Research Article

New Hermite–Hadamard Type Inequalities for ψ -Riemann–Liouville Fractional Integral via Convex Functions

Yining Sun and Run Xu

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China

Correspondence should be addressed to Run Xu; xurun_2005@163.com

Received 17 December 2019; Accepted 19 March 2020; Published 14 April 2020

Academic Editor: Elmetwally Elabbasy

Copyright © 2020 Yining Sun and Run Xu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper established some new Hermite–Hadamard type inequalities for ψ -Riemann–Liouville fractional integrals via convex functions. As applications, we applied the inequalities to special means of real numbers and constructed inequalities for the beta function.

1. Introduction

Hermite–Hadamard integral inequality provides the estimate for the integral average of any convex function defined on a compact interval. In recent years, many mathematicians extended it to s -convex functions, quasi-convex functions, ψ -Riemann–Liouville convex functions, η -convex functions, r -convex functions, (α, m) -convex functions, and Lipschitzian functions, see [1–7]. The most famous are trapezoid inequality and midpoint inequality [8, 9]: suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a differentiable function on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|),$$

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

(1)

In [10], Mehrez and Agarwal extended the interval of f' and established the following result: let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) , and $f': [((3a-b)/2), ((3b-a)/2)] \rightarrow \mathbb{R}$ be a continuous function on $[((3a-b)/2), ((3b-a)/2)]$. $q \geq 1$. If $|f'|^q$ is convex on

$[((3a-b)/2), ((3b-a)/2)]$, then the following inequality holds:

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{b-a}{8} \left(\left| f'\left(\frac{3a-b}{2}\right) \right|^q + \left| f'\left(\frac{3b-a}{2}\right) \right|^q \right)^{(1/q)}.$$

(2)

The classical integro-differential equation has been studied deeply, and a rigorous and systematic theoretical system has been established. In recent decades, the theoretical research of fractional integro-differential equation has been constantly improved. Many theoretical achievements have been obtained [2–4, 7, 11–15]. The research of Hermite–Hadamard type inequalities for fractional integrals has also been deepened.

In [3], Liu et al. established the Hermite–Hadamard type inequalities for ψ -Riemann–Liouville fractional integrals via convex functions: let $g: [a, b] \rightarrow \mathbb{R}$ be a positive convex integrable function, $0 \leq a < b$. $\psi(x)$ is an increasing positive monotone function and have continuous derivation on $(a, b]$, and $\alpha \in (0, 1)$. Then, the following inequality holds:

$$\begin{aligned}
g\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) \right. \\
&\quad \left. + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] \leq \frac{g(a) + g(b)}{2}.
\end{aligned} \tag{3}$$

Let $g: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) , $a < b$, $|g'|$ is convex on $[a, b]$. $\psi(x)$ is an increasing positive monotone function on $(a, b]$ and have continuous derivation on (a, b) , and $\alpha \in (0, 1)$. Then, the following inequalities hold:

$$\begin{aligned}
&\left| \frac{g(a) + g(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] \right| \\
&\leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) [|g'(a)| + |g'(b)|],
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{|g(b) - g(a)|}{2} + \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) [|g'(a)| + |g'(b)|],
\end{aligned} \tag{5}$$

where $I_{\psi^{-1}(a)^+}^{\alpha;\psi}$ and $I_{\psi^{-1}(b)^-}^{\alpha;\psi}$ are the left-sided and right-sided ψ -Riemann–Liouville fractional integral operators of order α , which are listed in Section 2.

The aim of this paper is to extend the range of argument of the function g' and establish some new Hermite–Hadamard type inequalities for ψ -Riemann–Liouville fractional integrals via convex functions on the basis of (3)–(5). Then, we apply them to special means of real numbers and construct inequalities for the beta function.

2. Preliminaries

In this section, we recall the definition of ψ -Riemann–Liouville fractional integral and some lemmas about $(\Gamma(\alpha+1)/2(b-a)^\alpha)[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (f \circ \psi)(\psi^{-1}(a))]$.

Definition 1 (see [16]). Let (a, b) ($-\infty \leq a < b \leq \infty$) be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on

$(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) . The left-sided and right-sided ψ -Riemann–Liouville fractional integrals of order α for function f with respect to another function ψ on $[a, b]$ are defined by

$$\begin{aligned}
I_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) dt, \quad (x > a), \\
I_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) dt, \quad (x < b),
\end{aligned} \tag{6}$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function.

Lemma 1 (see [3]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) ; suppose that $g' \in L[a, b]$, $\psi(x)$ is an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) , and $\alpha \in (0, 1)$. Then, we have

$$\begin{aligned}
&\frac{g(a) + g(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] \\
&= \frac{1}{2(b-a)^\alpha} \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} [(\psi(t) - a)^\alpha - (b - \psi(t))^\alpha] (g' \circ \psi)(t) \psi'(t) dt.
\end{aligned} \tag{7}$$

Lemma 2 (see [3]). Under the assumptions of Lemma 1, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\
&= \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} k(g' \circ \psi)(t) \psi'(t) dt + \frac{1}{2(b-a)^\alpha} \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} [(b - \psi(t))^\alpha - (\psi(t) - a)^\alpha] (g' \circ \psi)(t) \psi'(t) dt,
\end{aligned} \tag{8}$$

where

$$\begin{cases} k = \frac{1}{2}, & \psi^{-1}\left(\frac{a+b}{2}\right) \leq t < \psi^{-1}(b), \\ -\frac{1}{2}, & \psi^{-1}(a) < t < \psi^{-1}\left(\frac{a+b}{2}\right). \end{cases} \quad (9)$$

Lemma 3 (see [17]). Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$. Then,

$$a^{(q/p)} \leq \frac{q}{p} K^{((q-p)/p)} a + \frac{p-q}{p} K^{(q/p)}, \quad (10)$$

for any $K > 0$.

3. Main Results

First, we give an estimate for $(\Gamma(\alpha+1)/2(b-a)^\alpha)[I_{\psi^{-1}(a)}^{+ \alpha} \psi(g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)}^{- \alpha} (g \circ \psi)(\psi^{-1}(a))]$.

Lemma 4. Let $g: [a, b] \rightarrow \mathbb{R}$ be a positive function, $g \in L_1[a, b]$, $\psi(x)$ is an increasing and positive monotone function on $[a, b]$, with a continuous derivative $\psi'(x)$ on (a, b) , $0 \leq a < b$, and $\alpha \in (0, 1)$. If g is a convex function on $[a, b]$, then the following inequalities hold:

$$g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] \leq \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4}. \quad (11)$$

Proof. From the left side of (3), we can directly get the left side of (11). Then, we prove the right side of (11).

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] \\ &= \frac{\alpha}{2(b-a)^\alpha} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(v) (b - \psi(v))^{\alpha-1} (g \circ \psi)(v) dv + \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(v) (\psi(v) - a)^{\alpha-1} (g \circ \psi)(v) dv \right] \\ & \left(\text{let } \psi(v) = \frac{3}{4}t + \frac{a+b}{4} \right) \\ &= \frac{3\alpha}{8(b-a)^\alpha} \left[\int_{(3a-b)/3}^{(3b-a)/3} \left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} g\left(\frac{3}{4}t + \frac{a+b}{4}\right) dt + \int_{(3a-b)/3}^{(3b-a)/3} \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} g\left(\frac{3}{4}t + \frac{a+b}{4}\right) dt \right] \\ &\leq \frac{3\alpha}{8(b-a)^\alpha} \left[\int_{(3a-b)/3}^{(3b-a)/3} \left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} \left(\frac{1}{2}g\left(\frac{3}{2}t\right) + \frac{1}{2}g\left(\frac{a+b}{2}\right) \right) dt \right. \\ & \quad \left. + \int_{(3a-b)/3}^{(3b-a)/3} \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} \left(\frac{1}{2}g\left(\frac{3}{2}t\right) + \frac{1}{2}g\left(\frac{a+b}{2}\right) \right) dt \right] \\ &= \frac{3\alpha}{16(b-a)^\alpha} \left[\int_{(3a-b)/3}^{(3b-a)/3} \left[\left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} + \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} \right] g\left(\frac{3}{2}t\right) dt \right. \\ & \quad \left. + \int_{(3a-b)/3}^{(3b-a)/3} \left[\left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} + \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} \right] g\left(\frac{a+b}{2}\right) dt \right]. \end{aligned} \quad (12)$$

Let

$$\begin{aligned}
 N_1 &= \int_{(3a-b)/3}^{(3b-a)/3} \left(\left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} + \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} \right) g\left(\frac{3}{2}t\right) dt, \\
 N_2 &= \int_{(3a-b)/3}^{(3b-a)/3} \left(\left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} + \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} \right) g\left(\frac{a+b}{2}\right) dt,
 \end{aligned} \tag{13}$$

we have

$$\begin{aligned}
 N_1 &= \frac{1}{2^{\alpha-1}} \int_{(3a-b)/3}^{(3b-a)/3} \left(\left(\frac{3b-a}{2} - \frac{3}{2}t \right)^{\alpha-1} + \left(\frac{b-3a}{2} + \frac{3}{2}t \right)^{\alpha-1} \right) g\left(\frac{3}{2}t\right) dt \\
 &\quad \left(\text{let } \psi(v) = \frac{3}{2}t \right) \\
 &= \frac{1}{2^{\alpha-2} \cdot 3} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left(\left(\frac{3b-a}{2} - \psi(v) \right)^{\alpha-1} + \left(\psi(v) - \frac{3a-b}{2} \right)^{\alpha-1} \right) (g \circ \psi)(v) \psi'(v) dv \\
 &= \frac{\Gamma(\alpha)}{2^{\alpha-2} \cdot 3} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3b-a}{2} \right) \right) + I_{\psi^{-1}((3a-b)/2)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3a-b}{2} \right) \right) \right],
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 N_2 &= g\left(\frac{a+b}{2}\right) \int_{(3a-b)/3}^{(3b-a)/3} \left(\frac{3b-a}{4} - \frac{3}{4}t \right)^{\alpha-1} + \left(\frac{b-3a}{4} + \frac{3}{4}t \right)^{\alpha-1} dt \\
 &= \frac{4}{3} g\left(\frac{a+b}{2}\right) \int_{(3a-b)/4}^{(3b-a)/4} \left(\frac{3b-a}{4} - x \right)^{\alpha-1} + \left(\frac{b-3a}{4} + x \right)^{\alpha-1} dx \\
 &= \frac{8(b-a)^\alpha}{3\alpha} g\left(\frac{a+b}{2}\right).
 \end{aligned} \tag{15}$$

From (12)–(15), we have the inequality

$$\begin{aligned}
 &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1}(b) \right) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1}(a) \right) \right] \leq \frac{3\alpha}{16(b-a)^\alpha} (N_1 + N_2) \\
 &= \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3b-a}{2} \right) \right) + I_{\psi^{-1}((3a-b)/2)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3a-b}{2} \right) \right) \right] + \frac{1}{2} g\left(\frac{a+b}{2}\right).
 \end{aligned} \tag{16}$$

From (3), we have

$$\begin{aligned}
 0 &< \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3b-a}{2} \right) \right) + I_{\psi^{-1}((3b-a)/2)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3a-b}{2} \right) \right) \right] \\
 &\leq \frac{g((3a-b)/2) + g((3b-a)/2)}{4}.
 \end{aligned} \tag{17}$$

From (16) and (17), we obtain the right-hand side of inequality (11).

This completes the proof. $\square$

Remark 1. Based on (16), we have the inequality

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((a+b)/2)}{2} \\ & \leq \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi)\left(\psi^{-1}\left(\frac{3b-a}{2}\right)\right) + I_{\psi^{-1}((3b-a)/2)^-}^{\alpha;\psi} (g \circ \psi)\left(\psi^{-1}\left(\frac{3a-b}{2}\right)\right) \right]. \end{aligned} \quad (18)$$

From (17) and (18), we obtain a new inequality

$$0 < \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((a+b)/2)}{2} \leq \frac{g((3a-b)/2) + g((3b-a)/2)}{4}. \quad (19)$$

Theorem 1. Let $g: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$, $\psi(x)$ is an increasing and positive monotone function on $[a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) , and $\alpha \in (0, 1)$. $g': [((3a-b)/2), ((3b-a)/2)] \rightarrow \mathbb{R}$ is a continuous function on $[((3a-b)/2), ((3b-a)/2)]$. $q \geq 1$. If $|g'|^q$ is a convex function on $[((3a-b)/2), ((3b-a)/2)]$, then the following inequalities hold:

$g': [((3a-b)/2), ((3b-a)/2)] \rightarrow \mathbb{R}$ is a continuous function on $[((3a-b)/2), ((3b-a)/2)]$. $q \geq 1$. If $|g'|^q$ is a convex function on $[((3a-b)/2), ((3b-a)/2)]$, then the following inequalities hold:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ & \leq \frac{b-a}{2} \left[\frac{1}{\alpha+1} + \frac{1}{2} \right] \left(\left| g'\left(\frac{3b-a}{2}\right) \right| + \left| g'\left(\frac{3a-b}{2}\right) \right| \right). \end{aligned} \quad (20)$$

Proof. By Lemma 2, substituting $((3a-b)/2)$ for a and $((3b-a)/2)$ for b , we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2^{\alpha+1}(b-a)^\alpha} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi)\left(\psi^{-1}\left(\frac{3b-a}{2}\right)\right) + I_{\psi^{-1}((3b-a)/2)^-}^{\alpha;\psi} (g \circ \psi)\left(\psi^{-1}\left(\frac{3a-b}{2}\right)\right) \right] \\ & = g\left(\frac{a+b}{2}\right) + \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((a+b)/2)} -\frac{1}{2} (g' \circ \psi)(v) \psi'(v) dv + \int_{\psi^{-1}((a+b)/2)}^{\psi^{-1}((3b-a)/2)} \frac{1}{2} (g' \circ \psi)(v) \psi'(v) dv \\ & \quad + \frac{1}{2^{\alpha+1}(b-a)^\alpha} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left[\left(\frac{3b-a}{2} - \psi(v) \right)^\alpha - \left(\psi(v) - \frac{3a-b}{2} \right)^\alpha \right] (g' \circ \psi)(v) \psi'(v) dv. \end{aligned} \quad (21)$$

From (16) and (21), we have

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\
& \leq \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi)\left(\psi^{-1}\left(\frac{3b-a}{2}\right)\right) + I_{\psi^{-1}((3b-a)/2)^-}^{\alpha;\psi} (g \circ \psi)\left(\psi^{-1}\left(\frac{3a-b}{2}\right)\right) \right] - \frac{1}{2}g\left(\frac{a+b}{2}\right) \\
& = \frac{1}{2} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((a+b)/2)} \left(-\frac{1}{2}\right) (g' \circ \psi)(v) \psi'(v) dv + \int_{\psi^{-1}((a+b)/2)}^{\psi^{-1}((3b-a)/2)} \frac{1}{2} (g' \circ \psi)(v) \psi'(v) dv \\
& \quad + \frac{1}{2^{\alpha+1}(b-a)^\alpha} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left[\left(\frac{3b-a}{2} - \psi(v)\right)^\alpha - \left(\psi(v) - \frac{3a-b}{2}\right)^\alpha \right] (g' \circ \psi)(v) \psi'(v) dv \\
& \leq \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((a+b)/2)} \frac{1}{4} |(g' \circ \psi)(v) \psi'(v)| dv + \int_{\psi^{-1}((a+b)/2)}^{\psi^{-1}((3b-a)/2)} \frac{1}{4} |(g' \circ \psi)(v) \psi'(v)| dv \\
& \quad + \frac{1}{2^{\alpha+2}(b-a)^\alpha} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left| \left[\left(\frac{3b-a}{2} - \psi(v)\right)^\alpha - \left(\psi(v) - \frac{3a-b}{2}\right)^\alpha \right] (g' \circ \psi)(v) \psi'(v) \right| dv \tag{22} \\
& \leq \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \frac{1}{4} |(g' \circ \psi)(v) \psi'(v)| dv + \frac{1}{2^{\alpha+2}(b-a)^\alpha} \left[\int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left(\frac{3b-a}{2} - \psi(v)\right)^\alpha |(g' \circ \psi)(v) \psi'(v)| dv \right. \\
& \quad \left. + \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left(\psi(v) - \frac{3a-b}{2}\right)^\alpha |(g' \circ \psi)(v) \psi'(v)| dv \right] \\
& \left(\text{let } \psi(v) = \frac{3b-a}{2} - 2t(b-a) \right) \\
& \leq \frac{b-a}{2} \int_0^1 \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt + \frac{b-a}{2} \int_0^1 t^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\
& \quad + \frac{b-a}{2} \int_0^1 (1-t)^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt.
\end{aligned}$$

If $q = 1$ and the function $|g'|$ is a convex function on $[(3a-b)/2, (3b-a)/2]$, then we have

$$\left| g' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| \leq t \left| g' \left(\frac{3a-b}{2} \right) \right| + (1-t) \left| g' \left(\frac{3b-a}{2} \right) \right|, \tag{23}$$

for any $t \in [0, 1]$.

Therefore, we obtain

$$\begin{aligned} \int_0^1 \left| g' \left(t \frac{3a-b}{2} + (1-t) \frac{3b-a}{2} \right) \right| dt &\leq \int_0^1 t \left| g' \left(\frac{3a-b}{2} \right) \right| dt + \int_0^1 (1-t) \left| g' \left(\frac{3b-a}{2} \right) \right| dt \\ &= \frac{|g'((3a-b)/2)| + |g'((3b-a)/2)|}{2}, \end{aligned} \quad (24)$$

$$\begin{aligned} &\int_0^1 (1-t)^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt + \int_0^1 t^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\ &\leq \int_0^1 (1-t)^\alpha \left[t \left| g' \left(\frac{3a-b}{2} \right) \right| + (1-t) \left| g' \left(\frac{3b-a}{2} \right) \right| \right] dt + \int_0^1 t^\alpha \left[t \left| g' \left(\frac{3a-b}{2} \right) \right| + (1-t) \left| g' \left(\frac{3b-a}{2} \right) \right| \right] dt \\ &= \left| g' \left(\frac{3a-b}{2} \right) \right| \left[\int_0^1 (1-t)^\alpha t + t^{\alpha+1} dt \right] + \left| g' \left(\frac{3b-a}{2} \right) \right| \left[\int_0^1 (1-t)^{\alpha+1} + t^\alpha (1-t) dt \right] \\ &= \frac{1}{\alpha+1} \left| g' \left(\frac{3a-b}{2} \right) \right| + \frac{1}{\alpha+1} \left| g' \left(\frac{3b-a}{2} \right) \right| \\ &= \frac{1}{\alpha+1} \left[\left| g' \left(\frac{3a-b}{2} \right) \right| + \left| g' \left(\frac{3b-a}{2} \right) \right| \right]. \end{aligned} \quad (25)$$

From (22), (24), and (25), if $q = 1$, we have

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ &\leq \frac{b-a}{2} \cdot \frac{\alpha+3}{2\alpha+2} \left[\left| g' \left(\frac{3a-b}{2} \right) \right| + \left| g' \left(\frac{3b-a}{2} \right) \right| \right]. \end{aligned} \quad (26)$$

Suppose that $q > 1$, $|g'|^q$ is convex on $[(3a-b)/2, (3b-a)/2]$; using Hölder's inequality ($q > 1, p = (q/(q-1))$) and Jensen inequality, we have

$$\begin{aligned} &\int_0^1 \left| g' \left(t \frac{3a-b}{2} + (1-t) \frac{3b-a}{2} \right) \right| dt \\ &\leq \left(\int_0^1 1^{(q/(q-1))} dt \right)^{1-(1/q)} \left(\int_0^1 \left| g' \left(t \frac{3a-b}{2} + (1-t) \frac{3b-a}{2} \right) \right|^q dt \right)^{(1/q)} \leq \left(\int_0^1 t \left| g' \left(\frac{3a-b}{2} \right) \right|^q + (1-t) \left| g' \left(\frac{3b-a}{2} \right) \right|^q dt \right)^{(1/q)} \\ &= \left(\frac{|g'((3a-b)/2)|^q}{2} + \frac{|g'((3b-a)/2)|^q}{2} \right)^{(1/q)} \\ &\leq \frac{1}{2} \left(\left| g' \left(\frac{3a-b}{2} \right) \right| + \left| g' \left(\frac{3b-a}{2} \right) \right| \right), \end{aligned} \quad (27)$$

$$\begin{aligned}
& \int_0^1 t^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\
&= \int_0^1 (t^\alpha)^{1-(1/q)} \left[(t^\alpha)^{(1/q)} \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| \right] dt \\
&\leq \left(\int_0^1 t^\alpha dt \right)^{1-(1/q)} \left(\int_0^1 t^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right|^q dt \right)^{(1/q)} \\
&\leq \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\int_0^1 t^\alpha \left(\left| g' \left(\frac{3a-b}{2} \right| \right)^q + (1-t) \left| g' \left(\frac{3b-a}{2} \right| \right)^q \right) dt \right)^{(1/q)} \\
&= \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\frac{1}{\alpha+2} \left| g' \left(\frac{3a-b}{2} \right| \right|^q + B(\alpha+1, 2) \left| g' \left(\frac{3b-a}{2} \right| \right|^q \right)^{(1/q)},
\end{aligned} \tag{28}$$

$$\begin{aligned}
& \int_0^1 (1-t)^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\
&= \int_0^1 ((1-t)^\alpha)^{1-(1/q)} \left[((1-t)^\alpha)^{(1/q)} \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| \right] dt \\
&\leq \left(\int_0^1 (1-t)^\alpha dt \right)^{1-(1/q)} \left(\int_0^1 (1-t)^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right|^q dt \right)^{(1/q)} \\
&\leq \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\int_0^1 (1-t)^\alpha \left(\left| g' \left(\frac{3a-b}{2} \right| \right)^q + (1-t) \left| g' \left(\frac{3b-a}{2} \right| \right)^q \right) dt \right)^{(1/q)} \\
&= \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(B(2, \alpha+1) \left| g' \left(\frac{3a-b}{2} \right| \right|^q + \frac{1}{\alpha+2} \left| g' \left(\frac{3b-a}{2} \right| \right|^q \right)^{(1/q)},
\end{aligned} \tag{29}$$

where B is the beta function.

From (27)–(29), using Jensen inequality, we have

$$\begin{aligned}
& \int_0^1 \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt + \int_0^1 t^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\
&+ \int_0^1 (1-t)^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\
&\leq \frac{1}{2} \left(\left| g' \left(\frac{3a-b}{2} \right| \right) + \left| g' \left(\frac{3b-a}{2} \right| \right) \right) \\
&+ \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} 2 \left[\frac{1}{2} \left(\frac{1}{\alpha+2} + B(2, \alpha+1) \right) \left(\left| g' \left(\frac{3b-a}{2} \right| \right|^q + \left| g' \left(\frac{3a-b}{2} \right| \right|^q \right) \right)^{(1/q)} \right] \\
&\leq \frac{1}{2} \left(\left| g' \left(\frac{3a-b}{2} \right| \right) + \left| g' \left(\frac{3b-a}{2} \right| \right) \right) \\
&+ \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\frac{1}{2} \right)^{(1/q)-1} \left(\frac{1}{\alpha+2} + B(2, \alpha+1) \right)^{(1/q)} 2^{(1/q)-1} \left(\left| g' \left(\frac{3b-a}{2} \right| \right) + \left| g' \left(\frac{3a-b}{2} \right| \right) \right) \\
&= \left[\left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\frac{1}{\alpha+2} + B(2, \alpha+1) \right)^{(1/q)} + \frac{1}{2} \right] \left(\left| g' \left(\frac{3b-a}{2} \right| \right) + \left| g' \left(\frac{3a-b}{2} \right| \right) \right) \\
&= \left[\left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\frac{1}{\alpha+1} \right)^{(1/q)} + \frac{1}{2} \right] \left(\left| g' \left(\frac{3b-a}{2} \right| \right) + \left| g' \left(\frac{3a-b}{2} \right| \right) \right) \\
&= \left[\frac{1}{\alpha+1} + \frac{1}{2} \right] \left(\left| g' \left(\frac{3b-a}{2} \right| \right) + \left| g' \left(\frac{3a-b}{2} \right| \right) \right).
\end{aligned} \tag{30}$$

From (22) and (30), if $q > 1$, we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ & \leq \frac{b-a}{2} \left[\frac{1}{\alpha+1} + \frac{1}{2} \right] \left(\left| g'\left(\frac{3b-a}{2}\right) \right| + \left| g'\left(\frac{3a-b}{2}\right) \right| \right). \end{aligned} \quad (31)$$

In conclusion, we get (20). The proof is complete.

In the context of $q = 1$, comparing (5) with (20), the conclusion of (20) is easier to get. If we get the value of $|g'((3b-a)/2)|, |g'((3a-b)/2)|$, we get the estimation for

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) \right. \\ & \left. + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right). \end{aligned} \quad (32)$$

But (5) needs to calculate the value of $g'(a), g'(b), |g'(a)|, |g'(b)|$.

Based on Theorem 1, we give Theorem 2. $\square$

Theorem 2. Under the assumptions of Theorem 1, if $q > 1$, then the following inequalities hold:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ & \leq \frac{b-a}{2} \left[\left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} + \frac{1}{2} \right] \left(\left| g'\left(\frac{3b-a}{2}\right) \right| + \left| g'\left(\frac{3a-b}{2}\right) \right| \right). \end{aligned} \quad (33)$$

Proof. By Hölder's inequality $((1/p) + (1/q) = 1, p = (q/(q-1)))$ and Jensen inequality, we have

$$\begin{aligned} & \int_0^1 t^\alpha \left| g'\left(\frac{3b-a}{2}(1-t) + \frac{3a-b}{2}t\right) \right| dt \\ & \leq \left(\int_0^1 (t^\alpha)^{(q/(q-1))} dt \right)^{1-(1/q)} \left(\int_0^1 \left| g'\left(\frac{3b-a}{2}(1-t) + \frac{3a-b}{2}t\right) \right|^q dt \right)^{(1/q)} \\ & \leq \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\int_0^1 t \left| g'\left(\frac{3a-b}{2}\right) \right|^q + (1-t) \left| g'\left(\frac{3b-a}{2}\right) \right|^q dt \right)^{(1/q)} \\ & = \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\frac{1}{2} \left| g'\left(\frac{3a-b}{2}\right) \right|^q + \frac{1}{2} \left| g'\left(\frac{3b-a}{2}\right) \right|^q \right)^{(1/q)}, \end{aligned} \quad (34)$$

$$\begin{aligned} & \int_0^1 (1-t)^\alpha \left| g'\left(\frac{3b-a}{2}(1-t) + \frac{3a-b}{2}t\right) \right| dt \\ & \leq \left(\int_0^1 ((1-t)^\alpha)^{(q/(q-1))} dt \right)^{1-(1/q)} \left(\int_0^1 \left| g'\left(\frac{3b-a}{2}(1-t) + \frac{3a-b}{2}t\right) \right|^q dt \right)^{(1/q)} \\ & \leq \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\int_0^1 t \left| g'\left(\frac{3a-b}{2}\right) \right|^q + (1-t) \left| g'\left(\frac{3b-a}{2}\right) \right|^q dt \right)^{(1/q)} \\ & = \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\frac{1}{2} \left| g'\left(\frac{3a-b}{2}\right) \right|^q + \frac{1}{2} \left| g'\left(\frac{3b-a}{2}\right) \right|^q \right)^{(1/q)}. \end{aligned} \quad (35)$$

From (27), (34), and (35), using Jensen inequality, we have

$$\begin{aligned} & \int_0^1 \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt + \int_0^1 t^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\ & + \int_0^1 (1-t)^\alpha \left| g' \left(\frac{3b-a}{2} (1-t) + \frac{3a-b}{2} t \right) \right| dt \\ & \leq \left[\left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} + \frac{1}{2} \right] \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right). \end{aligned} \quad (36)$$

From (22) and (36), we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ & \leq \frac{b-a}{2} \left[\left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} + \frac{1}{2} \right] \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right). \end{aligned} \quad (37)$$

The proof is complete. $\square$

Corollary 1. From Theorems 1-2, we get the following inequalities for $q > 1$:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ & \leq \frac{b-a}{2} \min\{N_1, N_2\} \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right), \end{aligned} \quad (38)$$

where $N_1 = (1/(\alpha+1)) + (1/2)$ and $N_2 = (1 + (\alpha q/(q-1)))^{(1-1/q)} + (1/2)$.

Now, we get the estimation for

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] \\ & - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4}, \end{aligned} \quad (39)$$

in different ways from above.

Theorem 3. Under the assumptions of Theorem 1, if $|g'|^q$ is a convex function on $[(3a-b)/2, (3b-a)/2]$, $q \geq 1$, then the following inequalities hold:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] \\ & - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\ & \leq \frac{b-a}{2\alpha+2} \left(\left| g' \left(\frac{3a-b}{2} \right) \right| + \left| g' \left(\frac{3b-a}{2} \right) \right| \right). \end{aligned} \quad (40)$$

Proof. By Lemma 1, substituting $((3a-b)/2)$ for a and $((3b-a)/2)$ for b , we have

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2^{\alpha+1}(b-a)^{\alpha}} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3b-a}{2} \right) \right) + I_{\psi^{-1}((3b-a)/2)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3a-b}{2} \right) \right) \right] \\
&= \frac{g((3a-b)/2) + g((3b-a)/2)}{2} - \frac{1}{2^{\alpha+1}(b-a)^{\alpha}} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left[\left(\psi(v) - \frac{3a-b}{2} \right)^{\alpha} - \left(\frac{3b-a}{2} - \psi(v) \right)^{\alpha} \right] (g' \circ \psi)(v) \psi'(v) dv.
\end{aligned} \tag{41}$$

From (16) and (41), we have

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1}(b) \right) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1}(a) \right) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\
&\leq \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^{\alpha}} \left[I_{\psi^{-1}((3a-b)/2)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3b-a}{2} \right) \right) + I_{\psi^{-1}((3b-a)/2)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1} \left(\frac{3a-b}{2} \right) \right) \right] \\
&\quad - \frac{g((3a-b)/2) + g((3b-a)/2)}{4} \\
&= \frac{1}{2^{\alpha+1}(b-a)^{\alpha}} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left[\left(\psi(v) - \frac{3a-b}{2} \right)^{\alpha} - \left(\frac{3b-a}{2} - \psi(v) \right)^{\alpha} \right] (g' \circ \psi)(v) \psi'(v) dv \\
&\leq \left| -\frac{1}{2^{\alpha+1}(b-a)^{\alpha}} \int_{\psi^{-1}((3a-b)/2)}^{\psi^{-1}((3b-a)/2)} \left[\left(\psi(v) - \frac{3a-b}{2} \right)^{\alpha} - \left(\frac{3b-a}{2} - \psi(v) \right)^{\alpha} \right] (g' \circ \psi)(v) \psi'(v) dv \right| \\
&\quad \left(\text{let } \psi(v) = \frac{3a-b}{2} + 2t(b-a) \right) \\
&= \frac{b-a}{2} \left| \int_0^1 [t^{\alpha} - (1-t)^{\alpha}] g' \left(\frac{3a-b}{2}(1-t) + \frac{3b-a}{2}t \right) dt \right| \\
&\leq \frac{b-a}{2} \left[\int_0^1 t^{\alpha} \left| g' \left(\frac{3a-b}{2}(1-t) + \frac{3b-a}{2}t \right) \right| dt + \int_0^1 (1-t)^{\alpha} \left| g' \left(\frac{3a-b}{2}(1-t) + \frac{3b-a}{2}t \right) \right| dt \right].
\end{aligned} \tag{42}$$

Suppose that $q = 1$; the function $|g'|$ is a convex function on $[(3a-b)/2, (3b-a)/2]$, so we have

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1}(b) \right) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi) \left(\psi^{-1}(a) \right) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\
&\leq \frac{b-a}{2} \left[\int_0^1 t^{\alpha} \left[(1-t) \left| g' \left(\frac{3a-b}{2} \right) \right| + t \left| g' \left(\frac{3b-a}{2} \right) \right| \right] dt \right. \\
&\quad \left. + \int_0^1 (1-t)^{\alpha} \left[(1-t) \left| g' \left(\frac{3a-b}{2} \right) \right| + t \left| g' \left(\frac{3b-a}{2} \right) \right| \right] dt \right] \\
&\leq \frac{b-a}{2} \left[\int_0^1 t^{\alpha} (1-t) \left| g' \left(\frac{3a-b}{2} \right) \right| dt + \int_0^1 t^{\alpha+1} \left| g' \left(\frac{3b-a}{2} \right) \right| dt \right. \\
&\quad \left. + \int_0^1 (1-t)^{\alpha+1} \left| g' \left(\frac{3a-b}{2} \right) \right| dt + \int_0^1 (1-t)^{\alpha} t \left| g' \left(\frac{3b-a}{2} \right) \right| dt \right] \\
&= \frac{b-a}{2} \left(B(2, \alpha+1) + \frac{1}{\alpha+2} \right) \left(\left| g' \left(\frac{3a-b}{2} \right) \right| + \left| g' \left(\frac{3b-a}{2} \right) \right| \right).
\end{aligned} \tag{43}$$

Then, we suppose that $q > 1$; the function $|g'|^q$ is a convex function on $[(3a-b)/2, (3b-a)/2]$, so we have

$$\begin{aligned}
 & \int_0^1 t^\alpha \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right| dt \\
 & \leq \int_0^1 (t^\alpha)^{1-(1/q)} \left[(t^\alpha)^{(1/q)} \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right| \right] dt \\
 & \leq \left(\int_0^1 t^\alpha dt \right)^{1-(1/q)} \left(\int_0^1 t^\alpha \left| g' \left(t \left(\frac{3b-a}{2} \right) + (1-t) \left(\frac{3a-b}{2} \right) \right) \right|^q dt \right)^{(1/q)} \\
 & \leq \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\int_0^1 t^\alpha \left(\left| g' \left(\frac{3b-a}{2} \right) \right|^q + (1-t) \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right) dt \right)^{(1/q)} \\
 & \leq \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\frac{1}{\alpha+2} \left| g' \left(\frac{3b-a}{2} \right) \right|^q + B(2, \alpha+1) \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right)^{(1/q)},
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 & \int_0^1 (1-t)^\alpha \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right| dt \\
 & \leq \int_0^1 ((1-t)^\alpha)^{1-(1/q)} \left[((1-t)^\alpha)^{(1/q)} \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right| \right] dt \\
 & \leq \left(\int_0^1 (1-t)^\alpha dt \right)^{1-(1/q)} \left(\int_0^1 (1-t)^\alpha \left| g' \left(t \left(\frac{3b-a}{2} \right) + (1-t) \left(\frac{3a-b}{2} \right) \right) \right|^q dt \right)^{(1/q)} \\
 & \leq \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\int_0^1 (1-t)^\alpha \left(\left| g' \left(\frac{3b-a}{2} \right) \right|^q + (1-t) \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right) dt \right)^{(1/q)} \\
 & \leq \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(B(2, \alpha+1) \left| g' \left(\frac{3b-a}{2} \right) \right|^q + \frac{1}{\alpha+2} \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right)^{(1/q)}.
 \end{aligned} \tag{45}$$

From (42), (44), and (45), using Jensen inequality, for $q > 1$, we have

$$\begin{aligned}
 & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g^\circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g^\circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\
 & \leq \left(\frac{1}{2} \right)^{(1/q)} (b-a) \left(\frac{1}{\alpha+1} \right)^{1-(1/q)} \left(\frac{1}{\alpha+2} + B(2, \alpha+1) \right)^{(1/q)} \left(\left| g' \left(\frac{3a-b}{2} \right) \right|^q + \left| g' \left(\frac{3b-a}{2} \right) \right|^q \right)^{(1/q)} \\
 & \leq \frac{b-a}{2\alpha+2} \left(\left| g' \left(\frac{3a-b}{2} \right) \right| + \left| g' \left(\frac{3b-a}{2} \right) \right| \right).
 \end{aligned} \tag{46}$$

In conclusion, we get (40). The proof is complete. Based on Theorem 3, we give Theorems 4 and 5. $\square$

Theorem 4. Under the assumptions of Theorem 1, if $|g'|^q$ is a convex function on $[(3a-b)/2, (3b-a)/2]$, $q > 1$, then the following inequalities hold:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\ & \leq \frac{b-a}{2} \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right). \end{aligned} \quad (47)$$

Proof. From the convex function of $|g'|^q$ and Holder inequality, we have

$$\begin{aligned} & \int_0^1 t^\alpha \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right| dt \\ & \leq \left(\int_0^1 (t^\alpha)^{(q/(q-1))} dt \right)^{1-(1/q)} \left(\int_0^1 \left| g' \left(t \left(\frac{3b-a}{2} \right) + (1-t) \left(\frac{3a-b}{2} \right) \right) \right|^q dt \right)^{(1/q)} \\ & \leq \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\int_0^1 t \left| g' \left(\frac{3b-a}{2} \right) \right|^q + (1-t) \left| g' \left(\frac{3a-b}{2} \right) \right|^q dt \right)^{(1/q)} \\ & \leq \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\frac{1}{2} \left| g' \left(\frac{3b-a}{2} \right) \right|^q + \frac{1}{2} \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right)^{(1/q)}, \end{aligned} \quad (48)$$

$$\begin{aligned} & \int_0^1 (1-t)^\alpha \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right| dt \\ & \leq \left(\int_0^1 ((1-t)^\alpha)^{(q/(q-1))} dt \right)^{1-(1/q)} \left(\int_0^1 \left| g' \left(t \left(\frac{3b-a}{2} \right) + (1-t) \left(\frac{3a-b}{2} \right) \right) \right|^q dt \right)^{(1/q)} \\ & \leq \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\int_0^1 t \left| g' \left(\frac{3b-a}{2} \right) \right|^q + (1-t) \left| g' \left(\frac{3a-b}{2} \right) \right|^q dt \right)^{(1/q)} \\ & \leq \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\frac{1}{2} \left| g' \left(\frac{3b-a}{2} \right) \right|^q + \frac{1}{2} \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right)^{(1/q)}. \end{aligned} \quad (49)$$

From (42), (47), and (49), using Jensen inequality, we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\ & \leq (b-a) \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\frac{1}{2} \right)^{(1/q)} \left(\left| g' \left(\frac{3b-a}{2} \right) \right|^q + \left| g' \left(\frac{3a-b}{2} \right) \right|^q \right)^{(1/q)} \\ & \leq \frac{b-a}{2} \left(1 + \frac{\alpha q}{q-1} \right)^{1-(1/q)} \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right). \end{aligned} \quad (50)$$

The proof is complete. $\square$

Theorem 5. Under the assumptions of Theorem 1, if $|g'|^q$ is a convex function on $[(3a-b)/2, (3b-a)/2]$, $q > 1$, then the following inequalities hold:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\ & \leq \frac{b-a}{4} \left(\frac{2p}{\alpha+1} + (1-p) \right)^{(1/p)} \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right), \end{aligned} \quad (51)$$

where $(1/p) + (1/q) = 1$.

Proof. By (42), using Holder inequality, Jensen inequality, and Lemma 3 (let $K = 1$, $(q/p) = p$), we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\ & \leq \frac{b-a}{2} \left| \int_0^1 [t^\alpha - (1-t)^\alpha] g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) dt \right| \\ & \leq \frac{b-a}{2} \left| \int_0^1 [t^\alpha - (1-t)^\alpha] dt \right| \left| \int_0^1 g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) dt \right| \\ & \leq \frac{b-a}{2} \left[\int_0^1 [t^\alpha + (1-t)^\alpha]^p dt \right]^{(1/p)} \left[\int_0^1 \left| g' \left(\frac{3a-b}{2} (1-t) + \frac{3b-a}{2} t \right) \right|^q dt \right]^{(1/q)} \\ & \leq \frac{b-a}{2} \left[\int_0^1 p(1-t)^\alpha + pt^\alpha + (1-p) dt \right]^{(1/p)} \left[\int_0^1 (1-t) \left| g' \left(\frac{3a-b}{2} \right) \right|^q + t \left| g' \left(\frac{3b-a}{2} \right) \right|^q dt \right]^{(1/q)} \\ & \leq \frac{b-a}{4} \left(\frac{2p}{\alpha+1} + (1-p) \right)^{(1/p)} \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right). \end{aligned} \quad (52)$$

The proof is complete. $\square$

Corollary 2. From Theorems 3–5, we get the following inequality for $q > 1$:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha;\psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4} \\ & \leq \min\{N_3, N_4, N_5\} \left(\left| g' \left(\frac{3b-a}{2} \right) \right| + \left| g' \left(\frac{3a-b}{2} \right) \right| \right), \end{aligned} \quad (53)$$

where $N_3 = ((b-a)/(2\alpha+2))$, $N_4 = ((b-a)/2)(1 + (\alpha q/(q-1)))^{1-(1/q)}$, $N_5 = ((b-a)/4)((2p/(\alpha+1)) + (1-p))^{(1/p)}$.

4. Applications

Example 1. We consider the following the arithmetic means, geometric means, logarithmic means, and generalized logarithmic means for positive real numbers a, b , and $a \neq b$:

$$A(a, b) = \frac{a+b}{2}, \quad a, b \in R,$$

$$G(a, b) = \sqrt{ab}, \quad a, b \in R,$$

$$L(a, b) = \frac{b-a}{\ln|b| - \ln|a|}, \quad |a| \neq |b|, \quad ab \neq 0,$$

$$L_n(a, b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{(1/n)}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad a, b \in R, \quad a \neq b. \quad (54)$$

Now, using the results of Section 3, we have some conclusions to the special means of real numbers.

(1) Let $a, b \in \mathbb{R}$, $0 < a < b$, then we have

$$L^{-1}(a, b) - A^{-1}(a, b) \leq (b-a) \min\{N_1, N_2\} \cdot A\left(\left|\frac{3b-a}{2}\right|^{-2}, \left|\frac{3a-b}{2}\right|^{-2}\right), \quad (55)$$

where $N_1 = 1$, $N_2 = (1 + (q/(q-1)))^{1-(1/q)} + (1/2)$.

In fact, applying Corollary 1 with $g(x) = (1/x)$, $\psi(x) = x$, and $\alpha = 1$, we have

$$\begin{aligned} L^{-1}(a, b) - A^{-1}(a, b) &= \frac{\ln|b| - \ln|a|}{b-a} - \frac{2}{a+b} \\ &= \frac{1}{2(b-a)} \left[\int_a^b \frac{1}{v} dv + \int_a^b \frac{1}{v} dv \right] - \frac{2}{a+b} \\ &= \frac{\alpha}{2(b-a)^\alpha} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(v) (b - \psi(v))^{\alpha-1} (g \circ \psi)(v) dv \right. \\ &\quad \left. + \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(v) (\psi(v) - a)^{\alpha-1} (g \circ \psi)(v) dv \right] - g\left(\frac{a+b}{2}\right) \\ &= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right) \\ &\leq \frac{b-a}{2} \min\{N_1, N_2\} \left(\left| g'\left(\frac{3b-a}{2}\right) \right| + \left| g'\left(\frac{3a-b}{2}\right) \right| \right) \\ &= \frac{b-a}{2} \min\{N_1, N_2\} \left(\left| \frac{3b-a}{2} \right|^{-2} + \left| \frac{3a-b}{2} \right|^{-2} \right) \\ &= (b-a) \min\{N_1, N_2\} A\left(\left|\frac{3b-a}{2}\right|^{-2}, \left|\frac{3a-b}{2}\right|^{-2}\right). \end{aligned} \quad (56)$$

Then, we obtain the result.

(2) Let $a, b \in \mathbb{R}$, $0 < a < b$, then we have

$$\left| L_n^n(a, b) - A^n(a, b) \right| \leq n(b-a) \min\{N_1, N_2\} \cdot A\left(\left|\frac{3b-a}{2}\right|^{n-1}, \left|\frac{3a-b}{2}\right|^{n-1}\right), \quad (57)$$

where $N_1 = 1$, $N_2 = (1 + (q/(q-1)))^{1-(1/q)} + (1/2)$.

Applying Corollary 1 with $g(x) = x^n$, $\psi(x) = x$, and $\alpha = 1$, by similar calculation, we obtain the result.

(3) Let $a, b \in \mathbb{R}$, $0 < a < b$, then we have

$$|G^{-2}(a, b) - A^{-2}(a, b)| \leq 2(b-a) \min\{N_1, N_2\} \cdot A\left(\left|\frac{3b-a}{2}\right|^{-3}, \left|\frac{3a-b}{2}\right|^{-3}\right), \quad (58)$$

where $N_1 = 1$, and $N_2 = (1 + (q/(q-1)))^{1-(1/q)} + (1/2)$.

Applying Corollary 1 with $g(x) = (1/x^2)$, $\psi(x) = x$, $\alpha = 1$, by similar calculation, we obtain the result.

From Corollary 2, we get the estimate for $B(\alpha, \rho)$.

Example 2. Let $\alpha > 0$, $a = 0$, $b = 1$, $g(x) = x^{\rho-1}$ ($x \in [0, 1]$, $\rho \geq 3$), $\psi(x) = x$, $\Gamma(\alpha)$ is the gamma function, and $B(\alpha, \rho)$ is the beta function. Then, $|g'|$ is convex on $[(1/2), (3/2)]$. So, we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) \\ &= \frac{\alpha}{2} \int_0^1 (1-x)^{\alpha-1} x^{\rho-1} dx = \frac{\alpha}{2} B(\alpha, \rho), \\ & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \\ &= \frac{\alpha}{2} \int_0^1 x^{\alpha+\rho-2} dx = \frac{\alpha}{2(\alpha+\rho-1)}. \end{aligned} \quad (59)$$

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right), \\ & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] - \frac{g((3a-b)/2) + g((3b-a)/2) + 2g((a+b)/2)}{4}. \end{aligned} \quad (62)$$

In the context of $q = 1$, both (5) and (20) estimate for

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) \right. \\ & \left. + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right] - g\left(\frac{a+b}{2}\right). \end{aligned} \quad (63)$$

In Theorem 1, we extend the range of argument of the function $|g'|$, and get a different conclusion. $|g'|$ is convex on $[((3a-b)/2), ((3b-a)/2)]$. If we get the value of $|g'|((3b-a)/2)$, $|g'|((3a-b)/2)$, we get the conclusion of (20). In [3], $|g'|$ is convex on $[a, b]$, and if we have the value of $|g'(a)|$, $|g'(b)|$, $g(a)$ and $g(b)$, we have the conclusion of (5).

Data Availability

The data used to support the findings of this study are available within the article.

From Corollary 2, the following inequality holds:

$$\begin{aligned} & B(\alpha, \rho) + \frac{1}{(\alpha+\rho-1)} - \frac{(-1/2)^{\rho-1} + (3/2)^{\rho-1} + 2(1/2)^{\rho-1}}{2\alpha} \\ & \leq \frac{2}{\alpha(\rho-1)} \min\{N_3, N_4, N_5\} \left(\left(\frac{1}{2}\right)^{\rho-2} + \left(\frac{3}{2}\right)^{\rho-2} \right), \end{aligned} \quad (60)$$

where $N_3 = (1/(2\alpha+2))$, $N_4 = (1/2)(1 + (\alpha q/(q-1)))^{1-(1/q)}$, and $N_5 = (1/4)((2p/(\alpha+1)) + (1-p))^{(1/p)}$.

5. Conclusion

This paper extends the range of argument of the function g' and establishes some new Hermite–Hadamard type inequalities for ψ -Riemann–Liouville fractional integrals via convex functions. If g is a convex function on $[a, b]$, we obtain an estimate for

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha; \psi} (g \circ \psi)(\psi^{-1}(a)) \right]. \quad (61)$$

If $|g'|^q$ is a convex function on $[((3a-b)/2), ((3b-a)/2)]$, we obtain estimates for

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

YS carried out the main results and completed the corresponding proof. RX participated in the proof and helped to complete Section 4. All authors read and approved the final manuscript.

Acknowledgments

This research was supported by National Science Foundation of China (nos. 11671227 and 11971015) and the Natural Science Foundation of Shandong Province (ZR2019MA034).

References

- [1] Y. Shuang and F. Qi, "Integral inequalities of Hermite-Hadamard type for extended s -convex functions and applications," *Mathematics*, vol. 6, no. 11, p. 223, 2018.
- [2] I. İşcan, "New general integral inequalities for quasi-geometrically convex functions via fractional integrals," *Journal of Inequalities and Applications*, vol. 2013, p. 491, 2013.
- [3] K. Liu, J. R. Wang, and D. O'Regan, "On the Hermite-Hadamard type inequality for ψ -Riemann-Liouville fractional integrals via convex functions," *Journal of Inequalities and Applications*, vol. 2019, p. 27, 2019.
- [4] Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, "Hermite-Hadamard-type inequalities for functions whose derivatives are η -convex via fractional integrals," *Journal of Inequalities and Applications*, vol. 2019, p. 44, 2019.
- [5] M. K. Bakula, M. E. Özdemir, and J. Pecaric, "Hadamard type inequalities for m -convex and (α, m) -convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 4, p. 12, 2018.
- [6] S.-R. Hwang, K.-C. Hsu, and K.-L. Tseng, "Hadamard-type inequalities for Lipschitzian functions in one and two variables with applications," *Journal of Mathematical Analysis and Applications*, vol. 405, no. 2, pp. 546–554, 2013.
- [7] J. Wang, J. Deng, and M. Fečkan, "Hermite-Hadamard-type inequalities for r -convex functions based on the use of Riemann-Liouville fractional integrals," *Ukrainian Mathematical Journal*, vol. 65, no. 2, pp. 193–211, 2013.
- [8] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.
- [9] U. S. Kirmaci and M. E. Özdemir, "On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 153, no. 2, pp. 361–368, 2004.
- [10] K. Mehrez and P. Agarwal, "New Hermite-Hadamard type integral inequalities for convex functions and their applications," *Journal of Computational and Applied Mathematics*, vol. 350, pp. 274–285, 2019.
- [11] R. Xu and Y. Zhang, "Generalized Gronwall fractional summation inequalities and their applications," *Journal of Inequalities and Applications*, vol. 2015, p. 242, 2015.
- [12] R. Xu and F. Meng, "Some new weakly singular integral inequalities and their applications to fractional differential equations," *Journal of Inequalities and Applications*, vol. 2016, no. 1, pp. 1–16, 2016.
- [13] R. Xu, "Some new nonlinear weakly singular integral inequalities and their applications," *Journal of Mathematical Inequalities*, vol. 11, no. 4, pp. 1007–1018, 2017.
- [14] R. Xu and X. Ma, "Some generalized nonlinear Gamidov type integral inequalities with maxima in two variables and their weakly singular analogues," *Journal of Mathematical Inequalities*, vol. 13, no. 2, 2019.
- [15] Y. Luo and R. Xu, "Some new weakly singular integral inequalities with discontinuous functions for two variables and their applications," *Advances in Difference Equations*, vol. 2019, p. 387, 2019.
- [16] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.
- [17] F. C. Jang and F. W. Meng, "Explicit bounds in some new nonlinear integral inequalities with delay," *Journal of Computational and Applied Mathematics*, vol. 205, no. 1, pp. 479–486, 2007.