Research Article

Analysis of a Deterministic and a Stochastic SIS Epidemic Model with Double Epidemic Hypothesis and Specific Functional Response

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The purpose of this paper is to investigate the stability of a deterministic and stochastic SIS epidemic model with double epidemic hypothesis and specific nonlinear incidence rate. We prove the local asymptotic stability of the equilibria of the deterministic model. Moreover, by constructing a suitable Lyapunov function, we obtain a sufficient condition for the global stability of the disease-free equilibrium. For the stochastic model, we establish global existence and positivity of the solution. Thereafter, stochastic stability of the disease-free equilibrium in almost sure exponential and $p$th moment exponential is investigated. Finally, numerical examples are presented.

1. Introduction

Epidemiology is the study of the spread of infectious diseases with the objective to trace factors that are responsible for or contribute to their occurrence. Mathematical modeling has become an important tool in analyzing the epidemiological characteristics of infectious diseases and can provide useful control measures (see, for example, [1–5]).

In classical epidemic models, the susceptible individuals can be infected with only a disease. In the real world, the susceptible individuals can be infected by two or more kinds of diseases at the same time such as HBV coinfection with HCV and HDV and HIV coinfection with HBV, HCV, and TB. Recently, the authors of [6–9] investigated the epidemic model SIS (where infection with the disease does not confer permanent immunity against reinfection so that those who survived the infection revert to the class of wholly susceptible individuals [10]) with double epidemic hypothesis which has two epidemic diseases caused by two different viruses. In this paper, we consider a deterministic SIS model with double epidemic hypothesis described by the following differential system:
\[
\begin{align*}
\dot{S}(t) &= A - \mu S(t) - \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 S(t) + \gamma_1 I_1(t) + \mu S(t) I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 S(t) + \gamma_2 I_2(t) + \mu S(t) I_2(t)} + r_1 I_1(t) + r_2 I_2(t), \\
\dot{I}_1(t) &= \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 S(t) + \gamma_1 I_1(t) + \mu S(t) I_1(t)} - (\mu + a_1 + r_1) I_1(t), \\
\dot{I}_2(t) &= \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 S(t) + \gamma_2 I_2(t) + \mu S(t) I_2(t)} - (\mu + a_2 + r_2) I_2(t),
\end{align*}
\] (1)

where \( S(t) \) represents the number of susceptible at time \( t \), \( I_1(t) \) and \( I_2(t) \) are the total population of the infected with virus \( V_1 \) and \( V_2 \) at time \( t \), respectively, \( A \) represents the recruitment rate of the population, \( \mu \) is the natural death rate of the population, \( r_i \) is the treatment cure rate of the disease caused by virus \( V_i \), \( a_i \) is the disease-related death rate, and \( \beta_i \) is the infection coefficient, \( i = 1, 2 \). The incidence rate of disease is modeled by the specific functional response \( \beta_i S I_i/(1 + \alpha_i S + \gamma_i I_i + \mu S I_i) \), where \( \alpha_i, \gamma_i, \mu_i \) are saturation factors measuring the psychological or inhibitory effect. This specific functional response was introduced by Hattaf et al. [11], and here, it becomes to be a bilinear incidence rate if \( \alpha_i = \gamma_i = \mu_i = 0 \), a saturated incidence rate if \( \alpha_i = \mu_i = 0 \) or \( \gamma_i = \mu_i = 0 \), a Beddington–DeAngelis functional response [12, 13] if \( \mu_i = 0 \), and a Crowley–Martin functional response [14] if \( \alpha_i \gamma_i = \mu_i, i = 1, 2 \).

In the reality, epidemic systems are inevitably affected by environmental white noise. Therefore, it is necessary to study how the noise influences the epidemic models. Consequently, many authors have studied stochastic epidemic models, see, e.g., [15–17]. For this, we consider the case in which the rates \( \beta_i \) (\( i = 1, 2 \)) are subject to random fluctuations, namely, \( \beta_i dt \) is replaced by \( \beta_i dt + \sigma_i dB_i(t) \), where \( B_i(t) \) (\( i = 1, 2 \)) are independent standard Brownian motions, and \( \sigma_i > 0 \) represents the intensity of \( B_i(t) \) for \( i = 1, 2 \). Therefore, the corresponding stochastic system to (1) can be described by the following Itô equations:

\[
\begin{align*}
\mathrm{d}S(t) &= \left[ A - \mu S(t) - \frac{\beta_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} - \frac{\beta_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} + r_1 I_1(t) + r_2 I_2(t) \right] \mathrm{d}t - \frac{\sigma_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} \mathrm{d}B_1(t) - \frac{\sigma_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} \mathrm{d}B_2(t), \\
\mathrm{d}I_1(t) &= \left[ \frac{\beta_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} - (\mu + a_1 + r_1) I_1(t) \right] \mathrm{d}t + \frac{\sigma_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} \mathrm{d}B_1(t), \\
\mathrm{d}I_2(t) &= \left[ \frac{\beta_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} - (\mu + a_2 + r_2) I_2(t) \right] \mathrm{d}t + \frac{\sigma_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} \mathrm{d}B_2(t),
\end{align*}
\] (2)

with \( f_i(S, I_i) = 1 + \alpha_i S + \gamma_i I_i + \mu S I_i, i = 1, 2 \).

The rest of this paper is organized in the following manner. In Section 2, we present a local stability analysis of the equilibria and a global stability analysis of the disease-free equilibrium for the deterministic model (1). In Section 3, we prove that the stochastic model (2) has a unique global positive solution, and we give sufficient conditions for the almost sure exponential stability and the \( p \)th moment exponential stability of the disease-free equilibrium. Numerical examples will be presented in Section 4. Finally, we close the paper with a brief conclusion.

2. Deterministic SIS Epidemic Model

For biological reasons, we assume that the initial conditions of system (1) satisfy

\[
\begin{align*}
S(0) &\geq 0, \\
I_1(0) &\geq 0, \\
I_2(0) &\geq 0.
\end{align*}
\] (3)

Thus, system (1) is positive [18], that is, \( S(t) \geq 0, I_1(t) \geq 0, \) and \( I_2(t) \geq 0 \) for all \( t \geq 0 \). In fact, by Proposition 2.1 in [19], we have

\[
\begin{align*}
\dot{S} &= A + r_1 I_1 + r_2 I_2 \geq 0, &\text{for } S = 0 \text{ and } I_1, I_2 \geq 0, \\
\dot{I}_1 &= 0 \geq 0, &\text{for } I_1 = 0 \text{ and } S, I_2 \geq 0, \\
\dot{I}_2 &= 0 \geq 0, &\text{for } I_2 = 0 \text{ and } S, I_1 \geq 0.
\end{align*}
\] (4)

By summing all the equations of system (1), we find that the total population size \( N(t) = S(t) + I_1(t) + I_2(t) \) satisfies the inequality

\[
\dot{N}(t) = A - \mu N(t) - a_1 I_1(t) + a_2 I_2(t) \leq A - \mu N(t),
\] (5)
which ensures that $\dot{N}(t) < 0$ if $N(t) > A/\mu$. The standard comparison theorem [20] can be used to deduce that

$$N(t) \leq \frac{A}{\mu} - \left(\frac{A}{\mu} - N(0)\right)e^{-\mu t}. \quad (6)$$

Thus, the feasible solution set of the system equation of model (1) enters and remains in the region

$$\Gamma = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+; S + I_1 + I_2 \leq \frac{A}{\mu} \right\}. \quad (7)$$

Therefore, model (1) is well posed epidemiologically and mathematically [21]. Hence, it is sufficient to study the dynamics of model (1) in $\Gamma$.

It is easy to see that system (1) has a disease-free equilibrium state $E_0 = (A/\mu, 0, 0)$. Therefore, the basic reproduction number is

$$R_0 = \max\{R_{01}, R_{02}\}, \quad (8)$$

where

$$R_{01} = \frac{\beta_1 A}{(\mu + \alpha_1 A)(\mu + a_1 + r_1)} \quad (9)$$

$$R_{02} = \frac{\beta_2 A}{(\mu + \alpha_2 A)(\mu + a_2 + r_2)}$$

We mention that the expressions of $R_{01}$ and $R_{02}$ can also be obtained by applying the next generation matrix method provided by van den Driessche and Watmough [22].

Now, we investigate the local stability of the disease-free equilibrium $E_0$. The Jacobian matrix of system (1) at the equilibrium $E_0$ is as follows:

$$J_{E_0} = \begin{pmatrix}
-\mu & -\beta_1 A/\mu + a_1 + r_1 & -\beta_2 A/\mu + a_2 + r_1 \\
0 & \beta_1 A/\mu + a_1 & 0 \\
0 & 0 & \beta_2 A/\mu + a_2 - (\mu + a_2 + r_2)
\end{pmatrix} \quad (10)$$

The three eigenvalues of $J_{E_0}$ are $\lambda_1 = -\mu < 0$, $\lambda_2 = (\mu + a_1 + r_1)(R_{01} - 1)$, and $\lambda_3 = (\mu + a_2 + r_2)(R_{02} - 1)$. Hence, the equilibrium $E_0$ will be locally asymptotically stable if $R_0 < 1$ and unstable when $R_0 > 1$.

The following theorem discusses the global stability of the disease-free equilibrium $E_0$.

**Theorem 1.** If $R_0 \leq 1$, then the disease-free equilibrium $E_0$ of (1) is globally asymptotically stable in $\Gamma$.

**Proof.** Let $U$ be the Lyapunov function defined as

$$U(t) = I_1(t) + I_2(t). \quad (11)$$

Differentiating $U$ with respect to $t$ along the positive solutions of system (1), we get

$$\dot{U}(t) = \left[\frac{\beta_1 S}{1 + a_i S + \gamma_1 I_1 + \mu_1 S I_1} - (\mu + a_1 + r_1)\right]I_1 + \left[\frac{\beta_2 S}{1 + a_2 S + \gamma_2 I_2 + \mu_2 S I_2} - (\mu + a_2 + r_2)\right]I_2. \quad (12)$$

Therefore, $R_0 \leq 1$ ensures that $\dot{U}(t) \leq 0$. Suppose that $(S, I_1, I_2)$ is a solution of (1) contained entirely in the set $\Delta = \{(S, I_1, I_2) \in \Gamma; \dot{U}(t) = 0\}$. Then, $I_1 + I_2 = 0$. We discuss four cases:

**Case 1.** If $R_{01} < 1$ and $R_{02} < 1$, then

$$\dot{X}_i = \frac{\beta_i S}{1 + a_i S + \gamma_i I_i + \mu_i S I_i} - (\mu + a_i + r_i) \leq \frac{\beta_i A}{\mu + a_i A} - (\mu + a_i + r_i) = (\mu + a_i + r_i)(R_{0i} - 1) < 0, \quad i = 1, 2. \quad (15)$$

the other hand, solutions of (1) contained in the plane $I_1 = I_2 = 0$ satisfy $\dot{S} = A - \mu S$, which implies that $S \rightarrow A/\mu$ as $t \rightarrow \infty$. From the second and third equations of (1), we have $X_1I_1 + X_2I_2 = 0$, which implies, according to (15), that $I_1 = I_2 = 0$. On
Case 2. If \( R_{01} < 1 \) and \( R_{02} = 1 \), then \( X_1 < 0 \) and

\[
X_2 = \frac{\beta_2 S}{1 + \alpha_2 S + \gamma_2 I_2 + \mu_3 S I_2} \frac{\beta_2 A}{\mu + \alpha_2 A}
\]

Then, \( X_1 I_1 + X_2 I_2 = 0 \) implies that \( I_1 = 0 \) and consequently \( X_2 I_2 = 0 \). Suppose that \( I_2 > 0 \); then, \( X_2 = 0 \). Hence, \( S \mu - A = A I_2 (\gamma_2 + \mu S) > 0 \); then, \( S > A/\mu \) which is a contradiction. Then, \( I_1 = I_2 = 0 \).

Case 3. The case \( R_{01} = 1 \) and \( R_{02} < 1 \) is analogue to the previous case.

Case 4. If \( R_{01} = 1 \) and \( R_{02} = 1 \), then \( X_1 I_1 + X_2 I_2 = 0 \) such that

\[
X_i = (\beta_i (S \mu - A) - \beta_i A I_2 (\gamma_i + \mu S))(1 + \alpha S + \gamma_2 I_2 + \mu_3 S I_2)(\mu + \alpha A) \leq 0, \quad i = 1, 2.
\]

Hence, \( X_i I_i = X_2 I_2 = 0 \), and by the same analysis in Case 2, we obtain that \( I_1 = I_2 = 0 \).

Theorem 2. If \( R_{01} > 1 \) and \( R_{02} < 1 \), then the equilibrium \( E_1^* \) is locally asymptotically stable.

Proof. The Jacobian matrix of system (1) at the equilibrium \( E_1^* \) is determined by

\[
J_{E_1^*} = \begin{pmatrix}
-m_1 & -m_2 + r_1 & -m_3 + r_2 \\
m_4 & m_2 - m_5 & 0 \\
0 & 0 & m_3 - m_6
\end{pmatrix},
\]

where

\[
m_1 = \mu + \frac{\beta_1 T_1}{(1 + \alpha_1 S_1 + \gamma_1 T_1 + \mu_1 S_1 T_1)^2},
\]

\[
m_2 = \frac{\beta_1 S_1 (1 + \alpha_1 S_1)}{(1 + \alpha_1 S_1 + \gamma_1 T_1 + \mu_1 S_1 T_1)^2},
\]

\[
m_3 = \frac{\beta_2 S_1}{1 + \alpha_2 S_1},
\]

\[
m_4 = \frac{\beta_1 T_1}{(1 + \alpha_1 S_1 + \gamma_1 T_1 + \mu_1 S_1 T_1)^2},
\]

\[
m_5 = \mu + a_1 + r_1,
\]

\[
m_6 = \mu + a_2 + r_2.
\]

Clearly, \( \lambda_1 = \beta_2 S_1 / (1 + \alpha_2 S_1) - (\mu + a_2 + r_2) \) is an eigenvalue of \( J_{E_1^*} \). Since \( S_1' > A/\mu \) because \( A - \mu S_1^* = (\mu + a_1)T_1^* > 0 \) and the function \( f_2: x \in \mathbb{R}_+ \rightarrow \beta_2 x / (1 + a_2 x) \) is increasing, then \( \lambda_1 < f_2(A/\mu) = (\mu + a_1 + r_2) = (\beta_2 A / (\mu + a_1)) - (\mu + a_2 + r_2) = (\mu + a_2 + r_2)(R_{02} - 1) \). Hence, \( \lambda_1 < 0 \) if \( R_{02} < 1 \). The other two eigenvalues of \( J_{E_1^*} \) are determined by the following equation:

\[
\lambda^2 + \alpha_1 \lambda + \alpha_0 = 0,
\]

where

\[
a_1 = m_1 + m_5 - m_2,
\]

\[
a_0 = (\mu + a_1)m_4 + \mu(m_5 - m_2).
\]

Since \( m_5 - m_2 = ((\beta S_1 T_1 (\gamma + \mu S_1)) / ((1 + \alpha S_1 + \gamma T_1 + \mu S_1 T_1)^2)) > 0 \), then \( \alpha_1 > 0 \) and \( \alpha_0 > 0 \). Thus, by the Routh–Hurwitz criterion, the eigenvalues \( \lambda \) (\( j = 2 \)) of \( J_{E_1^*} \) have negative real part. Therefore, the equilibrium \( E_1^* \) of system (1) is asymptotically stable if \( R_{01} > 1 \) and \( R_{02} < 1 \).

Furthermore, if \( R_{02} > 1 \), then system (1) has the disease-free equilibrium for \( I_1, E_2^* = (S_2^*, 0, T_2) \), where

\[
E_2^* = \frac{A - (\mu + a_2)T_2^*}{\mu},
\]

\[
E_2^* = \frac{2 \alpha_2 (\mu + a_2 A) (R_{02} - 1)}{(\mu + a_2)(\beta_2 - a_2 \alpha_2 + \alpha_2 (\gamma_2 \mu + \mu A) + \sqrt{\Delta_2})},
\]

with \( \alpha_2 = \mu + a_2 + r_2 \) and
\[ \Delta_2 = [(\mu + a_2)(\beta_2 - \alpha_2 \omega_2) + \omega_2(y_2 \mu + \mu_2 A)]^2 - 4\mu_1(\mu + a_2)\omega_2[\beta_2 A - (\mu + \alpha_2 A) \omega_2] \\
= [(\mu + a_2)(\beta_2 - \alpha_2 \omega_2) + \omega_2(y_2 \mu - \mu_2 A)]^2 + 4\mu_1\omega_2^2(\mu + a_2 + \gamma_2 A). \]  
(24)

**Theorem 3.** If \( R_{01} < 1 \) and \( R_{02} > 1 \), then the equilibrium \( E^*_2 \) is locally asymptotically stable.

**Proof.** It is analogue to the previous proof.

Next, we investigate the local stability of system (1) at both-endemic equilibrium \( E^* = (S^*, I^*_1, I^*_2) \). To obtain conditions for the existence of the equilibrium \( E^* \), system (1) is rearranged to get \( I^*_1 \) and \( I^*_2 \) which gives

\[ I^*_1 = \frac{(\beta_1 - \alpha_1 \omega_1) S^* - \alpha_1}{\omega_1(y_1 + \mu_1 S^*)}, \]
\[ I^*_2 = \frac{(\beta_2 - \alpha_2 \omega_2) S^* - \alpha_2}{\omega_2(y_2 + \mu_2 S^*)}. \]

(25)

We have \( I^*_1 < 0 \) if \( \beta_1 - \alpha_1 \omega_1 > 0 \) for \( i = 1, 2 \), and \( S^* > \max_{i=1,2}\{\frac{\alpha_1}{\omega_1(y_1 + \mu_1 S^*)}\} \). In addition, \( S^* \) is given by the following cubic equation:

\[ C_0 S^3 + C_1 S^2 + C_2 S^* - C_3 = 0, \]

(26)

where

\[ C_0 = \mu_1 \mu_2 \omega_1 \omega_2 > 0, \]
\[ C_1 = \omega_1 \omega_2 [\mu(y_1 \mu_2 + y_2 \mu_1) - A \mu_1 \mu_2] \]
\[ + \mu_1 \omega_1 (\mu + a_2)(\beta_2 - \alpha_2 \omega_2) \\
+ \mu_2 \omega_2 (\mu + a_1)(\beta_1 - \alpha_1 \omega_1), \]
\[ C_2 = \omega_1 \omega_2 [\mu y_1 y_2 - A (y_1 \mu_2 + y_2 \mu_1)] \]
\[ + \omega_1 (\mu + a_2) [y_1 (\beta_2 - \alpha_2 \omega_2) - \mu_2 \omega_2] \\
+ \omega_2 (\mu + a_1) [y_2 (\beta_1 - \alpha_1 \omega_1) - \mu_1 \omega_1], \]
\[ C_3 = \omega_1 \omega_2 [Ay_1 y_2 + y_1 (\mu + a_2) + y_2 (\mu + a_1)] > 0. \]

With the help of Descartes’ rule of signs [24], equation (26) has a unique positive real root \( S^* \) if any one of the following holds:

(i) \( C_1 > 0 \) and \( C_2 > 0 \)
(ii) \( C_1 > 0 \) and \( C_2 < 0 \)
(iii) \( C_1 < 0 \) and \( C_2 < 0 \)

Hence, system (1) has a unique positive equilibrium \( E^* \) if \( \beta_i - \alpha_i \omega_i > 0 \) for \( i = 1, 2 \), one of the conditions (i), (ii), and (iii) hold true, and \( S^* > \max_{i=1,2}\{\frac{\alpha_i}{\omega_i(y_i + \mu_i S^*)}\} \).

The Jacobian matrix of system (1) at the equilibrium \( E^* \) is determined by

\[ J_{E^*} = \begin{pmatrix} -p_1 - p_2 + r_1 & -p_3 + r_2 \\
- p_4 & p_2 - p_5 & 0 \\
p_6 & 0 & p_3 - p_7 \end{pmatrix}. \]

(28)

where

\[ p_1 = \mu + \frac{\beta_1 I^*_1 (1 + \gamma_1 I^*_1)}{(1 + a_1 S^* + \gamma_1 I^*_1 + \mu_1 S^* I^*_1)^2} \]
\[ + \frac{\beta_1 I^*_1 (1 + \gamma_1 I^*_1)}{(1 + a_2 S^* + \gamma_2 I^*_2 + \mu_2 S^* I^*_2)^2} \]
\[ p_2 = \frac{\beta_1 I^*_1 (1 + \gamma_1 I^*_1)}{(1 + a_1 S^* + \gamma_1 I^*_1 + \mu_1 S^* I^*_1)^2} \]
\[ + \frac{\beta_1 I^*_1 (1 + \gamma_1 I^*_1)}{(1 + a_2 S^* + \gamma_2 I^*_2 + \mu_2 S^* I^*_2)^2} \]
\[ p_3 = \frac{\beta_1 I^*_1 (1 + \gamma_1 I^*_1)}{(1 + a_1 S^* + \gamma_1 I^*_1 + \mu_1 S^* I^*_1)^2} \]
\[ + \frac{\beta_1 I^*_1 (1 + \gamma_1 I^*_1)}{(1 + a_2 S^* + \gamma_2 I^*_2 + \mu_2 S^* I^*_2)^2} \]
\[ p_4 = \mu + a_1 + r_1, \]
\[ p_5 = \mu + a_2 + r_2. \]

**Theorem 4.** The endemic equilibrium \( E^* \) is locally asymptotically stable if it exists.

**Proof.** The characteristic equation of Jacobian matrix \( J_{E^*} \) can be written as

\[ \lambda^3 + Q_2 \lambda^2 + Q_1 \lambda + Q_0 = 0, \]

(30)

where

\[ Q_2 = p_1 + (p_5 - p_2) + (p_7 - p_3), \]
\[ Q_1 = (\mu + p_4)(p_7 - p_3) + (\mu + p_6)(p_5 - p_2) \]
\[ + (p_5 - p_2)(p_7 - p_3) + (\mu + a_1)p_4 + (\mu + a_2)p_6, \]
\[ Q_0 = \mu(p_5 - p_2)(p_7 - p_3) + (\mu + a_1)(p_7 - p_3)p_4 \]
\[ + (\mu + a_2)(p_5 - p_2)p_6. \]

(31)

Note that

\[ p_5 - p_2 = \frac{\beta_1 S^* I^*_1 (y_1 + \mu_1 S^*)}{(1 + a_1 S^* + \gamma_1 I^*_1 + \mu_1 S^* I^*_1)^2} > 0, \]
\[ p_7 - p_3 = \frac{\beta_1 S^* I^*_1 (y_2 + \mu_2 S^*)}{(1 + a_2 S^* + \gamma_2 I^*_2 + \mu_2 S^* I^*_2)^2} > 0. \]

(32)

Then, it is easy to show that \( Q_2 > 0, Q_1 > 0, Q_0 > 0, \) and \( Q_2 Q_1 > Q_0 \). Thus, by the Routh–Hurwitz criterion, all roots \( \lambda_i (i = 1, 2, 3) \) of (30) have negative real part. Therefore, the equilibrium \( E^* \) of system (1) is asymptotically stable. \( \square \)
3. Stochastic SIS Epidemic Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We consider the following stochastic differential system:

$$\text{dx}(t) = f(x(t), t)\text{dt} + g(x(t), t)\text{dB}(t), \quad t \geq 0, \quad (33)$$

where $x(t) \in \mathbb{R}^n$, $x(0) = x_0$ represents the initial value, and $f: \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n$ and $g: \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^{m \times n}$ are locally Lipschitz functions in $x$. $\{B(t)\}_{t \geq 0}$ is an $m$-dimensional standard Wiener process defined on the above probability space.

Let us suppose that $f(0, t) = g(0, t) = 0$ for all $t \geq 0$ so that zero of $\mathbb{R}^n$ is an equilibrium point of system (33).

**Definition 1** (see [25]). The trivial solution $x = 0$ of system (33) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$, we have

$$\limsup_{t \to \infty} \frac{1}{t} \ln|x(t)| < 0 \quad (a.s.) \quad (34)$$

Denote by $\mathcal{C}^{2,1}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n)$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^n \times [0, +\infty)$ such that they are continuously twice differentiable in $x$ and once in $t$. Denote by $E(X)$ the mathematical expectation of a random variable $X$. If $V$ acts on a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n)$, then

$$\mathcal{L}V(x, t) = V_t(x, t) + V_x(x, t)f(x, t)
+ \frac{1}{2}\text{trace} \left( g(x, t)\nabla^2 V_{xx}(x, t)g(x, t) \right), \quad (35)$$

where $V_t(x, t) = \partial V/\partial t$, $V_x(x, t) = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n)$, and $V_{xx}(x, t) = (\partial^2 V/\partial x_i \partial x_j)$.

By Itô’s formula, we have

$$\text{d}V(x, t) = \mathcal{L}V(x, t)\text{dt} + V_x(x, t)g(x, t)\text{dB}(t). \quad (36)$$

**Lemma 1** (see [26]). Suppose there exists a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n)$ satisfying the inequalities

$$K_1|x|^p \leq V(x, t) \leq K_2|x|^p, \quad (37)$$

$$\mathcal{L}V(x, t) \leq -K_3|x|^p,$$

where $p > 0$ and $K_i (i = 1, 2, 3)$ are positive constants. Then, the equilibrium of system (33) is $p$th moment exponentially stable. When $p = 2$, it is usually said to be exponentially stable in mean square, and the equilibrium $x = 0$ is globally asymptotically stable.


The following theorem shows that the solution of our system (2) is global and positive.

**Theorem 5.** For any initial value $(S(0), I_1(0), I_2(0)) \in \Gamma$, there is a unique solution $(S(t), I_1(t), I_2(t))$ to (2) on $t \geq 0$, and this solution remains in $\Gamma$ with probability one.

**Proof.** Let $(S(0), I_1(0), I_2(0)) \in \Gamma$. The total population in system (2) verifies the equation

$$\text{d}N(t) = (A - \mu N(t) - a_1I_1(t) - a_2I_2(t))\text{dt}. \quad (38)$$

If $(S(s), I_1(s), I_2(s)) \in \mathbb{R}_+^3$ for all $s \in [0, t]$ (a.s.), then we get

$$\text{d}N(s) \leq (A - \mu N(s))\text{ds} \quad (a.s.) \quad (39)$$

Hence, by integration, we have

$$N(s) \leq \frac{A}{\mu} - \left(\frac{A}{\mu} - N(0)\right)e^{-\mu t} \quad (a.s.) \quad (40)$$

Then, $N(s) \leq A/\mu$ (a.s.), so

$$S(s), I_1(s), I_2(s) \in \left[0, \frac{A}{\mu}\right] \quad \text{for all } s \in [0, t] \quad (a.s.) \quad (41)$$

Since the coefficients of system (2) are locally Lipschitz continuous, then by the work of Mao [25] for any initial value $(S(0), I_1(0), I_2(0)) \in \Gamma$, there is a unique local positive solution $(S(t), I_1(t), I_2(t))$ on $[0, \tau_\varepsilon)$, where $\tau_\varepsilon$ is the explosion time. To show that this solution is global, we only need to prove $\tau_\varepsilon = \infty$ (a.s.).

Let $\varepsilon_0 > 0$ such that $S(0), I_1(0), I_2(0) > \varepsilon_0$. For $\varepsilon \leq \varepsilon_0$, we define the stopping time:

$$\tau_\varepsilon = \inf\{t \in [0, \tau_\varepsilon): S(t) \leq \varepsilon \text{ or } I_1(t) \leq \varepsilon \text{ or } I_2(t) \leq \varepsilon\}. \quad (42)$$

Then

$$\tau = \lim_{\varepsilon \to 0} \tau_\varepsilon = \inf\{t \in [0, \tau_\varepsilon): S(t) \leq 0 \text{ or } I_1(t) \leq 0 \text{ or } I_2(t) \leq 0\}. \quad (43)$$

Consider the function $U$ defined for $(S, I_1, I_2) \in \mathbb{R}_+^3$ by

$$U(S, I_1, I_2) = -\ln\left(\frac{\mu}{A}S\right) - \ln\left(\frac{\mu}{A}I_1\right) - \ln\left(\frac{\mu}{A}I_2\right). \quad (44)$$

Calculating the differential of $U$ along the solution trajectories of system (2) and using Itô’s formula, for all $t \geq 0$ and $s \in [0, t \wedge \tau_\varepsilon)$, we get

$$\text{d}U(S(s), I_1(s), I_2(s)) = \mathcal{L}U\text{d}s + \sigma_1\frac{I_1}{f_1(S, I_1)}\text{dB}_1(s)$$
$$+ \sigma_2\frac{I_2}{f_2(S, I_2)}\text{dB}_2(s), \quad (45)$$

where

$$\mathcal{L}U = \frac{\mu}{A}(S - 1) > 0. \quad (46)$$
\[
\mathcal{L}U = -\frac{A + r_1I_1 + r_2I_2}{S} + 3\mu + a_1 + r_1 + a_2 + r_2 + \beta_1 \frac{I_1 - S}{f_1(S,I_1)} + \beta_2 \frac{I_2 - S}{f_2(S,I_2)} \\
+ \frac{\sigma_1^2}{2} \frac{I_1^2 + S^2}{(f_1(S,I_1))^2} + \frac{\sigma_2^2}{2} \frac{I_2^2 + S^2}{(f_2(S,I_2))^2}
\]
\[
\leq 3\mu + a_1 + r_1 + a_2 + r_2 + \frac{\beta_1 I_1}{f_1(S,I_1)} + \frac{\beta_2 I_2}{f_2(S,I_2)} + \frac{\sigma_1^2}{2} \frac{I_1^2 + S^2}{(f_1(S,I_1))^2} + \frac{\sigma_2^2}{2} \frac{I_2^2 + S^2}{(f_2(S,I_2))^2}
\]

(46)

According to (41), we have \(S(s), I_1(s), I_2(s) \in (0, A/\mu)\) for all \(s \in [0, t \wedge \tau_s]\) (a.s.). Hence,
\[
\frac{I_1(s)}{f_1(S(s), I_1(s))} \leq \frac{A}{\mu},
\]
\[
\frac{S(s)}{f_1(S(s), I_1(s))} \leq \frac{A}{\mu},
\]
\[
i = 1, 2.
\]

Therefore,
\[
dU \leq \mathcal{M} ds + \sigma_1 \frac{I_1 - S}{f_1(S,I_1)} dB_1(s) + \sigma_2 \frac{I_2 - S}{f_2(S,I_2)} dB_2(s),
\]
(48)

\[
\mathbb{E}[U(S(t \wedge \tau_s), I_1(t \wedge \tau_s), I_2(t \wedge \tau_s))] \geq \mathbb{E}\left[U(S(t \wedge \tau_s), I_1(t \wedge \tau_s), I_2(t \wedge \tau_s))\chi_{\{t \leq \tau_s\}}\right]
\]
\[
\geq \mathbb{E}\left[U(S(\tau_s), I_1(\tau_s), I_2(\tau_s))\chi_{\{t \leq \tau_s\}}\right],
\]
(51)

where \(\chi_{\{t \leq \tau_s\}}\) is the indicator function of \(\{\tau_s \leq t\}\). Note that there are some components of \((S(\tau_s), I_1(\tau_s), I_2(\tau_s))\) equal to \(\varepsilon\). Therefore,
\[
U(S(\tau_s), I_1(\tau_s), I_2(\tau_s)) \geq -\ln\left(\frac{\mu}{A}\varepsilon\right) = \ln\left(\frac{A}{\mu\varepsilon}\right)
\]
(52)

Thus,
\[
\mathbb{E}[U(S(t \wedge \tau_s), I_1(t \wedge \tau_s), I_2(t \wedge \tau_s))] \geq \ln\left(\frac{A}{\mu\varepsilon}\right) \mathbb{P}(\tau_s \leq t).
\]
(53)

By combining (50) and (53), we get that, for all \(t > 0\),
\[
\mathbb{P}(\tau_s \leq t) \leq \frac{U(S(0), I_1(0), I_2(0)) + \mathcal{M} t}{\ln(A/\mu\varepsilon)}.
\]
(54)

Extending \(\varepsilon\) to 0, we obtain for all \(t > 0\), \(\mathbb{P}(\tau_s \leq t) = 0\). Hence, \(\mathbb{P}(\tau = \infty) = 1\). As \(\tau_s \geq \tau\), then \(\tau = \tau_s = \infty\) (a.s.) which completes the proof.

3.2. Almost Sure Exponential Stability. The goal of this section is to establish a sufficient condition for the almost sure exponential stability of the disease-free equilibrium \(E_0\) in \(\Gamma\). For this, we consider

\[
\Psi(t) = \left(\frac{A}{\mu} - S(t)\right) + I_1(t) + I_2(t),
\]
(55)

\[
V(S(t), I_1(t), I_2(t)) = \ln \Psi(t).
\]

Proposition 1. \(\Psi(t)\) almost surely converges exponentially to 0 if
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathcal{L}V(S(s), I_1(s), I_2(s)) ds < 0 \quad \text{(a.s.)}.
\]
(56)

Proof. By Itô’s formula, we have
\[
dV(S(s), I_1(s), I_2(s)) = \mathcal{L}V(S(s), I_1(s), I_2(s)) ds \\
+ \frac{2\sigma_1 S(s)I_1(s)}{\Psi(s)f_1(S(s), I_1(s))} dB_1(s) \\
+ \frac{2\sigma_2 S(s)I_2(s)}{\Psi(s)f_2(S(s), I_2(s))} dB_2(s).
\]
(57)

Integrating both sides from 0 to \(t\) yields that
\[ V(S(t), I_1(t), I_2(t)) = V(S(0), I_1(0), I_2(0)) + \int_0^t \mathcal{L}V(S(s), I_1(s), I_2(s)) \, ds + M_1(t) + M_2(t), \]

where \( M_i(t) = \int_0^t (2\sigma_i S(s)I_i(s) \Psi(s)f_i(S(s), I_i(s))) \, dB_i(s), \quad i = 1, 2, \) are continuous local martingales with \( M_i(0) = 0. \)

Moreover, we have
\[ \langle M_1, M_1 \rangle = \int_0^t \left( \frac{2\sigma_1 S(s)I_1(s)}{\Psi(s)f_1(S(s), I_1(s))} \right)^2 \, ds \leq C_f t. \]

Therefore, we obtain the following theorem.

The proposition is proved. \( \square \)

Then, we obtain the following theorem.

**Theorem 6.** If \((\beta_1 \sigma_1)^2 + (\beta_2 \sigma_2)^2 < (\mu/2)(\sigma_1 \sigma_2)^2,\) then the disease-free equilibrium \( E_0 \) of stochastic system (2) is almost surely exponentially stable in \( \Gamma. \)

By Itô’s formula, we have
\[ \mathcal{L}V(S(s), I_1(s), I_2(s)) = \frac{1}{\Psi} \left[ -A + \mu S + \frac{2\beta_1 SI_1}{f_1(S, I_1)} + \frac{2\beta_2 SI_2}{f_2(S, I_2)} - r_1 I_1 - r_2 I_2 - (\mu + a_1 + r_1) I_1 - (\mu + a_2 + r_2) I_2 \right] \]
\[ - \frac{2}{\Psi} \left( \frac{\sigma_1 SI_1}{f_1(S, I_1)} \right)^2 - \frac{2}{\Psi} \left( \frac{\sigma_2 SI_2}{f_2(S, I_2)} \right)^2 \]
\[ = \frac{1}{\Psi} \left[ \frac{2\beta_1 SI_1}{f_1(S, I_1)} + \frac{2\beta_2 SI_2}{f_2(S, I_2)} - \mu \left( \frac{A}{\mu} - S \right) - (\mu + a_1 + 2r_1) I_1 - (\mu + a_2 + 2r_2) I_2 \right] \]
\[ - \frac{2}{\Psi} \left( \frac{\sigma_1 SI_1}{f_1(S, I_1)} \right)^2 - \frac{2}{\Psi} \left( \frac{\sigma_2 SI_2}{f_2(S, I_2)} \right)^2. \]

Since
\[ \mu \left( \frac{A}{\mu} - S \right) + (\mu + a_1 + 2r_1) I_1 + (\mu + a_2 + 2r_2) I_2 \geq \mu \left( \frac{A}{\mu} - S \right) + \mu I_1 + \mu I_2 = \mu \Psi, \]

we have
\[ \mathcal{L}V(S(s), I_1(s), I_2(s)) \leq \frac{2\beta_1 SI_1}{\Psi f_1(S, I_1)} + \frac{2\beta_2 SI_2}{\Psi f_2(S, I_2)} - \mu - 2 \left( \frac{\sigma_1 SI_1}{\Psi f_1(S, I_1)} \right)^2 - 2 \left( \frac{\sigma_2 SI_2}{\Psi f_2(S, I_2)} \right)^2. \]
Set $X_1 = S \Psi f_1 (S, I_1)$ and $X_2 = S \Psi f_2 (S, I_2)$. Then, 
\[
\mathcal{L}V(S(s), I_1(s), I_2(s)) \leq 2\beta X_1 - 2\sigma_1^2 X_1 + 2\beta X_2 - 2\sigma_2^2 X_2 - \mu.
\]
(66)

Since $2\beta X_1 - 2\sigma_1^2 X_1 - (\mu/2) = -2\sigma_1^2 (\beta/\sigma_1^2 - X_1)^2 + (4\beta^2 - 2\mu^2/2\sigma_1^2)$, $i = 1, 2$, hence,
\[
\mathcal{L}V(S(s), I_1(s), I_2(s)) \leq \frac{4\beta^2 - \mu\sigma_1^2}{2\sigma_1^2} + \frac{4\beta^2 - \mu\sigma_1^2}{2\sigma_1^2} X_1^2
\]
\[
\frac{2(\beta_1\sigma_2)}{(\sigma_1\sigma_2)^2} + \frac{2(\beta_2\sigma_1)}{(\sigma_1\sigma_2)^2} - \mu(\sigma_1\sigma_2)^2
\]
(67)

Therefore,
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathcal{L}V(S(s), I_1(s), I_2(s)) ds \leq \frac{2(\beta_1\sigma_2)}{(\sigma_1\sigma_2)^2} + \frac{2(\beta_2\sigma_1)}{(\sigma_1\sigma_2)^2} - \mu(\sigma_1\sigma_2)^2 < 0 \text{ (a.s.)}
\]
(68)

This completes the proof. □

3.3. Moment Exponential Stability. In this section, we investigate the $p$th moment exponential stability of the disease-free equilibrium $E_0$ in $\Gamma$ of stochastic system (2).

We use Lemma 1 to prove the following theorem.

**Theorem 7.** Let $p \geq 2$. If

\[
\max\left\{ R_{01} + \frac{(p - 1)\sigma_1^2}{2(\mu + a_1 + r_1)} \left( \frac{A}{\mu + a_1} \right)^2, \frac{R_{02}}{2(\mu + a_2 + r_2)} \left( \frac{A}{\mu + a_2} \right)^2 \right\} < 1,
\]
(69)

then the disease-free equilibrium $E_0$ of stochastic system (2) is $p$th moment exponentially stable in $\Gamma$.

**Proof.** Let $p \geq 2$ and $(S(0), I_1(0), I_2(0)) \in \Gamma$. We define the Lyapunov function $V$ as follows:

\[
\mathcal{L}V = -\omega p \left( \frac{A}{\mu} - S \right)^{p-1} \left( A - \mu S - \sum_{i=1}^{2} \frac{\beta_i S I_i}{f_i(S, I_i)} + \sum_{i=1}^{2} r_i I_i \right) + \omega p \sum_{i=1}^{2} \left( \beta_i A \left( \frac{A}{\mu} - S \right) \right) I_i
\]
\[
+ \omega p \sum_{i=1}^{2} \left( \frac{\beta_i S f_i(S, I_i)}{f_i(S, I_i)} - (\mu + a_i + r_i) \right) I_i^p + \frac{p - 1}{2} \sum_{i=1}^{2} \frac{\sigma_i^2}{f_i(S, I_i)} I_i^2
\]
\[
\leq -\omega p \left( \frac{A}{\mu} - S \right)^{p-1} + \omega p \sum_{i=1}^{2} \frac{\beta_i A}{\mu + a_i} \left( \frac{A}{\mu} - S \right)^{p-1} I_i + \frac{p - 1}{2} \sum_{i=1}^{2} \sigma_i^2 \left( \frac{A}{\mu + a_i} \right) I_i^2
\]
\[
+ \frac{p - 1}{2} \sum_{i=1}^{2} \frac{\sigma_i^2}{\mu + a_i} I_i^2
\]
(71)
From Young’s inequality, for $\varepsilon > 0$, we have
\[
\left( \frac{A}{\mu} - S \right)^{p-1} I_i \leq \frac{p-1}{p} \varepsilon \left( \frac{A}{\mu} - S \right)^p + \frac{1}{p} e^{-p/p} I_i^p, \\
\left( \frac{A}{\mu} - S \right)^{p-2} I_i^2 \leq \frac{p-2}{p} \varepsilon \left( \frac{A}{\mu} - S \right)^p + \frac{2}{p} e^{(2-p)/p} I_i^2,
\]
then, we have $V \leq -Q_0 \left( \frac{A}{\mu} - S \right)^p - \sum_{i=1}^2 Q_i I_i^p$, \hspace{1cm} (73)
where
\[
Q_0 = \omega \left[ \mu p - (p-1) \left( \sum_{i=1}^2 \frac{\beta_i A}{\mu + \alpha_i A} + \frac{p-2}{2} \sum_{i=1}^2 \sigma_i^2 \left( \frac{A}{\mu + \alpha_i A} \right)^2 \right) \right].
\]

\[
Q_i = \mu + a_i + r_i - \frac{\beta_i A}{\mu + \alpha_i A} - \frac{p-1}{2} \sigma_i^2 \left( \frac{A}{\mu + \alpha_i A} \right)^2 - \omega \left( \frac{\beta_i A}{\mu + \alpha_i A} e^{-p/p} + (p-1) \sigma_i^2 \left( \frac{A}{\mu + \alpha_i A} \right)^2 e^{(2-p)/p} \right),
\]

\[
\max \left\{ R_{01} + \frac{\sigma_1^2}{2(\mu + a_1 + r_1)} \left( \frac{A}{\mu + a_1 A} \right)^2, R_{02} + \frac{\sigma_2^2}{2(\mu + a_2 + r_2)} \left( \frac{A}{\mu + a_2 A} \right)^2 \right\} \leq 1,
\]
then the disease-free equilibrium $E_0$ of stochastic system (2) is globally asymptotically stable in $\Gamma$.

4. Numerical Examples
In this section, we give some numerical examples in order to illustrate our theoretical results in Theorem 1 and Theorem 6.

Example 1. We consider the deterministic SIS system with parameters $A = 0.9$, $\mu = 0.3$, $\beta_1 = 0.25$, $\beta_2 = 0.2$, $a_1 = 0.2$, $a_2 = 0.3$, $r_1 = 0.3$, $r_2 = 0.2$, $\gamma_1 = 0.1$, $\mu_1 = 0.06$, $\gamma_2 = 0.02$, and $\phi_2 = 0.07$. By calculation, we have $R_0 = \max \{ R_{01}, R_{02} \} = \max \{ 0.5859, 0.5172 \} < 1$. Hence, according to Theorem 1, the disease-free equilibrium is globally asymptotically stable, which means that the disease dies out.

Example 2. In this example, we consider the stochastic SIS system with parameters the same as in Example 1 and $\sigma_1 = 0.9$, $\sigma_2 = 0.95$. Then, we have $0.08880625 = (\beta_1 \sigma_1)^2 + (\beta_2 \sigma_2)^2 < (\mu/2)(\sigma_1 \sigma_2)^2 = 0.10965375$. Thus, from Theorem 6, we can conclude that the disease-free equilibrium is almost surely exponentially stable.

Then, $\mathcal{L} \mathcal{V} \leq -Q_0 \left( \frac{A}{\mu} - S \right)^p - \sum_{i=1}^2 Q_i I_i^p$, \hspace{1cm} (73)
where
\[
Q_0 = \omega \left[ \mu p - (p-1) \left( \sum_{i=1}^2 \frac{\beta_i A}{\mu + \alpha_i A} + \frac{p-2}{2} \sum_{i=1}^2 \sigma_i^2 \left( \frac{A}{\mu + \alpha_i A} \right)^2 \right) \right].
\]

\[
Q_i = \mu + a_i + r_i - \frac{\beta_i A}{\mu + \alpha_i A} - \frac{p-1}{2} \sigma_i^2 \left( \frac{A}{\mu + \alpha_i A} \right)^2 - \omega \left( \frac{\beta_i A}{\mu + \alpha_i A} e^{-p/p} + (p-1) \sigma_i^2 \left( \frac{A}{\mu + \alpha_i A} \right)^2 e^{(2-p)/p} \right),
\]

\[
\max \left\{ R_{01} + \frac{\sigma_1^2}{2(\mu + a_1 + r_1)} \left( \frac{A}{\mu + a_1 A} \right)^2, R_{02} + \frac{\sigma_2^2}{2(\mu + a_2 + r_2)} \left( \frac{A}{\mu + a_2 A} \right)^2 \right\} \leq 1,
\]
then the disease-free equilibrium $E_0$ of stochastic system (2) is globally asymptotically stable in $\Gamma$.

5. Conclusion
In this paper, we have proposed and analyzed a new stochastic SIS epidemic model with double epidemic hypothesis and specific functional response by introducing random perturbations of white noise. Firstly, in the absence of noise, we have derived sufficient conditions for local asymptotic stability of the equilibria; also, we have proved the global stability for disease-free equilibrium. Next, we have established global existence and positivity of the solution for our stochastic model. In addition, we have given a sufficient condition for the almost sure exponential stability and $p$th moment exponential stability of the disease-free equilibrium of model (2). It is shown that the magnitude of the intensity of noise $\sigma_i (i = 1, 2)$ has an effective impact on stochastic stability of $E_0$.

Data Availability
No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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