Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 5293748, 10 pages
https://doi.org/10.1155/2020/5293748

Research Article

Hopf Bifurcation and Turing Instability Analysis for the Gierer–Meinhardt Model of the Depletion Type

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Received 12 August 2019; Accepted 12 September 2019; Published 22 January 2020

Academic Editor: Kousuke Kuto

The reaction diffusion system is one of the important models to describe the objective world. It is of great guiding importance for people to understand the real world by studying the Turing patterns of the reaction diffusion system changing with the system parameters. Therefore, in this paper, we study Gierer–Meinhardt model of the Depletion type which is a representative model in the reaction diffusion system. Firstly, we investigate the stability of the equilibrium and the Hopf bifurcation of the system. The result shows that equilibrium experiences a Hopf bifurcation in certain conditions and the Hopf bifurcation of this system is supercritical. Then, we analyze the system equation with the diffusion and study the impacts of diffusion coefficients on the stability of equilibrium and the limit cycle of system. Finally, we perform the numerical simulations for the obtained results which show that the Turing patterns are either spot or stripe patterns.

1. Introduction

As early as 1952s, the famous British mathematician Turing turned his attention to the field of biology and succeeded with a reaction diffusion system. Meanwhile, the principle of surface pattern generation in some organisms is illustrated in his well-known paper [1]. Turing also mathematically showed that in a reaction-diffusion system, the steady state is unstable under certain conditions and spontaneously generates spatial stationary patterns and the pattern is usually called the Turing patterns [2].

It is worth nothing that Turing patterns have long been widely found in nature and in many experimental systems, such as real chemical system [3–5], spiral galaxies in space [6], spiral wave electrical signals of myocardial tissue [7], biology systems [8, 9], hyper-points in nonlinear optical systems [10], etc. To this end, the exploration and analysis of the Turing patterns has attracted the attention of many scholars. For instance, in their seminal paper, Gierer and Meinhardt [8] proposed a kind of reaction diffusion system, which contains Turing patterns. For the Gierer–Meinhardt system, there are a lot of related research work on it. Ruan [11] has studied the instability of the homogeneous equilibrium and periodic solution under different diffusion coefficients. Kolokolnikov et al. [12] have analyzed the stability of a stripe for the Gierer–Meinhardt system and the effect of saturation. Ghergu [13] has considered steady state solutions in the Gierer–Meinhardt system with Dirichlet boundary condition. An et al. [14] have studied the explicit solution to the initial-boundary value problem of Gierer–Meinhardt model under certain conditions.

Although much work has been done in this research field, most researchers are mainly concerned with the Activator-Inhibitor model of Gierer–Meinhardt system. However, the Depletion model as a remarkable type of reaction diffusion system with its research value is also very intuitive. Therefore, we will mainly concentrate on Hopf bifurcation and Turing instability analysis for Depletion model in this paper. More specifically, Depletion model has the following form

$$\begin{aligned} \frac{\partial a}{\partial t} &= \rho_a a^2 s - \mu_a a + \sigma_a + D_a \frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial s}{\partial t} &= \rho_s a^2 - \mu_s s + \sigma_s + D_s \frac{\partial^2 s}{\partial x^2}, \end{aligned} \tag{1}$$

where $\rho(x)$ is the sources of distribution, $a(x,t)$ and $s(x,t)$ represent the density of the activator and consumed by activation, and $D_a, D_s, \mu_a, \mu_s, \rho_a, \rho_s, \sigma_a, \sigma_s$ are positive constants.

In addition, Gierer and Meinhardt set $\mu_s = \sigma_s = \rho_s$, $\rho_a = \mu_a = D_s = 1$, then system becomes

$$\begin{aligned} \frac{\partial a}{\partial t} &= sa^2 - a + \sigma_a + D_a \frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial s}{\partial t} &= -\mu_s sa^2 + \mu_s + \frac{\partial^2 s}{\partial x^2}. \end{aligned} \tag{2}$$

In this paper, we will follow the same settings as above, and for notational convenience, we use D, σ, μ instead of D_a, σ_a, μ_s, thus the system is expressed as

$$\begin{aligned} \frac{\partial a}{\partial t} &= sa^2 - a + \sigma + D \frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial s}{\partial t} &= \mu\left(1 - sa^2\right) + \frac{\partial^2 s}{\partial x^2}. \end{aligned} \tag{3}$$

The rest of paper is organized as follows: In Section 2, we analyze the existence and stability of the positive equilibrium and the Hopf bifurcation. Section 3 studies the Turing instability of the equilibrium and limit cycles, and theoretically give the sufficient conditions and periodic solutions for the equilibrium and spatial homogeneous Turing instability. Lastly, in Section 4, we give examples to illustrate the analytic conditions and the numerical simulations are presented to verify the theoretical analysis.

2. Stability Analysis of the Equilibrium

In this section, we mainly consider the system equation without diffusion. Thus, we can write system (4) as

$$\begin{aligned} \frac{da}{dt} &= sa^2 - a + \sigma, \\ \frac{ds}{dt} &= \mu\left(1 - sa^2\right). \end{aligned} \tag{4}$$

Obviously, there is a unique equilibrium $(a^*, s^*) = \left(1 + \sigma, 1/(1+\sigma)^2\right)$. The Jacobian matrix of (4) is

$$J = \begin{pmatrix} 2as - 1 & a^2 \\ -2as\mu & -a^2\mu \end{pmatrix}. \tag{5}$$

Therefore, the Jacobian matrix at the equilibrium (a^*, s^*) is

$$J(\mu) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} & (1+\sigma)^2 \\ -\dfrac{2\mu}{1+\sigma} & -(1+\sigma)^2\mu \end{pmatrix}, \tag{6}$$

and the corresponding characteristic equation is

$$\lambda^2 - \frac{1 - \sigma - \mu(1+\sigma)^3}{1+\sigma}\lambda + \mu(1+\sigma)^2 = 0. \tag{7}$$

It is easy to note that characteristic equation (7) has two eigenvalues

$$\begin{aligned} \lambda_{1,2} &= \frac{-Tr \pm \sqrt{Tr^2 - 4Det}}{2} \\ &= -\frac{\mu\left(1+\sigma^3\right) + \sigma - 1 \pm \sqrt{\left(\mu(1+\sigma)^3 + \sigma - 1\right)^2 - 4\mu(1+\sigma)^4}}{2(1+\sigma)}, \end{aligned} \tag{8}$$

and

$$\Delta = \frac{\mu^2(1+\sigma)^6 - 2(1+\sigma)^3(3+\sigma)\mu + (1-\sigma)^2}{(1+\sigma)^2}. \tag{9}$$

Let

$$\begin{aligned} \mu_0 &= \frac{1-\sigma}{(1+\sigma)^3}, \\ \mu_1 &= \frac{(3+\sigma) - 2\sqrt{2}\sqrt{1+\sigma}}{1-\sigma}\mu_0, \\ \mu_2 &= \frac{(3+\sigma) + 2\sqrt{2}\sqrt{1+\sigma}}{1-\sigma}\mu_0. \end{aligned} \tag{10}$$

We have the following Theorem 1.

Theorem 1. *System (4) has a unique positive equilibrium that is asymptotically stable if either condition $(H1)$ or condition $(H2)$ holds and is unstable if condition $(H3)$ holds.*

$$(H1)\ \sigma \geq 1.\ (H2)\ \mu >; \mu_0, \quad 0 < \sigma < 1;\ (H3)\ \mu < \mu_0, \quad 0 < \sigma < 1. \tag{11}$$

Furthermore. (i) The equilibrium (a^*, s^*) is a stable node if one of the following conditions is satisfied: (ia) $\sigma \geq 1, 0 < \mu \leq \mu_1$; (ib) $\sigma \geq 1, \mu \geq \mu_2$; (ic) $0 < \sigma < 1$, $\mu \geq \mu_2$. (ii) The equilibrium (a^*, s^*) is a stable foucs if one of the following conditions is satisfied: (iia) $\sigma \geq 1, \mu_1 < \mu < \mu_2$; (iib) $0 < \sigma < 1$, $\mu_0 < \mu < \mu_2$. (iii) The equilibrium (a^*, s^*) is an unstable node if $0 < \sigma < 1$ and $0 < \mu \leq \mu_1$. (iv) The equilibrium (a^*, s^*) is an unstable foucs if $0 < \sigma < 1$ and $\mu_1 < \mu \leq \mu_0$.

Proof. Assume condition $(H1)$ holds, then $\Delta < 0$ (≥ 0) if $\mu_1 < \mu < \mu_2 (0 < \mu \leq \mu_1$ or $\mu \geq \mu_2)$. Assume $0 < \sigma < 1$ holds, it is easy to verify that $\mu_1 < \mu_0 < \mu_2$ and we have $\Delta < 0$ (≥ 0) if $\mu_1 < \mu < \mu_0$ or $\mu_0 < \mu < \mu_2 (0 < \mu \leq \mu_1$ or $\mu \geq \mu_2)$.

From the above analysis, we note that the system (4) experiences a Hopf bifurcation at $\mu = \mu_0$ under condition $0 < \sigma < 1$.

Let $x = a - a^*$, $y = s - s^*$, then system (4) linearizes at equilibrium (a^*, s^*) to obtain the following system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = J(\mu)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x, y, \mu) \\ g_1(x, y, \mu) \end{pmatrix}, \tag{12}$$

where

$$f_1(x, y, \mu) = 2(1+\sigma)xy + x^2 y + \frac{x^2}{(1+\sigma)^2} + O(4), \tag{13}$$

$$g_1(x, y, \mu) = -2(1+\sigma)xy\mu - x^2 y\mu - \frac{x^2\mu}{(1+\sigma)^2} + O(4), \tag{14}$$

and $O(4)$ represents the remaining terms with order greater than or equal to 4.

According to the Hopf bifurcation theorem [15], we need nonzero transversally condition $(d/d\mu)Re(\lambda_{1,2})|_{\mu=\mu_0}$, in fact $(d/d\mu)Re(\lambda_{1,2})|_{\mu=\mu_0} = (1+\sigma)^2/2 > 0$. Nextly, we consider the Hopf bifurcation at the critical point $\mu = \mu_0$. For $\mu = \mu_0$, we can get $\lambda_{1,2}(\mu_0) = \pm i\omega_0$ and $\omega_0 = \sqrt{(1-\sigma)/(1+\sigma)} < 0$ and the Jacobian matrix is

$$J(\mu_0) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} & (1+\sigma)^2 \\ -\dfrac{2(1-\sigma)}{(1+\sigma)^4} & -\dfrac{1-\sigma}{1+\sigma} \end{pmatrix}. \tag{15}$$

It can be easily obtained that the eigenvector corresponding to the eigenvalue $i\omega_0$ of the matrix $J(\mu_0)$ is $\xi = \left(\left(-(1+\sigma)^3/2\right) - \left((1+\sigma)^3/2\omega_0\right)i, 1\right)^T$.

Setting

$$P = \begin{pmatrix} -\frac{(1+\sigma)^3}{2\omega_0} & -\frac{(1+\sigma)^3}{2} \\ 0 & 1 \end{pmatrix}, \tag{16}$$

and the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}$$

then (12) is converted into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_2(u,v,\mu_0) \\ g_2(u,v,\mu_0) \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned} f_2(u,v,\mu_0) = &-\frac{(1+\sigma)^3\omega_0}{4(1-\sigma)}u^2 + \frac{1}{2}(1+\sigma)^2 uv + \frac{3}{4}(1+\sigma)^2\omega_0 v^2 \\ &- \frac{(1+\sigma)^5\omega_0}{4(1-\sigma)}u^2 v - \frac{1}{2}(1+\sigma)^4 - \frac{(1+\sigma)^4\omega_0}{4}v^3 + O(4), \end{aligned} \tag{19}$$

$$\begin{aligned} g_2(u,v,\mu_0) = &-\frac{1}{4}(1+\sigma)^2 u^2 + \frac{1}{2}(1+\sigma)^2\omega_0 uv \\ &+ \frac{3}{4}\left(1-\sigma^2\right)v^2 - \frac{1}{4}(1+\sigma)^4 u^2 v - \frac{1}{2}(1+\sigma)^4\omega_0 uv^2 \\ &- \frac{1}{4}(1-\sigma)(1+\sigma)^3 v^3 + O(4). \end{aligned} \tag{20}$$

In order to determine the type of the Hopf bifurcation at equilibrium (a^*, s^*), according to [16], the type of bifurcation is determined by the following symbols.

$$\begin{aligned} \sigma' = &\frac{1}{16}\left(f_{2uuu} + g_{2uuv} + f_{2uvv} + g_{2vvv}\right) \\ &+ \frac{1}{16\omega_0}\left[f_{2uv}\left(f_{2uu} + f_{2vv}\right) - g_{2uv}\left(g_{2uu} + g_{2vv}\right)\right. \\ &\left. - f_{2uu}g_{2uu} + f_{2vv}g_{2vv}\right], \end{aligned} \tag{21}$$

where

$$f_{2uu} = -\frac{(1+\sigma)^3}{2\sqrt{1-\sigma^2}}, g_{2uu} = -\frac{1}{2}(1+\sigma)^2, \tag{22}$$

$$f_{2uv} = \frac{1}{2}(1+\sigma)^2, g_{2uv} = \frac{1}{2}(1+\sigma)\sqrt{1-\sigma^2}, \tag{23}$$

$$f_{2vv} = -\frac{3(-1+\sigma)(1+\sigma)^2}{2\sqrt{1-\sigma^2}}, g_{2vv} = -\frac{3}{2}(-1+\sigma)(1+\sigma), \tag{24}$$

$$f_{2uuu} = 0, g_{2uuu} = 0, \tag{25}$$

$$f_{2uuv} = -\frac{(1+\sigma)^5}{2\sqrt{1-\sigma^2}}, g_{2uuv} = -\frac{1}{2}(1+\sigma)^4, \tag{26}$$

$$f_{2uvv} = -(1+\sigma)^4, g_{2uvv} = -(1+\sigma)^3\sqrt{1-\sigma^2}, \tag{27}$$

$$f_{2vvv} = \frac{3(-1+\sigma)(1+\sigma)^4}{2\sqrt{1-\sigma^2}}, g_{2vvv} = \frac{3}{2}(-1+\sigma)(1+\sigma)^3. \tag{28}$$

To this end,

$$\begin{aligned} \sigma' = &\frac{1}{16}\left(g_{2uuv} + f_{2uvv} + g_{2vvv}\right) + \frac{1}{16\omega_0}\left[f_{2uv}\left(f_{2uu} + f_{2vv}\right)\right. \\ &\left. - g_{2uv}\left(g_{2uu} + g_{2vv}\right) - f_{2uu}g_{2uu} + f_{2vv}g_{2vv}\right] = \frac{-(1+\sigma)^4}{16(1-\sigma)} < 0. \end{aligned} \tag{29}$$

According to the above analysis, we can get the following Theorem 2. □

Theorem 2. *Assume $0 < \sigma < 1$, then system (4) at the equilibrium (a^*, s^*) experiences a Hopf bifurcation for $\mu = \mu_0$. Since $\sigma' < 0$, the Hopf bifurcation is supercritical and the bifurcated limit cycle is stable.*

3. Turing Instability Analysis

In this section, we will consider the system equation with diffusion and study the impacts of diffusion coefficients on the stability of equilibrium (a^*, s^*) and the limit cycle of system (3).

3.1. Instability Analysis of the Equilibrium. We first assume that condition $(H2)$ is established. Obviously, the equilibrium (a^*, s^*) is a stable for system (4) with Neumann boundary conditions

$$\begin{aligned} \frac{\partial a}{\partial x}(0,t) &= \frac{\partial a}{\partial x}(\pi,t) = 0, \\ \frac{\partial s}{\partial x}(0,t) &= \frac{\partial s}{\partial x}(\pi,t) = 0. \end{aligned} \tag{30}$$

We consider the diffusion system (3) in the Banach space $\mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$,where

$$\mathbb{H}^2([0,\pi]) = \left\{ w(\cdot,t)\frac{\partial^i w}{\partial x^i}(\cdot,t) \in L^2([0,\pi]),\ i = 0,1,2 \right\}. \tag{31}$$

It is easy to obtain that equilibrium (a^*, s^*) is a stable solution of (3) and (30). The equilibrium (a^*, s^*) is nonlinearly unstable for (3) and (30), if it is linearly unstable in $\mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$.

Let $u_1 = a - a^*, u_2 = s - s^*$, then diffusion system (3) is transformed into

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} \frac{1-\sigma}{1+\sigma} + d\partial_{xx} & (1+\sigma)^2 \\ -\frac{2\mu}{1+\sigma} & -\mu(1+\sigma)^2 + \partial_{xx} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \doteq L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{32}$$

with the boundary conditions

$$u_{1x}(0,t) = u_{2x}(0,t) = u_{1x}(\pi,t) = u_{2x}(\pi,t) = 0. \tag{33}$$

Let $(u_1, u_2) \in \mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$ be a solution of (32) and (33). Since (32) is linear, we can signify it as

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \sum_{k=0}^{\infty}\begin{pmatrix} A_k \\ S_k \end{pmatrix}e^{\lambda t + ikx}, \tag{34}$$

where $\lambda \in \mathbb{C}$ is the temporal spectrum, k is the wave number and A_k, S_k are real numbers for $k = 0, 1, 2 \cdots, n, \cdots$. Bring (34) into (32), we know

$$\sum_{k=0}^{\infty} \lambda \begin{pmatrix} A_k \\ S_k \end{pmatrix} e^{\lambda t + ikx} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{1-\sigma}{1+\sigma} + d\partial_{xx} & (1+\sigma)^2 \\ -\frac{2\mu}{1+\sigma} & -\mu(1+\sigma)^2 + \partial_{xx} \end{pmatrix} \begin{pmatrix} A_k \\ S_k \end{pmatrix} e^{\lambda t + ikx}. \tag{35}$$

Considering the like terms with the wave number of k, we have

$$(\lambda I - J_k(\mu)) \begin{pmatrix} A_k \\ S_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad k = 0, 1, 2, \cdots, n, \cdots, \tag{36}$$

where

$$J_k(\mu) = \begin{pmatrix} \frac{1-\sigma}{1+\sigma} - dk^2 & (1+\sigma)^2 \\ -\frac{2\mu}{1+\sigma} & -\mu(1+\sigma)^2 - k^2 \end{pmatrix}. \tag{37}$$

It is easy to verify that (36) has a nonzero solution if and only if

$$Det(\lambda I - J_k(\mu)) = 0. \tag{38}$$

We are searching conditions $\mathrm{Re}(\lambda(k)) > 0$ and $\lambda(k)$ to satisfy the equation

$$\lambda^2 - Tr(k)\lambda + Det(k) = 0, \quad k = 0, 1, 2, \cdots, n, \cdots, \tag{39}$$

where

$$Tr(k) = -(1+d)k^2 - \frac{\mu(1+\sigma)^3 + \sigma - 1}{1+\sigma}, \tag{40}$$

and

$$Det(k) = \frac{k^2\left(dk^2(1+\sigma) - (1-\sigma)\right) + dk^2\mu(1+\sigma)^3 + \mu(1+\sigma)^3}{1+\sigma}. \tag{41}$$

Under condition $(H2)$, we have $Tr(k) < 0$ for all $k = 0, 1, 2, \cdots, n, \cdots$ and $Det(0) > 0$.

In addition, the equilibrium (a^*, s^*) is still stable, if $Det(k) > 0$ for all $k = 0, 1, 2, \cdots, n, \cdots$. The equilibrium (a^*, s^*) will lose its stability and Turing pattern occurs, if for at least one $k \in N$ makes $Det(k) < 0$, namely, if $m^2 < (1-\sigma)/d(1+\sigma) \le (m+1)^2$ and $d < D_m$, then there exist at least one negative in $Det(1), Det(2) \ldots Det(m)$. Hence, the equilibrium (a^*, s^*) is unstable for system (3).

Basing on the above analysis, we have Theorem 3.

Theorem 3. *Assume condition $(H2)$ holds, let*

$$D_m = \min_{1 \le k \le m} \frac{k^2(1-\sigma) - \mu(1+\sigma)^3}{k^4(1+\sigma) + k^2\mu(1+\sigma)^3}, \tag{42}$$

then (a^*, s^*) is a stable equilibrium for system (3) if condition $(H4)$ holds and is an unstable equilibrium for system (3) if condition $(H5)$ holds.

$$(H4)\ d > \frac{1-\sigma}{1+\sigma}; (H5)\ m^2 < \frac{1-\sigma}{d(1+\sigma)} \le (m+1)^2, \quad d > D_m. \tag{43}$$

3.2. Instability Analysis of the Limit Cycle. In this subsection, we discuss the stability of the limit cycle in Theorem 3 under spatially inhomogeneous perturbations. Assume condition $(H3)$ is satisfied, then the supercritical Hopf bifurcation appears at $\mu = \mu_0$. Therefore, the limit cycle is stable under spatially homogeneous perturbation.

According to [17], let $u_1 = a - a^*, u_2 = s - s^*, \mu = \mu_0$, and $U = (u_1, u_2)^T$, then system (3) becomes

$$\begin{aligned} U_t &= \left[J(\mu_0) + D \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix} \right] U + F(U, \mu_0), \\ U_x(0,t) &= U_x(\pi, t) = (0,0)^T, \end{aligned} \tag{44}$$

where

$$J(\mu_0) = \begin{pmatrix} \frac{1-\sigma}{1+\sigma} & (1+\sigma)^2 \\ -\frac{2(1-\sigma)}{(1+\sigma)^4} & -\frac{1-\sigma}{1+\sigma} \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \tag{45}$$

$$F(U, \mu_0) = (f_1(u_1, u_2, \mu_0), g_1(u_1, u_2, \mu_0))^T, \tag{46}$$

and f_1 and g_1 are defined in (13) and (14), respectively.

According to [18], the form of $F(U, \mu_0)$ is as follows

$$F(U, \mu_0) = \frac{1}{2}Q(U,U) + \frac{1}{6}C(U,U,U) + O\left(|U|^4\right), \tag{47}$$

where Q and C are in the following form

$$\begin{aligned} Q(U,U) &= (Q_1(U,U), Q_2(U,U))^T, \\ C(U,U,U) &= (C_1(U,U,U), C_2(U,U,U))^T, \end{aligned} \tag{48}$$

with

$$\begin{aligned} Q_1(U,V) &= f_{2uu}u_1v_1 + f_{2uv}u_1v_2 + f_{2vu}u_2v_1 + f_{2vv}u_2v_2 \\ &= \frac{2}{(1+\sigma)^2}\left(u_1v_1 + (1+\sigma)^3u_1v_2 + (1+\sigma)^3u_2v_1\right), \end{aligned} \tag{49}$$

$$\begin{aligned} Q_2(U,V) &= g_{2uu}u_1v_1 + g_{2uv}u_1v_2 + g_{2vu}u_2v_1 + g_{2vv}u_2v_2 \\ &= \frac{2(\sigma-1)\left(u_1v_1 + u_1v_2(1+\sigma)^3 + (1+\sigma)^3u_2v_1\right)}{(1+\sigma)^5}, \end{aligned} \tag{50}$$

$$\begin{aligned} C_1(U,V,W) &= f_{2uuu}u_1v_1w_1 + f_{2uuv}u_1v_1w_2 + f_{2uvu}u_1v_2w_1 \\ &\quad + f_{2uvv}u_1v_2w_2 + f_{2vuu}u_2v_1w_1 + f_{2vuv}u_2v_1w_2 \\ &\quad + f_{2vvu}u_2v_2w_1 + f_{2vvv}u_2v_2w_2 \\ &= 2(u_1v_1w_2 + u_1v_2w_1 + u_2v_1w_1), \end{aligned} \tag{51}$$

$$\begin{aligned} C_2(U,V,W) &= g_{2uuu}u_1v_1w_1 + g_{2uuv}u_1v_1w_2 + g_{2uvu}u_1v_2w_1 \\ &\quad + g_{2uvv}u_1v_2w_2 + g_{2vuu}u_2v_1w_1 + g_{2vuv}u_2v_1w_2 \\ &\quad + g_{2vvu}u_2v_2w_1 + g_{2vvv}u_2v_2w_2 \\ &= \frac{2(\sigma-1)(u_1v_1w_2 + u_1v_2w_1 + u_2v_1w_1)}{(1+\sigma)^3}, \end{aligned} \tag{52}$$

for any $U = (u_1, u_2)^T$, $V = (v_1, v_2)^T$, $W = (w_1, w_2)^T$, and $U, V, W \in \mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$.

The linear operator L defined in (30) for $\mu = \mu_0$ is

$$LU = \left[J(\mu_0) + D\begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix}\right]U, \tag{53}$$

for $U \in \mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$.

Let L^* be the conjugate adjoint operator of L defined in $\mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$, then

$$L^*U = \left[J^*(\mu_0) + D\begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix}\right]U, \tag{54}$$

where

$$J^*(\mu_0) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} & \dfrac{2(-1+\sigma)}{(1+\sigma)^4} \\ (1+\sigma)^2 & \dfrac{-1+\sigma}{1+\sigma} \end{pmatrix}. \tag{55}$$

It is easy to note that $\langle L^*U, V\rangle = \langle U, LV\rangle$ for any $U, V \in \mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$ and define the inner product of $\langle U, V\rangle = 1/\pi \times \int_0^\pi \overline{U}^T V dx$ on $\mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$.

The linearized system of (44) at the equilibrium $(0,0)$ is

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{56}$$

with the boundary conditions

$$U_x(0,t) = U_x(\pi,t) = (0,0)^T. \tag{57}$$

Let $U = (u_1, u_2)^T \in \mathbb{H}^2([0,\pi]) \times \mathbb{H}^2([0,\pi])$ is a solution of (56) and (57). Since (56) is linear, we can denote it as

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ s_k \end{pmatrix} e^{\lambda t + ikx}, \tag{58}$$

where $\lambda \in \mathbb{C}$ is the temporal spectrum, k is the wave number and a_k, s_k are real numbers for $k = 0, 1, 2 \cdots, n, \cdots$. Plugging (58) into (56), we have

$$\sum_{k=0}^{\infty} \lambda \begin{pmatrix} a_k \\ s_k \end{pmatrix} e^{\lambda t + ikx} = \sum_{k=0}^{\infty} L_k \begin{pmatrix} a_k \\ s_k \end{pmatrix} e^{\lambda t + ikx}. \tag{59}$$

Nextly, we consider like terms with the wave number of k, we have

$$(\lambda I - L_k)\begin{pmatrix} a_k \\ s_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad k = 0, 1, 2, \cdots, n, \cdots, \tag{60}$$

where

$$L_k = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} - dk^2 & (1+\sigma)^2 \\ -\dfrac{2(1-\sigma)}{(1+\sigma)^4} & -\dfrac{1-\sigma}{1+\sigma} - k^2 \end{pmatrix}. \tag{61}$$

It is obvious to note that (60) has a nonzero solution if and only if

$$det(\lambda I - L_k) = 0. \tag{62}$$

We are searching conditions such that $\mathrm{Re}(\lambda(k)) > 0$ and $\lambda(k)$ satisfies equation

$$\lambda^2 - T_k\lambda + D_k = 0, \quad k = 0, 1, 2, \cdots, n, \cdots, \tag{63}$$

where

$$T_k = -(1+d)k^2, \tag{64}$$

and

$$D_k = \frac{k^2\left(dk^2(1+\sigma) - (1-\sigma)\right)}{1+\sigma} + \left(dk^2 + 1\right)\omega_0^2. \tag{65}$$

we have $T_0 = 0$ and $T_k < 0$ for all $k = 1, 2, \cdots, n, \cdots$ and $D_0 = \omega_0^2 > 0$, if condition $\mu = \mu_0$ holds. Then, it follows that for $k = 0$, L has eigenvalues with zero real parts. We need to proceed to the center manifold reduction.

Firstly, if $d > 1 - \sigma/1 + \sigma$ then $D_k > (d+1)\omega_0^2 > 0$ for all $k = 1, 2, \cdots, n, \cdots$. In addition, if $m^2 < (1-\sigma)/(d(1+\sigma)) \le (m+1)^2$ and $d > \overline{D}$, there exists at least one negative in $D_1, D_2 \cdots D_m$.

Let $Lq = i\omega_0 q$ and $L^*q^* = -i\omega_0 q^*$, by calculation we can get $q = \left((-(1+\sigma)^3(1-\sigma) - i\omega_0(1+\sigma)^4)/2(1-\sigma), 1\right)^T$ and $q^* = \left(\left((1-\sigma)/-\omega_0(1+\sigma)^4\right)i, (\omega_0(1+\sigma) - (1-\sigma)i)/2\omega_0(1+\sigma)\right)^T$, respectively. It is easy to verify that $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

According to [18], let

$$U = zq + \overline{zq} + w, \quad z = \langle q^*, U\rangle, w = (w_1, w_2)^T, \tag{66}$$

and

$$\begin{aligned} u_1 &= \left(\tfrac{-(1+\sigma)^3(1-\sigma) - i\omega_0(1+\sigma)^4}{2(1-\sigma)}\right)z + \left(\tfrac{-(1+\sigma)^3(1-\sigma) + i\omega_0(1+\sigma)^4}{2(1-\sigma)}\right)\overline{z} + w_1, \\ u_2 &= z + \overline{z} + w_2. \end{aligned} \tag{67}$$

The coordinate of the system (44) on (z, w) is converted to

$$\begin{cases} \dot{z} = i\omega_0 z + \left\langle q^*, \tilde{f}\right\rangle, \\ \dot{w} = Lw + H(z, \overline{z}, w), \end{cases} \tag{68}$$

where

$$\tilde{f} = F(zq + \overline{z}\,\overline{q} + w, \mu_0), H(z, \overline{z}, w) = \tilde{f} - \left\langle q^*, \tilde{f}\right\rangle q - \left\langle \overline{q}^*, \tilde{f}\right\rangle \overline{q}, \tag{69}$$

and

$$\tilde{f} = \frac{1}{2}Q(q,q)z^2 + Q(q,\overline{q})z\overline{z} + \frac{1}{2}Q(\overline{q},\overline{q})\overline{z}^2 + O\left(|z|^3, |z|\cdot|w|, |w|^2\right), \tag{70}$$

$$\begin{aligned} \left\langle q^*, \tilde{f}\right\rangle &= \frac{1}{2}\langle q^*, Q(q,q)\rangle z^2 + \langle q^*, Q(q,\overline{q})\rangle z\overline{z} \\ &\quad + \frac{1}{2}\langle q^*, Q(\overline{q},\overline{q})\rangle \overline{z}^2 + O\left(|z|^3, |z|\cdot|w|, |w|^2\right), \end{aligned} \tag{71}$$

$$\begin{aligned} \left\langle \overline{q}^*, \tilde{f}\right\rangle &= \frac{1}{2}\langle \overline{q}^*, Q(q,q)\rangle z^2 + \langle \overline{q}^*, Q(q,\overline{q})\rangle z\overline{z} \\ &\quad + \frac{1}{2}\langle \overline{q}^*, Q(\overline{q},\overline{q})\rangle \overline{z}^2 + O\left(|z|^3, |z|\cdot|w|, |w|^2\right). \end{aligned} \tag{72}$$

So $H(z, \overline{z}, w) = (1/2)z^2 H_{20} + z\overline{z}H_{11} + (1/2)\overline{z}^2 H_{02} + O\left(|z|^3, |z|\cdot|w|, |w|^2\right)$, where

$$\begin{aligned} H_{20} &= Q(q,q) - \langle q^*, Q(q,q)\rangle q - \langle \overline{q}^*, Q(q,q)\rangle \overline{q}, \\ H_{11} &= Q(q,\overline{q}) - \langle q^*, Q(q,\overline{q})\rangle q - \langle \overline{q}^*, Q(q,\overline{q})\rangle \overline{q}, \\ H_{02} &= Q(\overline{q},\overline{q}) - \langle q^*, Q(\overline{q},\overline{q})\rangle q - \langle \overline{q}^*, Q(\overline{q},\overline{q})\rangle \overline{q}. \end{aligned} \tag{73}$$

Besides, we have

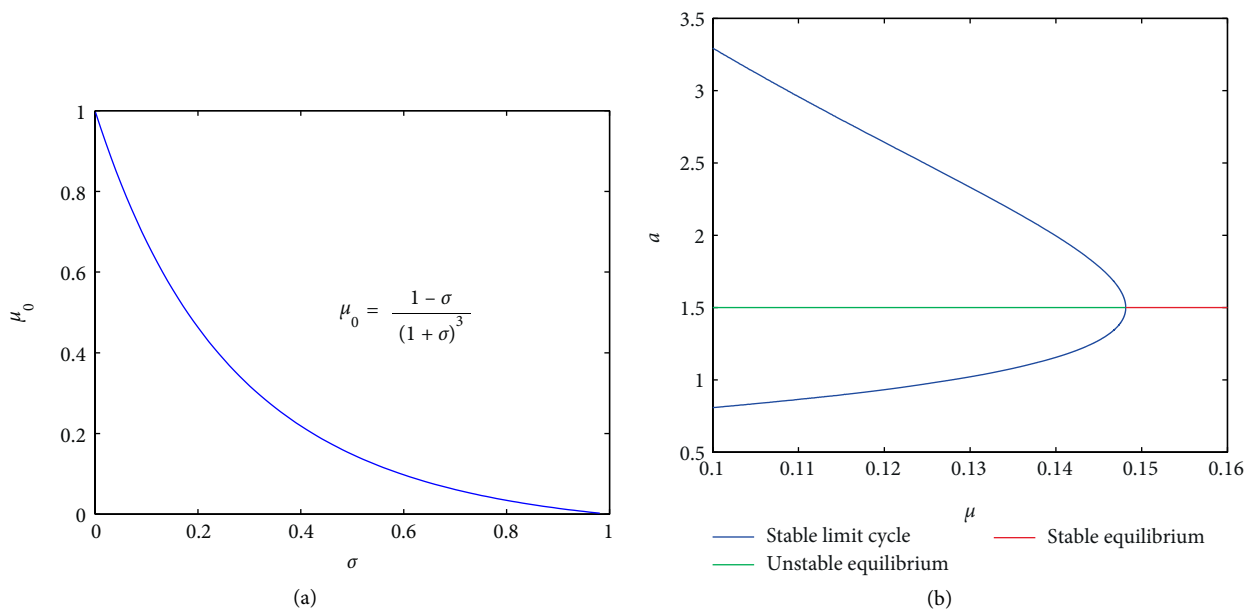

Figure 1: (a) The supercritical Hopf bifurcation diagrams of system (4) and (b) Hopf bifurcation on a two-dimensional system in polar coordinates of system (4).

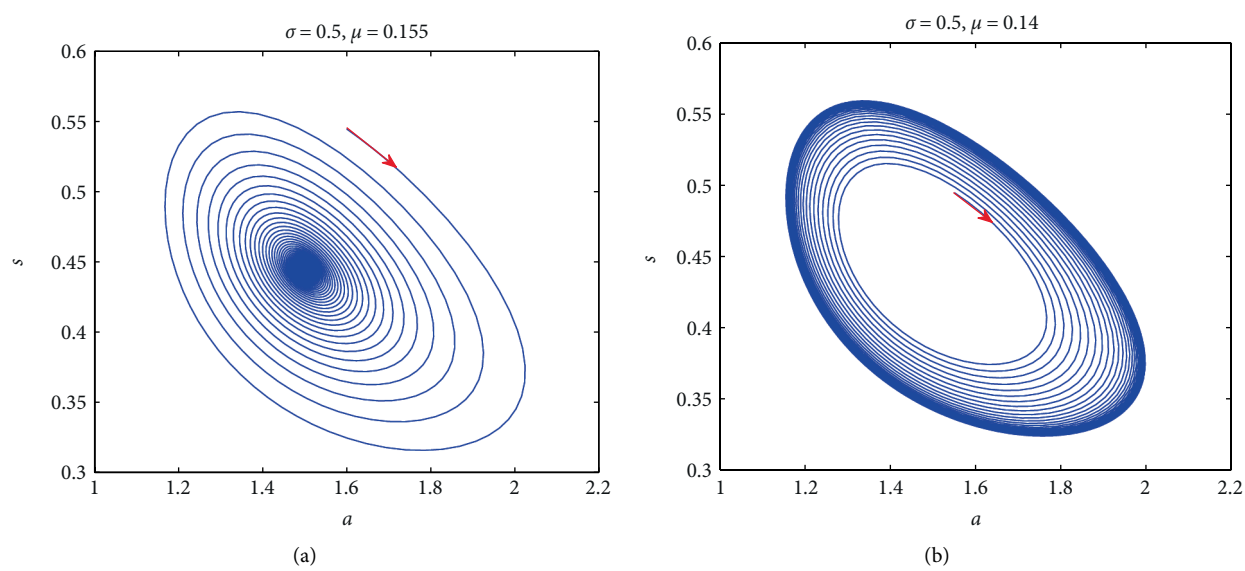

Figure 2: (a) The phase orbit starting from $(a_0, s_0) = (1.600,\ 0.5444)$ coverages to equilibrium $(a^*, s^*) = (1.500, 0.4444)$ for $\sigma = 0.5, \mu = 0.155$ and (b) equilibrium is unstable and phase orbit starting from $(a_0, s_0) = (1.550,\ 0.4944)$ coverages to stable limit cycle for $\sigma = 0.5, \mu = 0.14$.

$$H_{20} = H_{11} = H_{02} = (0,0)^T. \tag{74}$$

Hence, $H(z,\bar{z},w) = O\left(|z|^3, |z|\cdot|w|, |w|^2\right)$.

According to [18], the system possesses a central manifold, which we write as

$$w = \frac{1}{2}z^2 w_{20} + z\bar{z}w_{11} + \frac{1}{2}\bar{z}^2 w_{02} + O\left(|z|^3\right). \tag{75}$$

Since $Lw + H(z,\bar{z},w) = \dot{w} = (\partial w/\partial z)\dot{z} + (\partial w/\partial \bar{z})\dot{\bar{z}}$, then

$$\begin{aligned} w_{20} &= \left[2i\omega_0 - L\right]^{-1} H_{20} = \left[2i\omega_0 - J(\mu_0)\right]^{-1} H_{20} = (0,0)^T, \\ w_{11} &= -L^{-1}H_{11} = -J^{-1}(\mu_0)H_{11} = (0,0)^T, \\ w_{02} &= \left[-2i\omega_0 - L\right]^{-1} H_{02} = \left[-2i\omega_0 - J(\mu_0)\right]^{-1} H_{02} = (0,0)^T, \\ w &= O\left(|z|^3\right). \end{aligned} \tag{76}$$

In consequence, the diffusion system restricted to the center manifold is

$$\dot{z} = i\omega_0 z + \left\langle q^*, \tilde{f}\right\rangle = i\omega_0 z + \sum_{2\le i+j\le 3} \frac{g_{ij}}{i!\,j!} z^i \bar{z}^j + O\left(\left|z^4\right|\right), \tag{77}$$

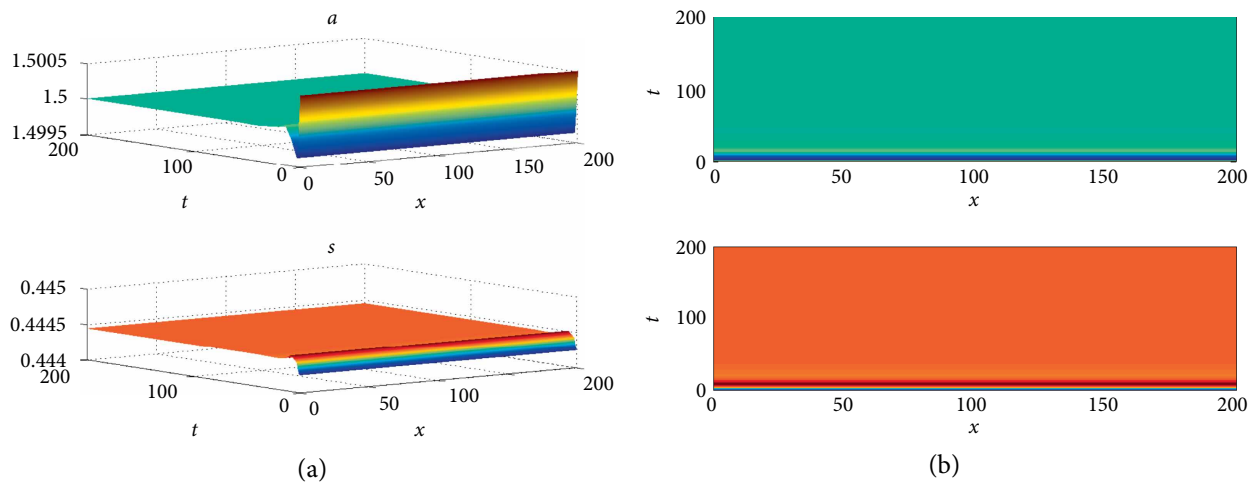

FIGURE 3: (a) The stable equilibrium $(a^*, s^*) = (1.500, 0.4444)$ of (4) is still stable for (3) for $\sigma = 0.5$, $\mu = 0.155$ and $d = 0.38$ and (b) the projection of (a) in (x, t) coordinates.

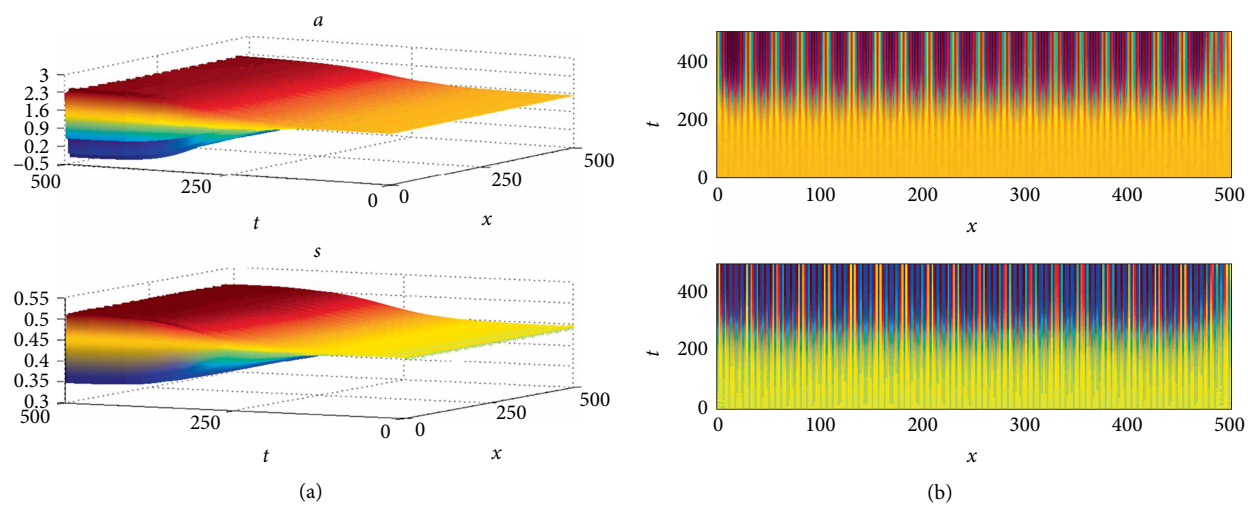

FIGURE 4: (a) The stable equilibrium $(a^*, s^*) = (1.500, 0.4444)$ of (4) becomes unstable for (3) for $\sigma = 0.5$, $\mu = 0.155$ and $d = D_m + 0.05$ and (b) the projection of (a) in (x, t) coordinates.

where

$$g_{20} = \langle q^*, Q(q,q)\rangle, g_{11} = \langle q^*, Q(q,\bar{q})\rangle,$$
$$g_{02} = \langle q^*, Q(\bar{q},\bar{q})\rangle, g_{21} = \langle q^*, C(q,q,\bar{q})\rangle. \quad (78)$$

The dynamics of the system (68) are determined by (77), and the normal form of Poincaré of (44) is expressed as

$$\dot{z} = (\alpha(\mu) + i\omega(\mu))z + z\sum_{j=1}^{M} c_j(\mu)(z\bar{z})^j, \quad (79)$$

where z is a complex variable, $M \geq 1$, and $c_j(\mu_0)$ is a complex-valued coefficient. So we can obtain

$$c_1(\mu_0) = \frac{g_{20}g_{11}[3\alpha(\mu_0) + i\omega(\mu_0)]}{2[\alpha^2(\mu_0) + \omega^2(\mu_0)]} + \frac{|g_{11}|^2}{\alpha(\mu_0) + i\omega(\mu_0)} + \frac{|g_{02}|^2}{2[\alpha(\mu_0) + 3i\omega(\mu_0)]} + \frac{g_{21}}{2}. \quad (80)$$

Owing to $\alpha(\mu_0) = 0$ and $\omega(\mu_0) = \omega_0 > 0$, then

$$Re(c_1(\mu_0)) = Re\left[\frac{g_{20}g_{11}}{2\omega_0}i + \frac{g_{21}}{2}\right]. \quad (81)$$

Since

$$g_{20} = \langle q^*, Q(q,q)\rangle = \frac{(1+\sigma)(3\omega_0 - i)}{2\omega_0}, \quad (82)$$

$$g_{11} = \langle q^*, Q(q,\bar{q})\rangle = \frac{(2\sigma^2 + \sigma - 1)(i - \omega_0)}{2\omega_0}, \quad (83)$$

$$g_{21} = \langle q^*, C(q,q,\bar{q})\rangle = \frac{(1+\sigma)^3(i - 3\omega_0)}{2\omega_0}, \quad (84)$$

we have $Re[c_1(\mu_0)] = -(1+\sigma)^4/4(1-\sigma) < 0$. As a consequence, the supercritical Hopf bifurcation occurs at $\mu = \mu_0$.

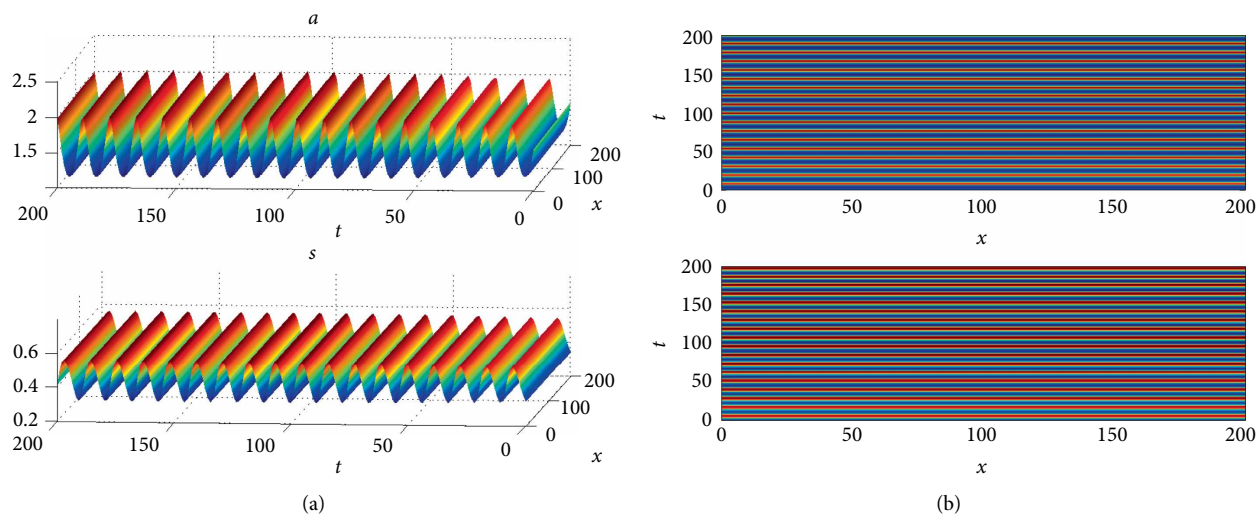

FIGURE 5: (a) The stable limit cycle of (4) is still stable for (3) for $\sigma = 0.5, \mu = 0.14$, and $d = 0.35$ and (b) the projection of (a) in (x, t) coordinates.

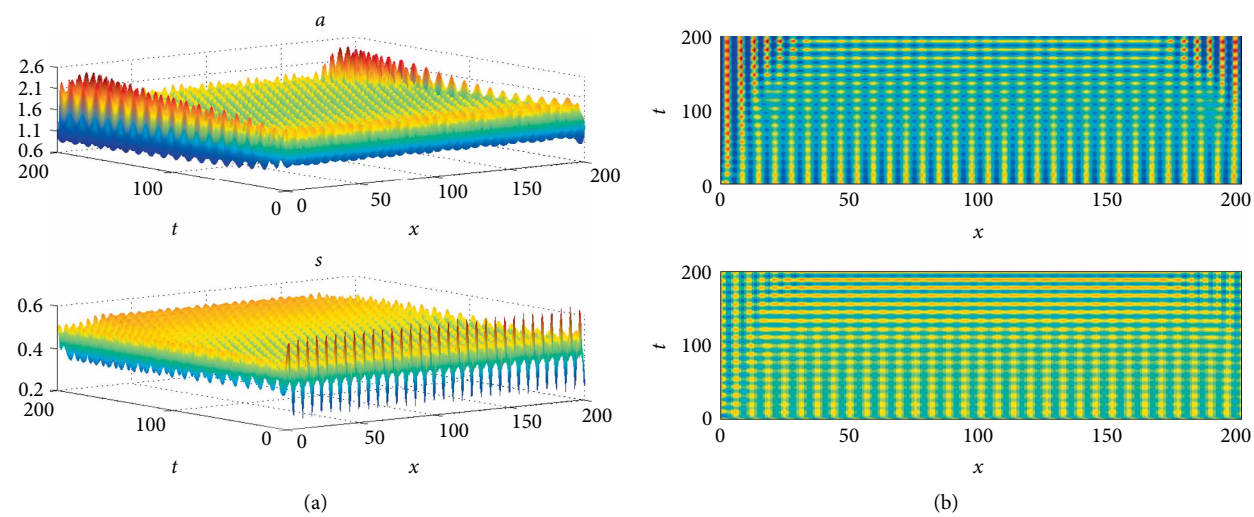

FIGURE 6: (a) The stable limit cycle of (4) becomes unstable for (3) for $\sigma = 0.5, \mu = 0.14$, and $d = \overline{D} + 0.05$ and (b) the projection of (a) in (x, t) coordinates.

We summarize the above analysis and get the following Theorem 4.

Theorem 4. *Assume condition $(H3)$ holds, the spatially homogeneous periodic solution of system (4) bifurcated from the equilibrium is stable. Let*

$$\overline{D} = \min_{1 \le k \le m} \frac{-1 + \sigma + k^2(1 - \sigma)}{k^4(1 + \sigma) + k^2(1 - \sigma)}. \tag{85}$$

The spatially homogeneous periodic solution for system (3) is stable if condition $(H6)$ holds and unstable if condition $(H7)$ holds.

$$(H6)\, d > \frac{1 - \sigma}{1 + \sigma}; (H7)\, m^2 < \frac{1 - \sigma}{d(1 + \sigma)} \le (m + 1)^2, \quad d > \overline{D}. \tag{86}$$

4. Numerical Simulations

In this section, we will use numerical simulations to illustrate the results in Sections 2 and 3.

A Hopf bifurcation occurs when a periodic solution or limit cycle, surrounding an equilibrium, arises or goes away as a parameter μ varies. When a stable limit cycle surrounds an unstable equilibrium, the bifurcation is called a supercritical Hopf bifurcation [19].

Firstly, we draw the supercritical Hopf bifurcation diagram of (4) in parameter space (σ, μ_0), and we also draw a Hopf bifurcation on a two-dimensional system in polar coordinates of (4), please see Figures 1(a) and 1(b), respectively.

Furthermore, we set $\sigma = 0.5$, then $\mu_0 = 0.1481$, the equilibrium $(a^*, s^*) = (1.500, 0.4444)$. Let $\mu = 0.155$ (here $\mu > \mu_0$), so that condition $(H2)$ in Theorem 1 is satisfied, and

the equilibrium (a^*, s^*) is asymptotically stable. Let $\mu = 0.14$ (here $\mu < \mu_0$), so that condition $(H3)$ in Theorem 1 is satisfied, and the equilibrium (a^*, s^*) is unstable and phase orbit converges to the stable limit cycle. Figure 2(a) shows the equilibrium (a^*, s^*) is asymptotically stable from initial values $(a_0, s_0) = (1.600, 0.5444)$. Figure 2(b) shows that phase orbit starting from $(a_0, s_0) = (1.550, 0.4944)$ converges to the stable limit cycle.

Furthermore, we introduce the effect of diffusion on the equilibrium and the spatial homogeneous periodic solution.

Let $\sigma = 0.5, \mu = 0.155, d = 0.38$, then $(a^*, s^*) = (1.5000, 0.4444)$, $\mu_0 = 0.1481, (1-\sigma)/(1+\sigma) = 0.3333$,the conditions $(H2)$ and $(H4)$ of Theorem 3 hold. Thus, the stable equilibrium $(a^*, s^*) = (1.500, 0.4444)$ of system (4) is still stable for system (3). Please see Figures 3(a) and 3(b). Let $\sigma = 0.5$, $\mu = 0.155$, $d = D_m + 0.05$, then $(a^*, s^*) = (1.500, 0.4444)$, $\mu_0 = 0.1481$, $D_m = 0$, the conditions $(H2)$ and $(H5)$ of Theorem 3 hold. As a result, the stable equilibrium $(a^*, s^*) = (1.500, 0.4444)$ of system (4) becomes unstable for system (3) because of diffusion. Please see Figures 4(a) and 4(b).

Let $\sigma = 0.5, \mu = 0.14, d = 0.35$, then the equilibrium $(a^*, s^*) = (1.500, 0.4444), \mu_0 = 0.1481, (1-\sigma)/(1+\sigma) = 0.3333$, the conditions $(H3)$ and $(H6)$ of Theorem 4 hold. Hence, the stable limit cycle of system (4) is still stable for system (3). Please see Figures 5(a) and 5(b). Let $\sigma = 0.5, \mu = 0.14, d = \overline{D} + 0.05$, then the equilibrium $(a^*, s^*) = (1.500, 0.4444), \mu_0 = 0.1481, \overline{D} = 0$, the conditions $(H3)$ and $(H7)$ of Theorem 4 hold. Thus, the stable limit cycle of system (4) becomes unstable for system (3) because of diffusion.Please see Figures 6(a) and 6(b).

5. Conclusions

Turing pattern dynamics of Gierer–Meinhardt model of the Depletion type is demonstrated in this paper. Through the mathematical analysis, we note that the system (4) undergoes a Hopf bifurcation at the equilibrium (a^*, s^*) for $\mu = \mu_0$ and the Hopf bifurcation is supercritical. Under the conditions of diffusion, some conditions of the Turing instability are obtained. To this end, the system (3) will have diffusion-driven instability and some spot or stripe patterns will be possibly formed. In addition, to further verify the validity of theoretical analysis, the numerical simulation methods are employed. From the outcome of numerical simulation, the complex dynamics does happen in Gierer–Meinhardt model of the Depletion type. In particular, we can note that the parameters are different and the patterns formed will be different. To sum up, the results will help to understand the formation of biological patterns and the method provides us with an understanding of the dynamical complexity of space and time in the Depletion model. More interesting and complex behavior about such model will further be explored in the future.

Data Availability

The data in this paper are generated in numerical simulations.

Conflicts of Interest

The author declares no conflicts of interest.

Acknowledgments

This work was supported by National Natural Science Foundation of China (No.71403022) and Beijing Municipal Social Science Foundation (No.18YJB011).

References

[1] A. M. Turing, "The chemical basis of morphogenesis," *Philosophical Transactions of the Royal Society of London*, vol. 237, no. 641, pp. 37–72, 1952.

[2] R.-P. Zhang, W. Zhen, Y. Wang, and Z.-J. Han, "Application of reaction diffusion model in turing pattern and numerical simulation," *Acta Physica Sinica*, vol. 67, no. 5, Article ID 050503, 2018.

[3] V. V. Castets, E. Dulos, J. Boissonade, and P. De Kepper, "Experimental evidence of a sustained standing turing-type nonequilibrium chemical pattern," *Physical Review Letters*, vol. 64, no. 24, pp. 2953–2956, 1990.

[4] P. D. Kepper, V. Castets, E. Dulos, and J. Boissonade, "Turing-type chemical patterns in the chlorite-iodide-malonic acid reaction," *Physica D: Nonlinear Phenomena*, vol. 49, no. 1–2, pp. 161–169, 1991.

[5] Q. Ouyang and H. L. Swinney, "Transition from a uniform state to hexagonal and striped turing patterns," *Nature*, vol. 352, no. 6336, pp. 610–612, 1991.

[6] W. Landry and B. T. Werner, "Computer simulations of self-organized wind ripple patterns," *Physica D: Nonlinear Phenomena*, vol. 77, no. 1–3, pp. 238–260, 1994.

[7] E. Pampaloni, S. Residori, S. Soria, and F. T Arecchi, "Phase locking in nonlinear optical patterns," *Physical Review Letters*, vol. 78, no. 6, pp. 1042–1045, 1997.

[8] A. Gierer and H. Meinhardt, "A theory of biological pattern formation," *Kybernetik*, vol. 12, no. 1, pp. 30–39, 1972.

[9] J. Wang, X. Hou, and Z. Jing, "Stripe and spot patterns in a Gierer–Meinhardt activator–inhibitor model with different sources," *International Journal of Bifurcation and Chaos*, vol. 25, no. 8, Article ID 1550108, 2015.

[10] H. Y. Liu, C. Y. Yang, and G. N. Tang, "Effects of the aging and systole of heart on the dynamics of spiral wave," *Acta Physica Sinica*, vol. 62, no. 1, Article ID 010505, 2013.

[11] S. Ruan, "Diffusion-driven instability in the Gierer–Meinhardt model of morphogenesis," *Natural Resource Modeling*, vol. 11, no. 2, pp. 131–141, 1998.

[12] T. Kolokolnikov, W. Sun, M. Ward, and J. Wei, "The stability of a stripe for the Gierer–Meinhardt model and the effect of saturation," *Siam Journal on Applied Dynamical Systems*, vol. 5, no. 2, pp. 313–363, 2006.

[13] M. Ghergu, "Steady-state solutions for Gierer–Meinhardt type systems with Dirichlet boundary condition," *Transactions of the American Mathematical Society*, vol. 361, no. 8, pp. 3953–3976, 2009.

[14] X. W. An, Z. He, and X. F. Song, "The explicit solution to the initial-boundary value problem of Gierer–Meinhardt model," *Applied Mathematics Letters*, vol. 80, pp. 59–63, 2018.

[15] W. Stephen and D. S. Mazel, "Introduction to applied nonlinear dynamical systems and chaos," *Computers in Physics*, vol. 4, no. 5, pp. 562–563, 1990.

[16] J. E. Marsden and M. McCracken, *The Hopf Bifurcation and its Applications*, Springer, NY, 1976.

[17] M. G. Crandall and P. H. Rabinowitz, "The Hopf bifurcation theorem in infinite dimensions," *Archive for Rational Mechanics and Analysis*, vol. 67, no. 1, pp. 53–72, 1977.

[18] B. Hassard, N. Kazarinoff, and Y. H. Wan, *Theory and Application of Hopf Bifurcation*, Springer, NY, 1981.

[19] F. Kaiser, "Limit cycle model for brain waves," *Biological Cybernetics*, vol. 27, no. 3, pp. 155–163, 1977.