Positive Solutions for BVP of Fractional Differential Equation with Integral Boundary Conditions

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In this paper, we consider a class of boundary value problems of nonlinear fractional differential equation with integral boundary conditions. By applying the monotone iterative method and some inequalities associated with Green’s function, we obtain the existence of minimal and maximal positive solutions and establish two iterative sequences for approximating the solutions to the above problem. It is worth mentioning that these iterative sequences start off with zero function or linear function, which is useful and feasible for computational purpose. An example is also included to illustrate the main result of this paper.

1. Introduction

Fractional calculus has widespread applications in many fields of science and engineering, for example, viscoelasticity, continuum mechanics, bioengineering, rheology, electrical networks, control theory of dynamical systems, and optics and signal processing [1, 2].

In the past decades, the existence of solutions or positive solutions for boundary value problems (BVPs for short) of nonlinear fractional differential equations attracted considerable attention from many authors, see [3–19] and the references therein.

Recently, the monotone iterative method has been applied to study BVPs of nonlinear fractional differential equations. For example, in [20], Cui et al. discussed the BVP

\[ \begin{cases} (D_q^0 u)(t) + q(t) f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, & u(1) = 0, \end{cases} \] (1)

where \( 2 < q \leq 3 \) and \( D_q^0 \) denotes the standard Riemann–Liouville fractional derivative of order \( q \). The authors obtained the existence of maximal and minimal solutions and the uniqueness result for BVP (1). In 2014, Sun and Zhao [21] investigated the following BVP with integral boundary conditions:

\[ \begin{cases} (D_q^0 u)(t) + q(t) f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s) u(s) ds, \end{cases} \] (2)

where \( 2 < q \leq 3 \) and \( D_q^0 \) is the standard Riemann–Liouville fractional derivative of order \( q \). By means of the monotone iterative method, they proved the existence of a positive solution and established an iterative sequence for approximating the solution to BVP (2). For relevant results, one can refer to [22–25].

Motivated by the aforementioned works, in this paper, we consider the following BVP of nonlinear fractional differential equation with integral boundary conditions:

\[ \begin{cases} (C D_q^0 u)(t) + f(t, u(t)) = 0, & t \in [0, 1], \\ u''(0) = 0, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s) u(s) ds, \\ \gamma u(1) + \delta (C D_0^0 u)(1) = \int_0^1 h_2(s) u(s) ds, \end{cases} \] (3)

where \( C D_q^0 \) and \( C D_0^0 \) denote the standard Caputo fractional derivatives of order \( q \) and order \( \sigma \), respectively. Throughout
this paper, we always assume that $2 < q \leq 3, 0 < \sigma \leq 1$ and $\alpha, \beta, \gamma$, and $\delta$ are nonnegative constants satisfying $0 < \rho = (\alpha + \beta) \gamma + (\alpha \delta \Gamma(2 - \sigma)) < \beta \gamma + (\delta \Gamma(q) \Gamma(q - \sigma))$, and $f: [0, 1] \times [0, +\infty) \to [0, +\infty)$ and $h_i (i = 1, 2): [0, 1] \to [0, +\infty)$ are continuous.

The main tool used is the following theorem [26].

**Theorem 1.** Let $K$ be a normal cone of a Banach space $E$ and $v_0 \leq \omega_0$. Suppose that

(a$_1$): $T: [v_0, \omega_0] \to E$ is completely continuous

(a$_2$): $T$ is monotone increasing on $[v_0, \omega_0]$

(a$_3$): $v_0$ is a lower solution of $T$, that is, $v_0 \leq Tv_0$

(a$_4$): $\omega_0$ is an upper solution of $T$, that is, $T\omega_0 \leq \omega_0$

Then, the iterative sequences

$$v_n = Tv_{n-1}, \quad \omega_n = T\omega_{n-1}, \quad n = 1, 2, 3, \ldots$$

satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \omega_n \leq \cdots \leq \omega_1 \leq \omega_0,$$

and converge to, respectively, $v$ and $\omega \in [v_0, \omega_0]$, which are fixed points of $T$.

### 2. Preliminaries

First, we present the definitions of Riemann–Liouville fractional integral and fractional derivative and Caputo fractional derivative on a finite interval of the real line, which may be found in [1].

In this section, we always assume that $\mathbb{N} = \{1, 2, 3, \ldots\}, \mu, \nu > 0$ and $[\mu]$ denotes the integer part of $\mu$.

**Definition 1.** The Riemann–Liouville fractional integral $I_{0+}^\mu, u$ of order $\mu$ on $[0, 1]$ is defined by

$$(I_{0+}^\mu, u)(t) := \frac{1}{\Gamma(\mu)} \int_0^t u(s) \, ds$$

**Definition 2.** The Riemann–Liouville fractional derivative $D_{0+}^\mu, u$ of order $\mu$ on $[0, 1]$ is defined by

$$(D_{0+}^\mu, u)(t) := \frac{d^n}{dt^n} \left( I_{0+}^{n-\mu}, u \right)(t)$$

$$= \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_0^t u(s) \, ds$$

where $n = [\mu] + 1$.

**Definition 3.** Let $D_{0+}^\mu, \Delta u(s)(t) \equiv (D_{0+}^\mu, u)(t)$ be the Riemann–Liouville fractional derivative of order $\mu$. Then, the Caputo fractional derivative $C D_{0+}^\mu, u$ of order $\mu$ on $[0, 1]$ is defined by

$$(C D_{0+}^\mu, u)(t) := \left( D_{0+}^\mu, \left[ u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} s^k \right] \right)(t),$$

where $n = [\mu] + 1, \mu \notin \mathbb{N}$.

**Lemma 1** (see [2]). Let $\nu > \mu$. Then, the equation $(C D_{0+}^\mu, I_{0+}^\nu, u)(t) = (I_{0+}^{\nu-\mu}, u)(t)$, $t \in [0, 1]$, is satisfied for $u \in C[0, 1]$.

**Lemma 2** (see [1]). Let $n$ be given by (9). Then, the following relations hold:

1. For $k \in \{0, 1, 2, \ldots, n - 1\}$, $C D_{0+}^\mu, t^k = 0$.
2. If $\nu > n$, then $C D_{0+}^\mu, t^{\nu-n-1} = (\Gamma(\nu)/\Gamma(\nu-\mu)) t^{\nu-\mu-1}$.

For convenience, we denote

$$P_i = \frac{1}{\rho} \int_0^1 (as + \beta) h_i(s) \, ds,$$

$$Q_i = \frac{1}{\rho \Gamma(2 - \sigma)} \int_0^1 [\gamma \Gamma(2 - \sigma)(1-s) + \delta] h_i(s) \, ds, \quad i = 1, 2.$$ (10)

**Lemma 3.** Let $(1 - Q_1)(1 - P_2) \neq P_1 Q_2$. Then, for any $y \in C[0, 1]$, the BVP

$$C D_{0+}^\mu, u(t) + y(t) = 0, \quad t \in [0, 1],$$

$$u'(0) = 0,$$

$$au(0) - \beta u'(0) = \int_0^1 h_1(s) u(s) \, ds,$$

$$y u(1) + \delta (C D_{0+}^\mu, u)(1) = \int_0^1 h_2(s) u(s) \, ds,$$

has a unique solution

$$u(t) = \int_0^1 H(t, s) y(s) \, ds, \quad t \in [0, 1].$$ (12)

Here,

$$H(t, s) = G(t, s) + \sum_{i=1}^2 \phi_i(t) \int_0^1 G(r, s) h_i(r) \, dr,$$ (13)

where

$i, s \in [0, 1] \times [0, 1]$,
In view of the equation in (11), Theorem 3.24 [1], and
Proof. In view of the equation in (11), Theorem 3.24 [1], and
and so,
\[ G(t, s) = \frac{at + \beta}{\rho} \left\{ \frac{y(1 - s)^{\sigma - 1}}{\Gamma(\sigma)} + \frac{\delta(1 - s)^{\sigma - 1}}{\Gamma(\sigma)} \right\} \left\{ \frac{(t - s)^{\sigma - 1}}{\Gamma(\sigma)} \right\}, \quad 0 \leq s \leq t \leq 1, \]
\[ \phi_1(t) = \frac{\Gamma(2 - \sigma)Q_2(at + \beta) + (1 - P_2)\left[y\Gamma(2 - \sigma)(1 - t) + \delta\right]}{\rho^1(2 - \sigma)[(1 - Q_1)(1 - P_2) - P_1Q_2]}, \quad t \in [0, 1], \]
\[ \phi_2(t) = \frac{\Gamma(2 - \sigma)(1 - Q_1)(at + \beta) + P_1y\Gamma(2 - \sigma)(1 - t) + \delta}{\rho^1(2 - \sigma)[(1 - Q_1)(1 - P_2) - P_1Q_2]}, \quad t \in [0, 1]. \]

Proof. In view of the equation in (11), Theorem 3.24 [1], and

By (15), Lemma 1, and Lemma 2, we obtain

\[ u(t) = -(I_{0+}^{\sigma} y)(t) + u(0) + u'(0)t, \quad t \in [0, 1]. \]

By (15), Lemma 1, and Lemma 2, we obtain

\[ u(0) = \frac{1}{\rho} \left\{ \beta y(I_{0+}^{\sigma} y)(1) + \beta \delta(I_{0+}^{\sigma - 2\gamma} y)(1) + \frac{y\Gamma(2 - \sigma) + \delta}{\Gamma(2 - \sigma)} \int_0^1 h_1(s)u(s)ds + \beta \int_0^1 h_2(s)u(s)ds \right\}, \]
\[ u'(0) = \frac{1}{\rho} \left\{ \alpha y(I_{0+}^{\sigma} y)(1) + \alpha \delta(I_{0+}^{\sigma - 2\gamma} y)(1) - y \int_0^1 h_1(s)u(s)ds + \alpha \int_0^1 h_2(s)u(s)ds \right\}, \]

which together with (15) shows that

\[ u(t) = \int_0^t \left\{ \frac{at + \beta}{\rho} \left[ \frac{y(1 - s)^{\sigma - 1}}{\Gamma(\sigma)} + \frac{\delta(1 - s)^{\sigma - 1}}{\Gamma(\sigma)} \right] - \frac{(t - s)^{\sigma - 1}}{\Gamma(\sigma)} \right\} y(s)ds \]
\[ + \int_t^1 \left\{ \frac{at + \beta}{\rho} \left[ \frac{y(1 - s)^{\sigma - 1}}{\Gamma(\sigma)} + \frac{\delta(1 - s)^{\sigma - 1}}{\Gamma(\sigma)} \right] - \frac{(t - s)^{\sigma - 1}}{\Gamma(\sigma)} \right\} y(s)ds \]
\[ + \frac{y\Gamma(2 - \sigma)(1 - t) + \delta}{\rho^1(2 - \sigma)} \int_0^1 h_1(s)u(s)ds + \frac{at + \beta}{\rho} \int_0^1 h_2(s)u(s)ds \]
\[ = \int_0^1 G(t, s)y(s)ds + \frac{y\Gamma(2 - \sigma)(1 - t) + \delta}{\rho^1(2 - \sigma)} \int_0^1 h_1(s)u(s)ds + \frac{at + \beta}{\rho} \int_0^1 h_2(s)u(s)ds, \quad t \in [0, 1]. \]

From (18), we get

\[ (1 - Q_2) \int_0^1 h_1(s)u(s)ds - P_1 \int_0^1 h_2(s)u(s)ds = \int_0^1 h_1(s) \int_0^1 G(s, \tau)y(\tau)ds, \]
\[ -Q_2 \int_0^1 h_1(s)u(s)ds + (1 - P_2) \int_0^1 h_2(s)u(s)ds = \int_0^1 h_2(s) \int_0^1 G(s, \tau)y(\tau)ds, \]

and so,
\[
\int_0^1 h_1(s)u(s)ds = \frac{(1 - P_2)\int_0^1 h_1(s)\int_0^1 G(s, r)y(\tau)d\tau drds + P_1\int_0^1 h_2(s)\int_0^1 G(s, r)y(\tau)d\tau drds}{(1 - Q_1)(1 - P_2) - (P_1 Q_2)},
\]
\[
\int_0^1 h_2(s)u(s)ds = \frac{Q_2\int_0^1 h_1(s)\int_0^1 G(s, r)y(\tau)d\tau drds + (1 - Q_1)\int_0^1 h_2(s)\int_0^1 G(s, r)y(\tau)d\tau drds}{(1 - Q_1)(1 - P_2) - (P_1 Q_2)},
\]
which together with (18) implies that
\[
u(t) = \int_0^1 G(t, s)y(s)ds + \sum_{i=1}^2 \hat{\phi}_i(t) \int_0^1 G(t, \tau)h_i(\tau)d\tau drds
\]

In what follows, we let
\[
G(t, s) = \frac{(at + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma - 1} + \delta \Gamma(q)(1 - s)^{q - \sigma - 1}\right] - \rho \Gamma(q - \sigma)(t - s)^{q - 1}}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma - 1} + \delta \Gamma(q)(1 - s)^{q - \sigma - 1}\right]}
\]
\[
\geq \frac{\beta \gamma \Gamma(q - \sigma)(1 - s)^{\sigma} + \beta \delta \Gamma(q) - \rho \Gamma(q - \sigma)(1 - s)^{q}}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma} + \delta \Gamma(q)\right]}
\]
\[
\geq \frac{\beta \delta \Gamma(q) - \Gamma(q - \sigma)(\rho - \beta \gamma)}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma) + \delta \Gamma(q)\right]}(1 - s)^{\sigma}
\]
\[
= \eta(s),
\]
and if \( t \leq s \), then
\[
G(t, s) = \frac{(at + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma - 1} + \delta \Gamma(q)(1 - s)^{q - \sigma - 1}\right] - \rho \Gamma(q - \sigma)(t - s)^{q - 1}}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma - 1} + \delta \Gamma(q)(1 - s)^{q - \sigma - 1}\right]}
\]
\[
\geq \frac{\beta \gamma \Gamma(q - \sigma)(1 - s)^{\sigma} + \beta \delta \Gamma(q)}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma} + \delta \Gamma(q)\right]}
\]
\[
\geq \frac{\beta \gamma \Gamma(q - \sigma)(1 - s)^{\sigma} + \beta \delta \Gamma(q) - \rho \Gamma(q - \sigma)(1 - s)^{q}}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma)(1 - s)^{\sigma} + \delta \Gamma(q)\right]}
\]
\[
\geq \frac{\beta \delta \Gamma(q) - \Gamma(q - \sigma)(\rho - \beta \gamma)}{(\alpha + \beta)\left[\gamma \Gamma(q - \sigma) + \delta \Gamma(q)\right]}(1 - s)^{\sigma}
\]
\[
= \eta(s).
\]
By the definition of $\eta$ and the condition $0 < \rho < \beta[y + (\delta \ell(q)H(q - \sigma))]$, we may obtain the following remark.

Remark 1. $\eta(s) \geq 0$ and $\eta(s) \neq 0$ for $s \in [0, 1]$.

In the remainder of this paper, we always assume that the following condition is fulfilled:

\[(C1) Q_1 < 1, \quad P_2 < 1, \quad (1 - Q_1)(1 - P_2) > P_1 Q_2. \tag{25}\]

Now, we define
\[x(t) = 1 + \sum_{i=1}^{3} \int_{0}^{t} h_i(\tau)d\tau p_i(t), \quad t \in [0, 1]. \tag{26}\]

Lemma 5. $H(t, s)$ has the following property:

\[x(t)\eta(s)g(s) \leq H(t, s) \leq x(t)g(s), \quad (t, s) \in [0, 1] \times [0, 1]. \tag{27}\]

Proof. In view of the definition of $H(t, s)$ and Lemma 4, it is obvious.

3. Main Results

For convenience, we let
\[A = \frac{1}{\int_{0}^{1} g(s)ds}, \quad a = \max_{t \in [0, 1]} x(t). \tag{28}\]

Theorem 2. Assume that $f(t, 0) \equiv 0$ for $t \in [0, 1]$ and the following condition is satisfied:

\[(C2) f(t, u_1) \leq f(t, u_2) \leq A, \quad 0 \leq t \leq 1, \quad 0 \leq u_1 \leq u_2 \leq a. \tag{29}\]

Then, BVP (3) possesses a minimal positive solution $\upsilon$ and a maximal positive solution $\omega$.

Proof. Let $E = C[0, 1]$ be equipped with the norm
\[\|u\| = \max_{t \in [0, 1]} |u(t)|, \quad K = \{u \in E : u(t) \geq 0, \quad t \in [0, 1]\}. \tag{30}\]

Then, $K$ is a normal cone in Banach space $E$. Note that this induces an order relation “$\preceq$” in $E$ by defining $u \preceq v$ if and only if $v - u \in K$.

Now, we define $\upsilon_0(t) \equiv 0$ and $\omega_0(t) = x(t)$ for $t \in [0, 1]$. Let $T : [\upsilon_0, \omega_0] \longrightarrow K$ by
\[(Tu)(t) = \int_{0}^{t} H(t, s)f(s, u(s))ds, \quad u \in [\upsilon_0, \omega_0], \quad t \in [0, 1]. \tag{31}\]

Then, it is easy to know that $T : [\upsilon_0, \omega_0] \longrightarrow K$ is completely continuous, and fixed points of $T$ are nonnegative solutions of BVP (3).

Step 1. We assert that $T$ is monotone increasing on $[\upsilon_0, \omega_0]$.

Let $\upsilon, \nu \in [\upsilon_0, \omega_0]$ and $\upsilon \preceq \nu$. Then, $0 \leq \upsilon(t) \leq \nu(t) \leq a$ for $t \in [0, 1]$, which together with (C2) implies that
\[(Tu)(t) = \int_{0}^{1} H(t, s)f(s, u(s))ds \leq \int_{0}^{1} H(t, s)f(s, \nu(s))ds = (Tv)(t), \quad t \in [0, 1]. \tag{32}\]

This shows that $Tu \preceq Tv$.

Step 2. We prove that $\upsilon_0$ is a lower solution of $T$.

For any $t \in [0, 1]$, we have
\[(Tv_0)(t) = \int_{0}^{1} H(t, s)f(s, 0)ds \geq 0 = \upsilon_0(t), \tag{33}\]

which indicates that $\upsilon_0 \preceq Tv_0$.

Step 3. We show that $\omega_0$ is an upper solution of $T$.

In view of Lemma 5 and (C2), we get
\[(T\omega_0)(t) = \int_{0}^{1} H(t, s)f(s, \omega_0(s))ds \leq Ax(t) \int_{0}^{1} g(s)ds = \omega_0(t), \quad t \in [0, 1], \tag{34}\]

which implies that $T\omega_0 \preceq \omega_0$.

Step 4. We claim that BVP (3) possesses a minimal positive solution $\upsilon$ and a maximal positive solution $\omega$.

In fact, if we construct sequences $\{\upsilon_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ as in the following,
\[\upsilon_n = T\upsilon_{n-1}, \quad \omega_n = T\omega_{n-1}, \quad n = 1, 2, 3, \ldots, \tag{35}\]

then it follows from Theorem 1 that
\[\upsilon_0 \preceq \upsilon_1 \preceq \cdots \preceq \upsilon_n \preceq \cdots \preceq \omega_n \preceq \cdots \preceq \omega_1 \preceq \omega_0, \tag{36}\]

and $\{\upsilon_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ converge to, respectively, $\upsilon$ and $\omega \in [\upsilon_0, \omega_0]$, which are nonnegative solutions of BVP (3).

In view of Lemma 5, Remark 1, the definitions of $x(t)$ and $g(s)$, and the assumption $f(t, 0) \equiv 0$ for $t \in [0, 1]$, we get
\[(Tv_0)(t) = \int_{0}^{1} H(t, s)f(s, 0)ds \geq x(t) \int_{0}^{1} \eta(s)g(s)f(s, 0)ds \tag{37}\]

and so,
\[0 < (Tv_0)(t) \leq (Tv)(t) = \upsilon(t) \leq \omega(t), \quad t \in [0, 1], \tag{38}\]

which shows that $\upsilon$ and $\omega$ are positive solutions of the BVP (3).
Moreover, if \( u \in [v_0, \omega_0] \) is a positive solution of the BVP (3), then it follows from the fact \( T \) is monotone increasing on \([v_0, \omega_0] \) that \( v \leq u \leq \omega \). This indicates that \( v \) and \( \omega \) are minimal and maximal positive solutions of the BVP (3), respectively.

**Example 1.** Consider the following BVP:

\[
\begin{cases}
(\frac{\sqrt{t}}{3} u^n(t)) + \frac{\sqrt{t}}{3} \left[ u^2(t) \frac{14741}{13456} \sin(\pi t) \right] = 0, & t \in [0, 1], \\
u''(0) = 0, \\
u(0) - 3u'(0) = \frac{1}{2} \int_0^1 s u(s)ds, \\
2u(1) + u'(1) = \frac{1}{2} \int_0^1 s u(s)ds.
\end{cases}
\]

(39)

Since \( q = (5/2), \sigma = \alpha = \delta = 1, \beta = 3, \) and \( y = 2, \) a simple calculation shows that

\[
0 < \rho = (\alpha + \beta)y + \frac{a \delta}{\Gamma(2 - \sigma)} = 9 \beta \left[ y + \frac{\delta \Gamma(q)}{\Gamma(q - \sigma)} \right] = 21/2, \quad \lambda = \frac{15\sqrt{t}}{16}
\]

Moreover, in view of \( h_1(s) = s \) and \( h_2(s) = s/2 \) for \( s \in [0, 1], \) we get

\[
P_1 = \frac{11}{54}, \\
P_2 = \frac{11}{108}, \\
Q_1 = \frac{5}{54}, \\
Q_2 = \frac{5}{108}
\]

which indicates that (C1) is fulfilled. At the same time, we also obtain that

\[
a = \max_{t \in [0, 1]} x(t) = \frac{38}{29}
\]

Now, if we let \( f(t, u) = (\sqrt{t}/3)[u^2 + (14741/13456) \sin(\pi t)] \), \((t, u) \in [0, 1] \times [0, +\infty), \) then it is easy to know that \( f: [0, 1] \times [0, +\infty) \rightarrow [0, +\infty) \) is continuous, \( f(t, 0) \neq 0 \) for \( t \in [0, 1], \) and (C2) is fulfilled. Therefore, it follows from Theorem 2 that BVP (39) possesses a minimal positive solution \( v \) and a maximal positive solution \( \omega \). In addition, the two iterative schemes are as follows:

\[
v_0(t) \equiv 0, \quad t \in [0, 1],
\]

\[
v_{n+1}(t) = \int_0^t \left[ \frac{32t - 96}{3045} (t-s)^{7/2} + (t-s)^{(5/2)}s \right] ds + \frac{16t^2 - 48t}{609} (t-s)^{(5/2)} + \frac{48t + 320}{783} (1-s)^{(3/2)} + \frac{12t + 80}{261} (1-s)^{(1/2)} \left[ \nu_n^2(s) + \frac{14741}{13456} \sin(\pi s) \right] ds, \quad t \in [0, 1], n = 0, 1, 2, \ldots
\]

\[
\omega_0(t) = \frac{-3t + 38}{29}, \quad t \in [0, 1],
\]

\[
\omega_{n+1}(t) = \int_0^t \left[ \frac{32t - 96}{3045} (t-s)^{7/2} + (t-s)^{(5/2)}s \right] ds + \frac{16t^2 - 48t}{609} (t-s)^{(5/2)} + \frac{48t + 320}{783} (1-s)^{(3/2)} + \frac{12t + 80}{261} (1-s)^{(1/2)} \left[ \omega_n^2(s) + \frac{14741}{13456} \sin(\pi s) \right] ds, \quad t \in [0, 1], n = 0, 1, 2, \ldots
\]

(43)
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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