i = 1, 2$) is the prey population density in the i th patch at time t , $y_1(t)$ and $z_1(t)$ are predator populations density with immature and mature in the first patch at time t , $y_2(t)$ and $z_2(t)$ are predator populations density with immature and mature in the second patch at time t ; r_i and k_i are the Gompertz intrinsic growth rates and the carrying capacity in the i th patch, α_i represents the growth rate of immature to mature predators in the i th patch; ω_1 and γ_1 are the immature and mature predator death rates in the first patch, ω_2 and γ_2 are the immature and mature predator death rates in the second patch, E_i ($i = 1, 2$) is the harvesting effort of the mature population in the i th patch. Further, $\tau_i, \beta_i, d_i \in (0, 1)$ are the constant time to maturity, the conversion rates of predator, and dispersal rates of prey in the i th patch. The pulse diffusion occurs every τ period (τ is a positive constant).

Also, $p_i(\cdot) \in H$; here $H = \{f: R \rightarrow R | f(0) = 0, f'(x) > 0 \text{ and } f''(x) \leq 0 \text{ for all } x > 0\}$. Examples of functions found in the biological literature that satisfy H are as follows:

- (F1) $f_1(x) = ax$, with $a > 0$
- (F2) $f_2(x) = ax / (1 + bx)$, with $a, b > 0$
- (F3) $f_3(x) = a(1 - e^{-bx})$, with $a, b > 0$

Functions (F1) and (F2) are known as Holling type functional responses. Function (F3) is Ivlev type functional responses. Functions (F1) and (F2) also were regarded as incidence rate function. Function (F1) is a double linear incidence rate function. Function (F2) is saturated incidence rate function.

We only consider system (3) in the biological meaning region: $D = \{(x_1(t)y_1(t), z_1(t), x_1(t), y_2(t)z_2(t)) | x_1(t) \geq 0, y_1(t) \geq 0, z_1(t) \geq 0, x_2(t) \geq 0, y_2(t) \geq 0, z_2(t) \geq 0\}$ and assume that solutions of system (3) satisfy the initial conditions:

$$(\phi_1(s), \phi_2(s), \phi_3(s), \phi_4(s), \phi_5(s), \phi_6(s)) \in C([- \tau, 0], \mathbb{R}_+^6),$$

$$\phi_i(0) > 0, \quad i = 1, 2, 3, 4, 5, 6.$$
(4)

We can simplify model (3) organized by network and need to restrict our attention to the following model:

$$\left. \begin{cases} x_1'(t) = r_1 x_1(t) \ln \frac{k_1}{x_1(t)} - p_1(x_1(t)) z_1(t), \\ z_1'(t) = \alpha_1 e^{-\omega_1 \tau_1} z_1(t - \tau_1) + \beta_1 p_1(x_1(t)) z_1(t) - E_1 z_1(t) - \gamma_1 z_1^2(t), \\ x_2'(t) = r_2 x_2(t) \ln \frac{k_2}{x_2(t)} - p_2(x_2(t)) z_2(t), \\ z_2'(t) = \alpha_2 e^{-\omega_2 \tau_1} z_2(t - \tau_1) + \beta_2 p_2(x_2(t)) z_2(t) - E_2 z_2(t) - \gamma_2 z_2^2(t), \end{cases} \right\} t \neq n\tau, \quad (5)$$

$$\left. \begin{cases} \Delta x_1(t) = d_1(x_2(t) - x_1(t)), \\ \Delta x_2(t) = d_2(x_1(t) - x_2(t)), \\ \Delta z_1(t) = 0, \\ \Delta z_2(t) = 0, \end{cases} \right\} t = n\tau, n \in \mathbb{N},$$

with the initial condition

$$\begin{aligned} (\phi_1(s), \phi_3(s), \phi_4(s), \phi_6(s)) &\in C([- \tau, 0], \mathbb{R}_+^4), \\ \phi_i(0) &> 0, \quad i = 1, 3, 4, 6. \end{aligned} \quad (6)$$

3. Preliminaries

The solution $x(t) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t))^T$ of (3) is a piecewise continuous function $x: \mathbb{R}_+ \rightarrow \mathbb{R}_+^6$. Thus, $x(t)$ is continuous on $(n\tau, (n+1)\tau]$, for all $n \in \mathbb{Z}_+$ and $\lim_{t \rightarrow n\tau^+} x(t) = x(n\tau^+)$ exists. Obviously, the smoothness properties of x guarantee the global existence and uniqueness of the solution of (3) (see [30]). We assumed that $x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t) \geq 0$. If $z_1(t) = 0, z_2(t) = 0$, we can obtain the following subsystem of (5):

$$\left. \begin{cases} x_1'(t) = r_1 x_1(t) \ln \frac{k_1}{x_1(t)}, \\ x_2'(t) = r_2 x_2(t) \ln \frac{k_2}{x_2(t)}, \end{cases} \right\} t \neq n\tau, \quad (7)$$

$$\left. \begin{cases} \Delta x_1(t) = d_1(x_2(t) - x_1(t)), \\ \Delta x_2(t) = d_2(x_1(t) - x_2(t)), \end{cases} \right\} t = n\tau, n \in \mathbb{N}.$$

For simplicity, let $u_1 = x_1/k_1, u_2 = x_2/k_2, k = k_2/k_1$, so the system (7) can be written as follows:

$$\left. \begin{cases} u_1'(t) = r_1 u_1(t) \ln \frac{1}{u_1(t)}, \\ u_2'(t) = r_2 u_2(t) \ln \frac{1}{u_2(t)}, \end{cases} \right\} t \neq n\tau, \quad (8)$$

$$\left. \begin{cases} \Delta u_1(t) = d_1(ku_2(t) - u_1(t)), \\ \Delta u_2(t) = d_2\left(\frac{1}{k}u_1(t) - u_2(t)\right), \end{cases} \right\} t = n\tau, n \in \mathbb{N}.$$

Integrating and solving the first two equations of system (8) between pulses, we have the following:

$$\begin{aligned} u_i(t) &= u_i(n\tau^+) e^{-r_i(t-n\tau)}, \\ n\tau < t \leq (n+1)\tau, \quad i &= 1, 2. \end{aligned} \quad (9)$$

Then, considering the last two equations of system (8), we get the following stroboscopic map of system (8):

$$\begin{cases} u_{1,(n+1)\tau} = u_{1,n}^{b_1} + d_1(ku_{2,n}^{b_2} - u_{1,n}^{b_1}), \\ u_{2,(n+1)\tau} = u_{2,n}^{b_2} + d_2\left(\frac{1}{k}u_{1,n}^{b_1} - u_{2,n}^{b_2}\right). \end{cases} \quad (10)$$

Here, $u_{i,(n+1)\tau} = u_i[(n+1)\tau^+]$, $0 < b_1 = e^{-r_1\tau} < 1$, $0 < b_2 = e^{-r_2\tau} < 1$. Equation (10) is a difference equation. It describes the densities of the population in two patches at a pulse in terms of values at the previous pulse, in other words, stroboscopically sampling at its pulsing period. The dynamical behavior of system (10), coupled with (9), determines the dynamical behavior of system (8). To write system (8) as a map, we can define a map $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ such that

$$\begin{cases} F_1(u) = u_1^{b_1} + d_1(ku_2^{b_2} - u_1^{b_1}), \\ F_2(u) = u_2^{b_2} + d_2\left(\frac{1}{k}u_1^{b_1} - u_2^{b_2}\right). \end{cases} \quad (11)$$

We see that $F_i(n\tau)$ describes the population densities in time $n\tau$, and the sets of all iterations of the map F are equivalent to the set of all density sequence generated by system (10). Furthermore, we have the following.

Lemma 1 (see [23]). There exists a unique positive fixed point $q = (q_1, q_2)$ of F , and for every $u = (u_1, u_2) > 0$, $F^n \rightarrow q$ as $n \rightarrow \infty$.

That is, the fixed point $q = (q_1, q_2)$ of F is globally stable. The trajectory of system (8) will trend to the positive periodic solution $(\tilde{u}_1, \tilde{u}_2)$ with a period τ , i.e.,

$$\begin{aligned} \tilde{u}_i(t) &= q_i^{e^{-r_i(t-n\tau)}}, \\ n\tau < t \leq (n+1)\tau, \quad i &= 1, 2. \end{aligned} \quad (12)$$

Then, the trajectory of system (7) will trend to the positive periodic solution $(\tilde{x}_1, \tilde{x}_2)$ with a period τ , i.e.,

$$\begin{aligned} \tilde{x}_i(t) &= k_i q_i^{e^{-r_i(t-n\tau)}}, \\ n\tau < t \leq (n+1)\tau, \quad i &= 1, 2. \end{aligned} \quad (13)$$

Lemma 2. There exists a constant $M > 0$ such that $x_1(t) \leq M/\beta_1$, $y_1(t) \leq M$, $z_1(t) \leq M$, $x_2(t) \leq M/\beta_2$, $y_2(t) \leq M$, $z_2(t) \leq M$ for each solution of (3) with all t large enough.

Proof. First, we define $V(t) = \beta_1 x_1(t) + y_1(t) + z_1(t) + \beta_2 x_2(t) + y_2(t) + z_2(t)$; then, we have $\lambda_0 = \min\{\omega_1, r_1\}$. Then, it is obvious that

$$\begin{aligned} V'(t) &= \beta_1 r_1 x_1(t) \ln \frac{k_1}{x_1(t)} + \beta_2 r_2 x_2(t) \ln \frac{k_2}{x_2(t)} \\ &+ (\alpha_1 - E_1) z_1(t) + (\alpha_2 - E_2) z_2(t) - \omega_1 y_1(t) \\ &- \omega_2 y_2(t) - \gamma_1 z_1^2(t) - \gamma_2 z_2^2(t). \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{dV(t)}{dt} + \lambda_0 V(t) &\leq \beta_1 r_1 x_1(t) \ln \frac{k_1 e^{\lambda_0/r_1}}{x_1(t)} + \beta_2 r_2 x_2(t) \ln \frac{k_2 e^{\lambda_0/r_2}}{x_2(t)} \\ &+ (\alpha_1 - E_1 + \lambda_0) z_1(t) - \gamma_1 z_1^2(t) \\ &+ (\alpha_2 - E_2 + \lambda_0) z_2(t) - \gamma_2 z_2^2(t) \\ &\leq \beta_1 r_1 k_1 e^{(\lambda_0/r_1)} + \beta_2 r_2 k_2 e^{(\lambda_0/r_2)} \\ &+ \frac{(\alpha_1 - E_1 + \lambda_0)^2}{4\gamma_1} + \frac{(\alpha_2 - E_2 + \lambda_0)^2}{4\gamma_2} = M_0. \end{aligned} \quad (15)$$

Then, we can obtain $V(t) \leq M_0/\lambda_0 + (V(0^+) - M_0/\lambda_0)e^{-\lambda_0 t}$, when $t \rightarrow \infty$, $V(t) \leq M_0/\lambda_0$. So $V(t)$ is uniformly ultimately bounded. Hence, by the definition of $V(t)$, there exists a constant $M = M_0/\lambda_0$ such that $x_1(t) \leq M/\beta_1$, $y_1(t) \leq M$, $z_1(t) \leq M$, $x_2(t) \leq M/\beta_2$, $y_2(t) \leq M$, $z_2(t) \leq M$ for all t large enough. The proof is completed. \square

Lemma 3 (see [31]). Consider the following equation with delay:

$$x'(t) = ax(t - \tau) - bx(t) - cx^2(t). \quad (16)$$

We assumed that a, b, c , and τ are positive constants. $x(t) > 0$ for $t \in [-\tau, 0]$, we have the following:

- (i) If $a > b$, then $\lim_{t \rightarrow +\infty} x(t) = (a - b)/c$
- (ii) If $a < b$, then $\lim_{t \rightarrow +\infty} x(t) = 0$

4. Extinction of the Predator

From the previous section, we know there exists a predator eradicated periodic solution $(\tilde{x}_1, 0, 0, \tilde{x}_2, 0, 0)$ of system (3). In this section, we will prove that $(\tilde{x}_1, 0, 0, \tilde{x}_2, 0, 0)$ for the network organized system (3) is globally attractive.

Theorem 1. *If*

$$\begin{aligned} \alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(x_1^*) &< E_1, x_1^* = k_1 q_1^{e^{-r_1 \tau}}; \\ \alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(x_2^*) &< E_2, x_2^* = k_2 q_2^{e^{-r_2 \tau}} \end{aligned} \quad (17)$$

holds, the predator-extinction periodic solution $(\tilde{x}_1, 0, 0, \tilde{x}_2, 0, 0)$ of system (3) is globally attractive.

Proof. It is obvious from the global attraction of the periodic solution of $(\tilde{x}_1, 0, 0, \tilde{x}_2, 0, 0)$, system (3) is equivalent to the global attraction of the periodic solution $(\tilde{x}_1, 0, \tilde{x}_2, 0)$ of system (5). From (17), we can choose $\varepsilon_0 > 0$ sufficiently small such that

$$\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(x_1^* + \varepsilon_0) < E_1, \alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(x_2^* + \varepsilon_0) < E_2. \quad (18)$$

It follows from that the first and third equations of system (5) that

$$\begin{cases} x_1'(t) \leq r_1 x_1(t) \ln \frac{k_1}{x_1(t)}, \\ x_2'(t) \leq r_2 x_2(t) \ln \frac{k_2}{x_2(t)}. \end{cases} \quad (19)$$

So we consider the following comparison impulsive differential system:

$$\left\{ \begin{array}{l} v_1'(t) = r_1 v_1(t) \ln \frac{k_1}{v_1(t)}, \\ v_2'(t) = r_2 v_2(t) \ln \frac{k_2}{v_2(t)}, \end{array} \right\} t \neq n\tau, \tag{20}$$

$$\left\{ \begin{array}{l} \Delta v_1(t) = d_1(v_2(t) - v_1(t)), \\ \Delta v_2(t) = d_2(v_1(t) - v_2(t)), \end{array} \right\} t = n\tau, n \in \mathbb{N}.$$

By Lemma 1 and (11), we obtain the boundary periodic solution of system (20):

$$\tilde{v}_i(t) = k_i q_i^{e^{-r_i(t-n\tau)}}, \tag{21}$$

$$n\tau < t \leq (n+1)\tau, \quad i = 1, 2,$$

which is globally asymptotically stable. In view of the comparison theorem of the impulsive differential equation

(see [30]), we have $x_i(t) \leq v_i(t) \rightarrow \tilde{x}_i(t)$ as $t \rightarrow \infty$. Then, there exist n_2 with $n_2 > n_1$ and $t > n_2$ such that

$$\begin{aligned} x_1(t) &\leq v_1(t) < \tilde{x}_1(t) + \varepsilon_0, \\ x_2(t) &\leq v_2(t) < \tilde{x}_2(t) + \varepsilon_0, \\ n\tau &< t \leq (n+1)\tau, \\ n &> n_2. \end{aligned} \tag{22}$$

That is,

$$\begin{aligned} x_1(t) &\leq \tilde{x}_1(t) + \varepsilon_0 \leq x_1^* + \varepsilon_0 \triangleq \delta_1, \\ x_2(t) &\leq \tilde{x}_2(t) + \varepsilon_0 \\ &\leq x_2^* + \varepsilon_0 \triangleq \delta_2, \\ n\tau &< t \leq (n+1)\tau. \end{aligned} \tag{23}$$

From the second and fourth equations of system (5), we have the following:

$$\left\{ \begin{array}{l} z_1'(t) \leq \alpha_1 e^{-\omega_1 \tau_1} z_1(t - \tau_1) - (E_1 - \beta_1 p_1(\delta_1(t))) z_1(t) - \gamma_1 z_1^2(t), \quad t > n\tau + \tau_1, n > n_2, \\ z_2'(t) \leq \alpha_2 e^{-\omega_2 \tau_2} z_2(t - \tau_2) - (E_2 - \beta_2 p_2(\delta_2(t))) z_2(t) - \gamma_2 z_2^2(t), \quad t > n\tau + \tau_2, n > n_2. \end{array} \right. \tag{24}$$

Now, consider the following comparison differential system:

$$\left\{ \begin{array}{l} m_1'(t) = \alpha_1 e^{-\omega_1 \tau_1} m_1(t - \tau_1) - (E_1 - \beta_1 p_1(\delta_1(t))) m_1(t) - \gamma_1 m_1^2(t), \quad t > n\tau + \tau_1, n > n_2, \\ m_2'(t) = \alpha_2 e^{-\omega_2 \tau_2} m_2(t - \tau_2) - (E_2 - \beta_2 p_2(\delta_2(t))) m_2(t) - \gamma_2 m_2^2(t), \quad t > n\tau + \tau_2, n > n_2. \end{array} \right. \tag{25}$$

From (18), we have $\alpha_1 e^{-\omega_1 \tau_1} < E_1 - \beta_1 p_1(x_1^* + \varepsilon_0)$ and $\alpha_2 e^{-\omega_2 \tau_2} < E_2 - \beta_2 p_2(x_2^* + \varepsilon_0)$, by Lemma 3, $\lim_{t \rightarrow \infty} m_1(t) = 0$ and $\lim_{t \rightarrow \infty} m_2(t) = 0$. By the comparison theorem, we have $\lim_{t \rightarrow \infty} z_1(t) \leq \lim_{t \rightarrow \infty} m_1(t) = 0$ and $\lim_{t \rightarrow \infty} z_2(t) \leq \lim_{t \rightarrow \infty} m_2(t) = 0$. Because of the positivity of $z_1(t)$ and $z_2(t)$, we know that $\lim_{t \rightarrow \infty} z_1(t) = 0$ and $\lim_{t \rightarrow \infty} z_2(t) = 0$. Therefore, for a small $\varepsilon_1 > 0$, there exists a $n_3 (n_3 \tau > n_2 \tau + \tau_1, n_3 \tau > n_2 \tau + \tau_2)$ such that $z_1(t) < \varepsilon_1, z_2(t) < \varepsilon_1$ for all $t > n_3 \tau$. From system (5), we have the following:

$$\left\{ \begin{array}{l} x_1'(t) > r_1 x_1(t) \ln \frac{k_1 e^{(-p_1'(0)\varepsilon_1/r_1)}}{x_1(t)}, \\ x_2'(t) > r_2 x_2(t) \ln \frac{k_2 e^{(-p_1'(0)\varepsilon_1/r_2)}}{x_2(t)}, \end{array} \right\} t \neq n\tau, \tag{26}$$

$$\left\{ \begin{array}{l} \Delta x_1(t) = d_1(x_2(t) - x_1(t)), \\ \Delta x_2(t) = d_2(x_1(t) - x_2(t)), \end{array} \right\} t = n\tau, n \in \mathbb{N}.$$

Here, $p_i(\bullet) \in H$ and we have $p_i(x_i(t)) \leq p_i'(0)x_i(t)$. Consider the following comparison differential system:

$$\left\{ \begin{array}{l} s_1'(t) = r_1 s_1(t) \ln \frac{k_1 e^{(-p_1'(0)\varepsilon_1/r_1)}}{s_1(t)}, \\ s_2'(t) = r_2 s_2(t) \ln \frac{k_2 e^{(-p_1'(0)\varepsilon_1/r_2)}}{s_2(t)}, \end{array} \right\} t \neq n\tau, \tag{27}$$

$$\left\{ \begin{array}{l} \Delta s_1(t) = d_1(s_2(t) - s_1(t)), \\ \Delta s_2(t) = d_2(s_1(t) - s_2(t)), \end{array} \right\} t = n\tau, n \in \mathbb{N},$$

where $\tilde{s}_1(t) = k_1 e^{-p_1'(0)\varepsilon_1/r_1} (q_1')^{e^{-r_1(t-n\tau)}}$ and $\tilde{s}_2(t) = k_2 e^{-p_2'(0)\varepsilon_1/r_2} (q_2')^{e^{-r_2(t-n\tau)}}$, q_1' and q_2' can be confirmed homoplastically as q_1, q_2 . Let $\varepsilon_1 \rightarrow 0$ use the comparison theorem $x_i(t) \geq s_i(t) \rightarrow \tilde{x}_i(t)$ as $t \rightarrow \infty$; there for any $\varepsilon_2 > 0$ and t large enough, there exists a $n_4, n > n_4$ such that

$$\begin{aligned} x_1(t) &> \tilde{x}_1(t) - \varepsilon_2, \\ x_2(t) &> \tilde{x}_2(t) - \varepsilon_2. \end{aligned} \tag{28}$$

Thus, from (23) and (28), we get $x_1(t) \rightarrow \tilde{x}_1(t)$ and $x_2(t) \rightarrow \tilde{x}_2(t)$ as $t \rightarrow \infty$. The proof is complete. \square

5. Permanence

In this section, we will discuss the permanence of the system (3) organized by the network. To facilitate the discussion, we give the following lemma.

Lemma 4. *If $\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(q_1) > E_1 + \gamma_1 M$ and $\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(q_2) > E_2 + \gamma_2 M$, then there exist two positive constants g_1 and g_2 such that each positive solution of (5) satisfies $z_1(t) \geq g_1$ and $z_2(t) \geq g_2$ for t large enough.*

Proof. From the second and fourth equations of system (5), it can be rewritten as follows:

$$\begin{cases} z_1'(t) = [\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 P_1(x_1(t)) - (E_1 + \gamma_1 z_1(t))]z_1(t) - \alpha_1 e^{-\omega_1 \tau_1} \frac{d}{dt} \int_{t-\tau_1}^t z_1(u)du, \\ z_2'(t) = [\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 P_2(x_2(t)) - (E_2 + \gamma_2 z_2(t))]z_2(t) - \alpha_2 e^{-\omega_2 \tau_2} \frac{d}{dt} \int_{t-\tau_2}^t z_2(u)du. \end{cases} \quad (29)$$

We can define Q_1 and Q_2 as follows:

$$\begin{cases} Q_1(t) = z_1(t) + \alpha_1 e^{-\omega_1 \tau_1} \int_{t-\tau_1}^t z_1(u)du, \\ Q_2(t) = z_2(t) + \alpha_2 e^{-\omega_2 \tau_2} \int_{t-\tau_2}^t z_2(u)du. \end{cases} \quad (30)$$

Calculate the derivative of Q_1 and Q_2 along the solution of (5) as follows:

$$\begin{cases} Q_1'(t) = [\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 P_1(x_1(t)) - (E_1 + \gamma_1 z_1(t))]z_1(t), \\ Q_2'(t) = [\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 P_2(x_2(t)) - (E_2 + \gamma_2 z_2(t))]z_2(t). \end{cases} \quad (31)$$

By using Lemma 2 and combining with (30), we can obtain $Q_1(t) \leq M(1 + \alpha_1 \tau_1 e^{-\omega_1 \tau_1})$ and $Q_2(t) \leq M(1 + \alpha_2 \tau_2 e^{-\omega_2 \tau_2})$ as $t \rightarrow \infty$. Since $\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(q_1) > E_1 + \gamma_1 M$ and $\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(q_2) > E_2 + \gamma_2 M$, we can find a sufficiently small $\varepsilon > 0$ such that $\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(q_1 - \varepsilon) > E_1 + \gamma_1 M$ and $\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(q_2 - \varepsilon) > E_2 + \gamma_2 M$. We suppose $t_0 > 0$, such that $z_1(t) < z_1^*$ and $z_2(t) < z_2^*$ for all $t > t_0$. It follows the first two equations of system (5) that for all $t > t_0$,

$$\begin{cases} x_1'(t) > r_1 x_1(t) \ln \frac{k_1 e^{(-p_1'(0)z_1^*/r_1)}}{x_1(t)}, \\ x_2'(t) > r_2 x_2(t) \ln \frac{k_2 e^{(-p_2'(0)z_2^*/r_2)}}{x_2(t)}. \end{cases} \quad (32)$$

Here, $p_i(\bullet) \in H$, we have $p_i(x_i(t)) \leq p_i'(0)x_i(t)$. For all $t > t_0$, consider the following comparison differential system:

$$\begin{cases} l_1'(t) = r_1 l_1(t) \ln \frac{k_1 e^{p_1'(0)z_1^*/r_1}}{l_1(t)}, \\ l_2'(t) = r_2 l_2(t) \ln \frac{k_2 e^{(-p_2'(0)z_2^*/r_2)}}{l_2(t)}, \\ \Delta l_1(t) = d_1(l_2(t) - l_1(t)), \\ \Delta l_2(t) = d_2(l_1(t) - l_2(t)), \end{cases} \begin{cases} t \neq n\tau, \\ \\ \\ t = n\tau, \quad n \in \mathbb{N}. \end{cases} \quad (33)$$

By Lemma 1, we obtain the following global asymptotically periodic unique positive solution of system (33):

$$\begin{cases} \tilde{l}_1(t) = k_1 e^{-p_1'(0)z_1^*/r_1} (q_1'')^{e^{-r_1(t-n\tau)}}, \\ \tilde{l}_2(t) = k_2 e^{-p_2'(0)z_2^*/r_2} (q_2'')^{e^{-r_2(t-n\tau)}}, \end{cases} \quad (34)$$

where q_1'' and q_2'' can be confirmed homoplastically as q_1, q_2 . By the comparison theorem for an impulsive differential equation, we know there exists a $t_1, t_2 (t_1 > t_0 + \tau_1, t_2 > t_0 + \tau_2)$ such that the inequality $x_1(t) > l_1(t) > \tilde{l}_1(t) - \varepsilon$ holds for all $t \geq t_1$ and $x_2(t) > l_2(t) > \tilde{l}_2(t) - \varepsilon$ holds for all $t \geq t_2$. Let $z_1^* \rightarrow 0$ and $z_2^* \rightarrow 0$, then $x_1(t) > \tilde{x}_1(t) - \varepsilon$ holds for all $t \geq t_1$ and $x_2(t) > \tilde{x}_2(t) - \varepsilon$ holds for all $t \geq t_2$. Thus, $x_1(t) > q_1 - \varepsilon$ holds for $t \geq t_1$ and $x_2(t) > q_2 - \varepsilon$ holds for all $t \geq t_2$. We make a notation as $\sigma_1 = q_1 - \varepsilon$ and $\sigma_2 = q_2 - \varepsilon$, for convenience. So we have $\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(\sigma_1) > E_1 + \gamma_1 M$ and $\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(\sigma_2) > E_2 + \gamma_2 M$. Then from (31), we can obtain the following:

$$\begin{cases} Q_1'(t) > [\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 P_1(x_1(t)) - (E_1 + \gamma_1 z_1(t))]z_1(t), \quad t \geq t_1, \\ Q_2'(t) > [\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 P_2(x_2(t)) - (E_2 + \gamma_2 z_2(t))]z_2(t), \quad t \geq t_2. \end{cases} \quad (35)$$

Let $\bar{z}_1 = \min_{t \in [t_1, t_1 + \tau_1]} z_1(t)$ and $\bar{z}_2 = \min_{t \in [t_2, t_2 + \tau_2]} z_2(t)$; we show that $z_1(t) \geq \bar{z}_1$ for all $t \geq t_1$ and $z_2(t) \geq \bar{z}_2$ for all $t \geq t_2$. Suppose the contrary, then there are $T_0, T_1 > 0$ such that $z_1(t) \geq \bar{z}_1$ for $t \in [t_1, t_1 + \tau_1 + T_0]$, $z_1(t_1, t_1 + \tau_1 + T_0) =$

\bar{z}_1 and $z_1'(t_1, t_1 + \tau_1 + T_0) \leq 0$ and such that $z_2(t) \geq \bar{z}_2$ for $t \in [t_2, t_2 + \tau_2 + T_1]$, $z_2(t_2, t_2 + \tau_2 + T_1) = \bar{z}_2$, and $z_2'(t_2, t_2 + \tau_2 + T_1) \leq 0$. Thus, from the third and fourth equations of system (5) imply that

$$\begin{aligned} z_1'(t_1 + \tau_1 + T_0) &= \alpha_1 e^{-\omega_1 \tau_1} z_1(t_1 + T_0) + \beta_1 p_1(x_1(t_1 + \tau_1 + T_0)) z_1(t_1 + \tau_1 + T_0) - E_1 z_1(t_1 + \tau_1 + T_0) \\ &\quad - \gamma_1 z_1^2(t_1 + \tau_1 + T_0) \geq [\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 P_1(\sigma_1) - (E_1 + \gamma_1 M)] \bar{z}_1 > 0, \\ z_2'(t_2 + \tau_2 + T_1) &= \alpha_2 e^{-\omega_2 \tau_2} z_2(t_2 + T_1) + \beta_2 p_2(x_2(t_2 + \tau_2 + T_1)) z_2(t_2 + \tau_2 + T_1) - E_2 z_2(t_2 + \tau_2 + T_1) \\ &\quad - \gamma_2 z_2^2(t_2 + \tau_2 + T_1) \geq [\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 P_2(\sigma_2) - (E_2 + \gamma_2 M)] \bar{z}_2 > 0. \end{aligned} \quad (36)$$

This is a contradiction. Thus, $z_1(t) \geq \bar{z}_1$ for all $t \geq t_1$ and $z_2(t) \geq \bar{z}_2$ for all $t \geq t_2$. From (35), we have the following:

$$\begin{aligned} Q_1'(t) &\geq [\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 P_1(\sigma_1) - (E_1 + \gamma_1 M)] \bar{z}_1 > 0, \quad t \geq t_1, \\ Q_2'(t) &\geq [\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 P_2(\sigma_2) - (E_2 + \gamma_2 M)] \bar{z}_2 > 0, \quad t \geq t_2, \end{aligned} \quad (37)$$

which implies $Q_1(t) \rightarrow \infty$ and $Q_2(t) \rightarrow \infty$ as $t \rightarrow \infty$. It is a contradiction to $Q_1(t) \leq M(1 + \alpha_1 \tau_1 e^{-\omega_1 \tau_1})$ and $Q_2(t) \leq M(1 + \alpha_2 \tau_2 e^{-\omega_2 \tau_2})$. Therefore, the claim is complete. By the claim, we are left to consider two cases. First, $z_1(t) \geq z_1^*$ and $z_2(t) \geq z_2^*$ for t large enough. Second, $z_1(t)$ and $z_2(t)$ oscillate about z_1^* and z_2^* for t large enough. Define

$$g_1 = \min \left\{ \frac{z_1^*}{2}, g_1^* \right\}, g_2 = \min \left\{ \frac{z_2^*}{2}, g_2^* \right\}, \quad (38)$$

where $g_1^* = z_1^* e^{-(E_1 + \gamma_1 M)\tau_1}$ and $g_2^* = z_2^* e^{-(E_2 + \gamma_2 M)\tau_2}$. In the following, we shall show that $z_1(t) \geq g_1$ and $z_2(t) \geq g_2$ for t are large enough; the conclusion is evident in the first case. For the second case, let $t_1^*, t_2^* > 0$ and $\zeta_1, \zeta_2 > 0$ satisfy $z_1(t_1^*) = z_1(t_1^* + \zeta_1) = z_1^*$ and $z_1(t) < z_1^*$ for all $t \in (t_1^*, t_1^* + \zeta_1)$ and $z_2(t_2^*) = z_2(t_2^* + \zeta_2) = z_2^*$ and $z_2(t) < z_2^*$ for all $t \in (t_2^*, t_2^* + \zeta_2)$, where t_1^*, t_2^* are large enough such that $x_1(t) > \sigma_1$ for $t_1^* < t < t_1^* + \zeta_1$ and $x_2(t) > \sigma_2$ for $t_2^* < t < t_2^* + \zeta_2$. Thus, $z_1(t)$ and $z_2(t)$ are uniformly continuous. The positive solutions of (3) are ultimately bounded and not affected by impulses. Hence, there are $T_1', T_2' (0 < T_1' < \tau_1, 0 < T_2' < \tau_2)$ and T_1', T_2' are dependent on the choice of t_1^*, t_2^* such that $z_1(t) > z_1^*/2$ for all $t \in (t_1^*, t_1^* + T_1')$ and $z_2(t) > z_2^*/2$ for all $t \in (t_2^*, t_2^* + T_2')$. If $\zeta_1 < T_1', \zeta_2 < T_2'$, our aim is obtained. Let us consider the case $T_1' < \zeta_1 \leq \tau_1, T_2' < \zeta_2 \leq \tau_2$, since $z_1'(t) > -(E_1 + \gamma_1 M)z_1(t)$, $z_1(t_1^*) = z_1^*$ and $z_2'(t) > -(E_2 + \gamma_2 M)z_2(t)$, $z_2(t_2^*) = z_2^*$, it is clear that $z_1(t) \geq g_1$ for all $t \in (t_1^*, t_1^* + \tau_1)$ and $z_2(t) \geq g_2$ for all $t \in (t_2^*, t_2^* + \tau_2)$. If $\zeta_1 > \tau_1$ and $\zeta_2 > \tau_2$; then we have that $z_1(t) \geq g_1$ for all $t \in (t_1^*, t_1^* + \tau_1)$ and $z_2(t) \geq g_2$ for all $t \in (t_2^*, t_2^* + \tau_2)$. The same arguments can be continued and we can obtain $z_1(t) \geq g_1$ for all $t \in (t_1^* + \tau_1, t_1^* + \zeta_1)$ and $z_2(t) \geq g_2$ and for all $t \in (t_2^* + \tau_2, t_2^* + \zeta_2)$. Since the interval $[t_1^*, t_1^* + \zeta_1]$ and $[t_2^*, t_2^* + \zeta_2]$ are arbitrarily chosen (t_1^*, t_2^* to be enough), we get that $z_1(t) \geq g_1$ and $z_2(t) \geq g_2$ for t are

large enough. In view of the above discussion, the choice of g_1, g_2 is independent of the positive solution of system (5), which satisfies $z_1(t) \geq g_1$ and $z_2(t) \geq g_2$ for sufficiently large t . The proof is completed. \square

Theorem 2. *If $\alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(q_1) > E_1 + \gamma_1 M$ and $\alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(q_2) > E_2 + \gamma_2 M$, then the system (3) is permanent.*

Proof. From system (3) and Lemma 2, we have the following:

$$\left\{ \begin{array}{l} x_1'(t) \geq r_1 x_1(t) \ln \frac{k_1 e^{-p_1'(0)M/n}}{x_1(t)}, \\ x_2'(t) \geq r_2 x_2(t) \ln \frac{k_2 e^{-p_2'(0)M/r_2}}{x_2(t)}, \\ \Delta x_1(t) = d_1(x_2(t) - x_1(t)), \\ \Delta x_2(t) = d_2(x_1(t) - x_2(t)), \end{array} \right. \quad \left. \begin{array}{l} t \neq n\tau, \\ t = n\tau, n \in \mathbb{N}. \end{array} \right. \quad (39)$$

By the similar argument as those in the proof of Theorem 1, we have that $x_1(t) \geq f_1 - \varepsilon$ and $x_2(t) \geq f_2 - \varepsilon$, and $\varepsilon > 0$ is small enough, where

$$f_i = k_i e^{-p_i'(0)M/r_i} (\bar{q}_i)^{e^{-r_i \tau}}, \quad i = 1, 2. \quad (40)$$

\bar{q}_1 and \bar{q}_2 can be confirmed homoplastically to be q_1, q_2 . Using Lemma 2 and Lemma 4, the second and fifth equations of system (3) become as follows:

$$\left\{ \begin{array}{l} y_1'(t) \geq \alpha_1 (g_1 - M e^{-\omega_1 \tau_1}) - \omega_1 y_1(t), \\ y_2'(t) \geq \alpha_2 (g_2 - M e^{-\omega_2 \tau_2}) - \omega_2 y_2(t). \end{array} \right. \quad (41)$$

It is easy to obtain $y_1(t) \geq \rho_1 - \varepsilon$ and $y_2(t) \geq \rho_2 - \varepsilon$, $\varepsilon > 0$ is small enough, where $\rho_1 = \alpha_1 (g_1 - M e^{-\omega_1 \tau_1}) / \omega_1 - e^{-\omega_1 \tau_1} / \omega_1$ and $\rho_2 = \alpha_2 (g_2 - M e^{-\omega_2 \tau_2}) / \omega_2 - e^{-\omega_2 \tau_2} / \omega_2$. Hence, by Lemma 2, Lemma 4, and the above discussion, we obtain that system (3) is permanent. The proof is completed. \square

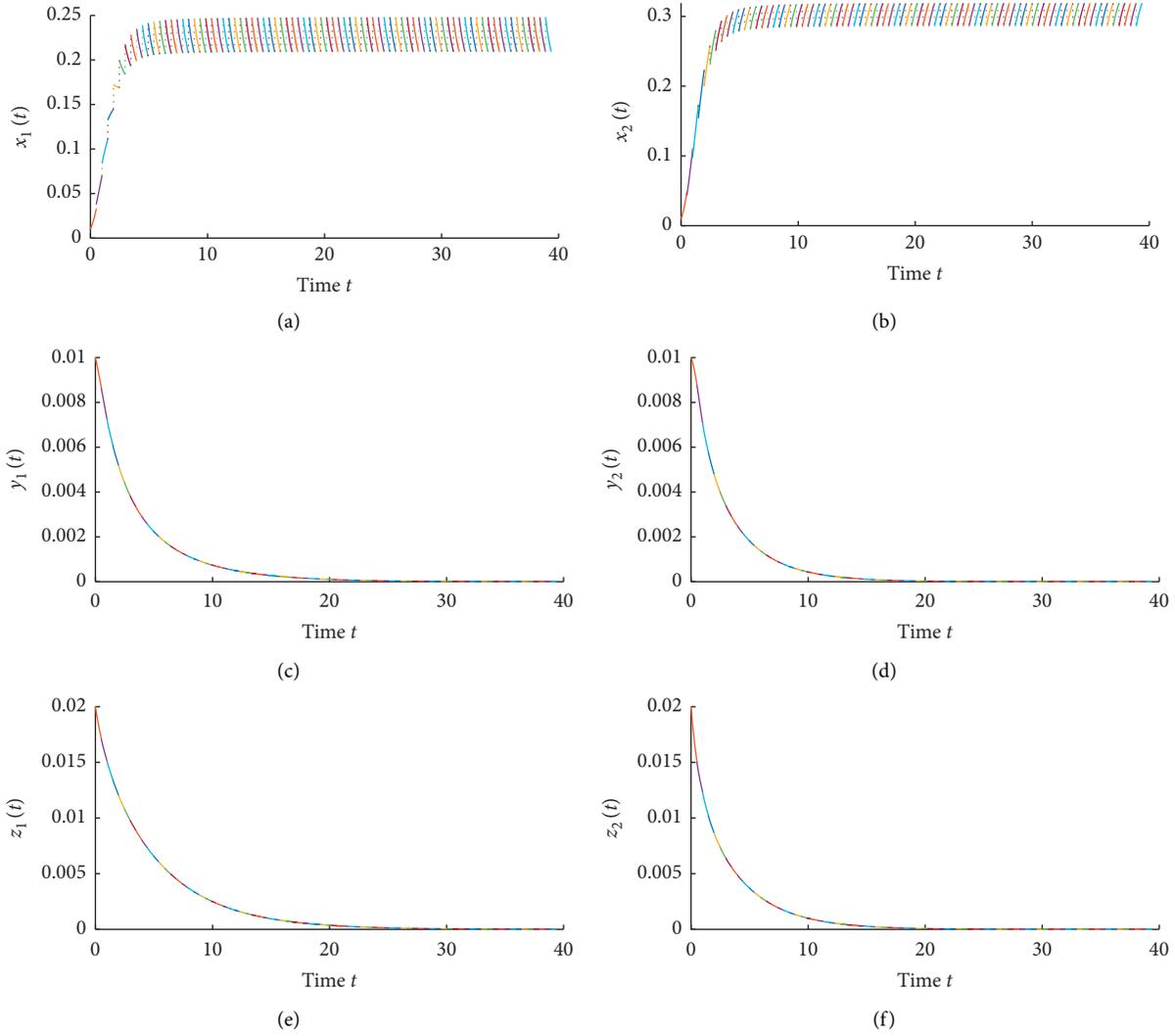


FIGURE 1: The time response in Case 1.

6. Illustrative Example and Discussion

Example 1. Consider the hybrid network model (3) of delayed predator-prey Gompertz system with impulsive diffusion between two patches, in which $p_1(x_1) = 1.5x_1$, $p_2(x_2) = 5x_2$, $\tau = 0.5$, and

$$r_1 = 1.1,$$

$$k_1 = 0.2,$$

$$\alpha_1 = 0.4,$$

$$\omega_1 = 0.5,$$

$$\beta_1 = 0.3,$$

$$\gamma_1 = 0.2,$$

$$\tau_1 = 0.8,$$

$$d_1 = 0.35;$$

$$r_2 = 1.2,$$

$$k_2 = 0.6,$$

$$\alpha_2 = 0.5,$$

$$\omega_2 = 0.4,$$

$$\beta_2 = 0.2,$$

$$\gamma_2 = 0.2,$$

$$\tau_2 = 0.8,$$

$$d_2 = 0.3. \tag{42}$$

From (11), we compute the fixed point $q = (1.7659, 0.7553)$.

Case 1. $E_1 = 0.6$ and $E_2 = 1$. In Theorem 1, we have $(x_1^*, x_2^*) = (0.2199, 0.5752)$ and

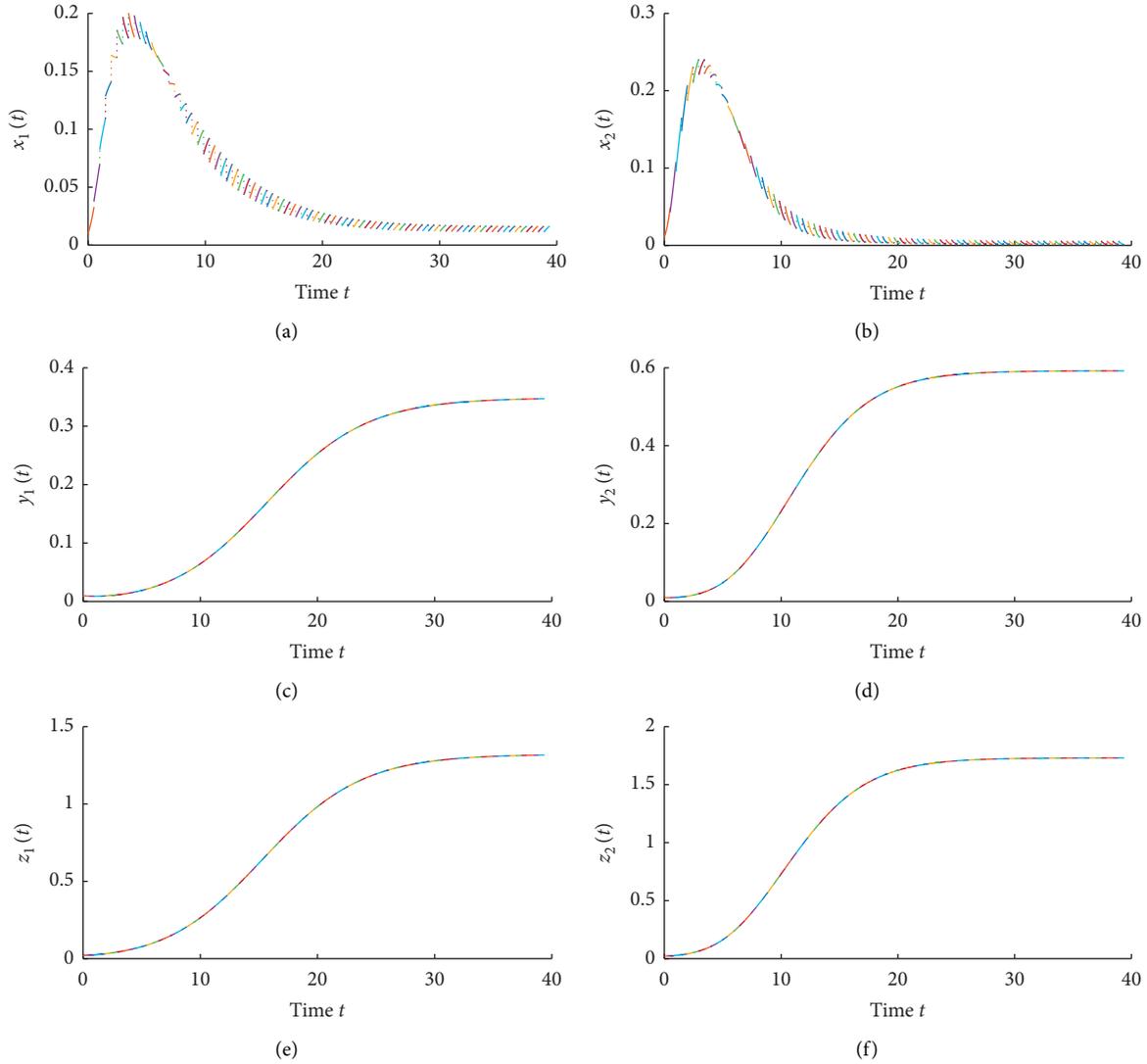


FIGURE 2: The time response in Case 2.

$$\begin{aligned} \alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(x_1^*) - E_1 &= -0.2329 \\ < 0; \alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(x_2^*) - E_2 &= -0.0618 < 0. \end{aligned} \quad (43)$$

It follows from Theorem 1 that the predator-extinction periodic solution $(\bar{x}_1, 0, 0, \bar{x}_2, 0, 0)$ of the system (3) is globally attractive. Figure 1 shows the attractivity of the solution with the initial condition $x_1(s) = 0.01, x_2(s) = 0.02$, and $s \in [-0.8, 0]$.

Case 2. $E_1 = 0.01$ and $E_2 = 0.02$. In Theorem 2, we have $M = 5.0261$ and

$$\begin{aligned} \alpha_1 e^{-\omega_1 \tau_1} + \beta_1 p_1(q_1) - E_1 - \gamma_1 M &= 0.0476 > 0; \\ \alpha_2 e^{-\omega_2 \tau_2} + \beta_2 p_2(q_2) - E_2 - \gamma_2 M &= 0.0931 > 0. \end{aligned} \quad (44)$$

It follows from Theorem 2 that system (3) is permanent. Figure 2 shows the permanent of the solution with the above initial condition.

In this paper, a delayed functional response predator-prey Gompertz system with impulsive diffusion between two patches defined on the network was investigated. The patches represent nodes of a network such that the prey population interacts locally in each patch and occurs diffusively over links connecting nodes. By extending system (3) to the network version, we analyzed that the predator-extinction solution of system (3) is globally attractive and obtained the permanence condition of system (3). We also observed that constant time delay and the growth rate of the immature predator can bring obvious effects on the dynamics of the system, and the stability and extinction (or prey and predators coexist) of the system are determined by their thresholds. Thus, from Theorems 1 and 2, we can easily guess that there must exist thresholds τ_1^* and τ_2^* . If $\tau_1 > \tau_1^*$ and $\tau_2 > \tau_2^*$, then the immature and mature predator tends to be extinct; if $\tau_1 < \tau_1^*$ and $\tau_2 < \tau_2^*$, then the system will be permanent. If immature predator growth rates are below (or above) their thresholds, then the predators become extinct

(or prey and predators coexist) in all patches. Hence, the immature predator growth rates and the predator's maturation times play an important role in delayed functional response predator-prey Gompertz systems with impulsive diffusion between the two patches. We hope that the results will provide a reliable tactic basis for biological resource protection. In addition, it is meaningful to generalize the results to the case of n -patch case, but it is difficult to discuss the properties of the n -dimension stroboscopic map. One may instigate this problem in the future.

Data Availability

No data or codes were generated or used during this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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