Research Article

Dynamics of a Discrete Allelopathic Phytoplankton Model with Infinite Delays and Feedback Controls

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A discrete allelopathic phytoplankton model with infinite delays and feedback controls is studied in this paper. By applying the discrete comparison theorem, a set of sufficient conditions which guarantees the permanence of the system is obtained. Also, by constructing some suitable discrete Lyapunov functionals, some sufficient conditions for the extinction of the system are obtained. Our results extend and supplement some known results and show that the feedback controls and toxic substances play a crucial role on the permanence and extinction of the system.

1. Introduction

Given a bounded sequence of real numbers \( f(k) \), let \( f^u \) and \( f^l \) denote sup \( f(k) \) and inf \( f(k) \), respectively.

Many real-world phenomena are studied through discrete mathematical models governed by difference equations which are more appropriate than the continuous ones when the populations have nonoverlapping generations; the study of the dynamic behaviors of discrete time models becomes the subject of intense research in mathematics biology, such topics as permanence and extinction, and existence of positive periodic solution (almost periodic solution), have been extensively studied by many scholars (see [1–23] and the references cited therein).

Recently, some scholars believed that a more appropriate competition model should be considered with nonlinear interinhibition terms. Qin et al. [1] and Wang et al. [2] considered the following discrete two species competitive system with nonlinear interinhibition terms:

\[
\begin{align*}
x_1(k+1) &= x_1(k) \exp \left\{ r_1(k) - a_1(k)x_1(k) - \frac{c_1(k)x_2(k)}{1 + x_2(k)} \right\}, \\
x_2(k+1) &= x_2(k) \exp \left\{ r_2(k) - a_2(k)x_2(k) - \frac{c_2(k)x_1(k)}{1 + x_1(k)} \right\}
\end{align*}
\]

where \( r_i(k), a_i(k), c_i(k) \) \((i = 1, 2)\) are assumed to be bounded positive sequences, and \( x_1(k) \) and \( x_2(k) \) represent the density of species \( x_1 \) and \( x_2 \) at the \( k \)th generation, respectively, \( r_i(k) \) \((i = 1, 2)\) is the intrinsic growth rate of two species. \( a_i(k) \) \((i = 1, 2)\) is the rate of intraspecific competition of the first and second species, respectively. \( c_i(k) \) \((i = 1, 2)\) is the rate of interspecific competition of the first and second species, respectively. Qin et al. [1] obtained sufficient conditions which ensure the permanence, existence, and global stability of positive periodic solutions of system (1). As for the almost periodic case, Wang and Liu [2] further...
investigated the existence, uniqueness, and uniformly asymptotic stability of positive almost periodic solution of system (1).

In recent years, many scholars have made great achievements in the competition system with the effect of toxic substance [3, 4, 24–26], Yue [3] investigated system (1) with the second species which could be toxic, while the other one is nontoxic. He studied system (2) and gave the sufficient conditions of the extinction of one species and the global attractive of the other one.

By applying the discrete comparison theorem, a set of sufficient conditions for the extinction of one species and the global attractive of the other one.

\[
x_1(k + 1) = x_1(k) \exp \left\{ r_1(k) - a_1(k)x_1(k) - \frac{c_2(k)x_2(k)}{1 + x_2(k)} - b_1(k)x_1(k)x_2(k) \right\},
\]
\[
x_2(k + 1) = x_2(k) \exp \left\{ r_2(k) - a_2(k)x_2(k) - \frac{c_1(k)x_1(k)}{1 + x_1(k)} \right\},
\]

(2)

Note that ecosystems are easily disturbed by human activities, such as planting and harvesting, which can give rise to changes of population density. In order to give a better description of such a system, scholars introduced feedback control variables into ecosystems. Many researchers have done research on the systems with feedback control variables [6–8, 11, 14, 16, 17]. Recently, Wang and Liu [6] studied the competitive system with the effect of toxic substance [3, 4, 24–26], Yue [3] investigated system (1) with the second species which could be toxic, while the other one is nontoxic. He studied system (2) and gave the sufficient conditions of the extinction of one species and the global attractive of the other one.

As we all know, due to seasonal fluctuations in the environment and hereditary factors, time delays have been introduced into the biological system (see [11, 14, 21, 27–36]). Zhao et al. [8] further considered a discrete Lotka–Volterra competition system with infinite delays and single feedback control variable as follows:

\[
x_1(n + 1) = x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n) - \frac{b_1(n)x_2(n)}{1 + x_2(n)} - c_1(n)u_1(n) \right\},
\]
\[
x_2(n + 1) = x_2(n) \exp \left\{ r_2(n) - \frac{b_2(n)x_1(n)}{1 + x_1(n)} - a_2(n)x_2(n) - c_2(n)u_2(n) \right\},
\]
\[
\Delta u_1(n) = -b_1(n)u_1(n) + d_1(n)x_1(n),
\]
\[
\Delta u_2(n) = -b_2(n)u_2(n) + d_2(n)x_2(n),
\]

(3)

where \(u_i(n), i = 1, 2\), denotes the feedback control variables. Wang and Liu [6] studied the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (3). Yu [7] further investigated the influence of feedback control variables on the extinction of the system. He gave a set of sufficient conditions for the extinction of one species.

By applying the discrete comparison theorem, a set of sufficient conditions which guarantees the permanence of the system is obtained. Also, by constructing some suitable discrete Lyapunov functionals, some sufficient conditions for the global attractivity and extinction of the system are obtained.

Through the above discussion, we find that, based on system (2), we can further consider the infinite delay and...
feedback control. So, we propose and study the following
discrete competition system with infinite delays and feed-
back control variables:

\[
x_1(k+1) = x_1(k)\exp\left\{r_1(k) - a_1(k)x_1(k) - \frac{b_1(k)\sum_{s=0}^{\infty}K_{12}(s)x_2(k-s)}{1 + \sum_{s=0}^{\infty}K_{11}(s)x_1(k-s)} - \gamma(k)x_1(k)x_2^2(k) - c_1(k)\sum_{s=0}^{\infty}H_1(s)u_1(k-s)\right\},
\]
\[
x_2(k+1) = x_2(k)\exp\left\{r_2(k) - \frac{b_2(k)\sum_{s=0}^{\infty}K_{21}(s)x_1(k-s)}{1 + \sum_{s=0}^{\infty}K_{22}(s)x_2(k-s)} - a_2(k)x_2(k) - c_2(k)\sum_{s=0}^{\infty}H_2(s)u_2(k-s)\right\},
\]
\[
u_1(k+1) = u_1(k)(1 - \epsilon_1(k)) + d_{11}(k)\sum_{s=0}^{\infty}L_1(s)x_1(k-s),
\]
\[
u_2(k+1) = u_2(k)(1 - \epsilon_2(k)) + d_{21}(k)\sum_{s=0}^{\infty}L_2(s)x_2(k-s).
\]

(5)

In system (4), \(x_i(k)\) \((i = 1, 2)\) is the density of \(x_i\) species at the \(k\)th generation, and \(u_i(k), i = 1, 2\), is the feedback control variable.

Throughout this paper, we assume that
\((C_i)\ r_i(k), a_i(k), b_i(k), c_i(k), d_i(k),\) and \(\epsilon_i(k)\) \((i = 1, 2)\)
are bounded sequences of real numbers defined on \(\mathbb{Z}\) such that
\[
\begin{align*}
    r_i^l &> 0, \\
    a_i^l &> 0, \\
    b_i^l &> 0, \\
    c_i^l &> 0, \\
    d_i^l &> 0, \\
    0 < e_i^l < e_i^u < 1,
\end{align*}
\]

\((C_2)\ K_{ij}(s), H_i(s),\) and \(L_i(s)\) \((i, j = 1, 2)\) are nonnegative bounded sequences such that
\[
\begin{align*}
    \sum_{s=0}^{\infty}K_{ij}(s) = 1, \\
    \sum_{s=0}^{\infty}H_i(s) = 1, \\
    \sum_{s=0}^{\infty}L_i(s) = 1, \quad i, j = 1, 2, \\
    \Theta_{ij} = \sum_{s=0}^{\infty}K_{ij}(s), \quad s < +\infty, \\
    \Lambda_i = \sum_{s=0}^{\infty}H_i(s), \quad s < +\infty, \\
    \Psi_i = \sum_{s=0}^{\infty}L_i(s), \quad s < +\infty.
\end{align*}
\]

According to the biological background of system (5), we only consider the solution of system (5) with the following initial conditions:
\[
\begin{align*}
    x_i(s) &= \Psi_i(s) \geq 0, \\
    \Psi_i(0) &> 0, \\
    \sup_{k \in \mathbb{Z}} \Psi_i(k) < +\infty, \\
    u_i(s) &= \Phi(s) \geq 0, \\
    \Phi_i(0) &> 0, \\
    \sup_{k \in \mathbb{Z}} \Phi_i(k) < +\infty,
\end{align*}
\]

\((i = 1, 2)\)

where \(s = \ldots, -n, -n + 1, \ldots, 1, 0\). It is easy to prove that the solution of system (5) which satisfies the initial conditions (8) is positive.

Here, we mention that this is the first time such kind of model be proposed and studied, and as far as system (5) is concerned, whether the feedback control variables and toxic substances have influence on the permanence and extinction of the system or not is an interesting problem. The aim of this paper is to investigate the permanence and extinction of system (5); specially, we will find out the answer to the above problem.

The paper is structured in the following way. Some useful Lemmas are presented in Section 2. In Sections 3 and 4, by using the methods of Zhao et al. [8], we investigate the permanence and extinction of system (5). Three examples are presented to show the feasibility of the main results in Section 5. We end this paper by a brief discussion.

2. Lemmas

Now, let us consider the following difference equation:
\[
y(k + 1) = ay(k) + b,
\]

(9)
where \( a \) and \( b \) are positive constants.

**Lemma 1** (see [37]). Assume that \( |a| < 1 \), for any initial value \( y(0) \), there exists a unique solution \( y(k) \) of equation (9), which can be expressed as follows:

\[
y(k) = a^k (y(0) - y^*) + y^*,
\]

where \( y^* = b/(1 - a) \). Thus, for any solution \( y(k) \) of system (9), we have

\[
\lim_{k \to \infty} y(k) = y^*.
\]

**Lemma 2** (see [37]). Let \( k \in N^+_k = \{k_0, k_0 + 1, \ldots, k_0 + l, \ldots\} \), \( r \geq 0 \). For any fixed \( k \), \( g(k, r) \) is a nondecreasing function with respect to \( r \), and for \( k \geq k_0 \), the following inequalities hold:

\[
y(k + 1) \leq g(k, y(k)), u(k + 1) \geq g(k, u(k)).
\]

If \( y(k_0) \leq u(k_0) \), then \( y(k) \leq u(k) \) for all \( k \geq k_0 \).

**Lemma 3** (see [38]). Assume that \( r(n) > 0 \), \( x(n) \) satisfies \( x(n) > 0 \) and

\[
x(n + 1) \leq x(n) \exp[r(n)(1 - ax(n))],
\]

for \( n \in [n_1, \infty) \), where \( a \) is a positive constant. Then,

\[
\limsup_{n \to \infty} x(n) \leq \frac{1}{a r^n} \exp(r^n - 1).
\]

**Lemma 4** (see [38]). Assume that \( r(n) > 0 \), \( x(n) \) satisfies \( x(n) > 0 \) and

\[
x(n + 1) \geq x(n) \exp[r(n)(1 - ax(n))],
\]

for \( n \in [n_1, \infty) \), \( \limsup_{n \to \infty} x(n) \leq x^* \) and \( x(n_1) > 0 \), where \( a \) and \( x^* \) are positive constants such that \( ax^* > 1 \). Then,

\[
\liminf_{n \to \infty} x(n) \geq \frac{1}{a} \exp(r^n - 1). \tag{14}
\]

**Lemma 5** (see [38]). Let \( x : Z \to R \) be a nonnegative bounded sequence, and let \( H: Z_+ \to R \) be a nonnegative sequence such that \( \sum_{n=0}^{\infty} H(n) = 1 \), then

\[
\liminf_{n \to \infty} x(n) \leq \liminf_{n \to \infty} \sum_{s=0}^{n} H(n - s) x(s) \leq \limsup_{n \to \infty} \sum_{s=0}^{n} H(n - s) x(s) \leq \limsup_{n \to \infty} x(n). \tag{17}
\]

3. **Permanence**

Concerned with the permanence of system (5), we have the following result.

**Theorem 1.** Assume that

\[
\frac{r_i^k - b_i^k M_j - c_i^k U_j}{M_i M_j} > 0, \quad i, j = 1, 2, i \neq j,
\]

\[
y^u < \frac{r_i^k - b_i^k M_j - c_i^k U_j}{M_i M_j}
\]

hold; then, for any positive solution \((x_1(k), x_2(k), u_1(k), u_2(k))\) of system (5), we have

\[
m_i \leq \liminf_{k \to \infty} x_i(k) < \limsup_{k \to \infty} x_i(k) \leq M_i,
\]

\[
B_i \leq \liminf_{k \to \infty} u_i(k) < \limsup_{k \to \infty} u_i(k) \leq U_i,
\]

where

\[
M_i = \frac{1}{a_i^k} \exp(r_i^k - 1),
\]

\[
U_i = \frac{d_i^k M_i}{a_i^k},
\]

\[
m_1 = \frac{r_i^k - b_i^k M_2 - c_i^k U_2}{a_i^k} \cdot \exp(r_i^k - b_i^k M_2 - c_i^k U_2 - a_i^k M_1),
\]

\[
m_2 = \frac{r_i^k - b_i^k M_1 - c_i^k U_1}{a_i^k} \cdot \exp(r_i^k - b_i^k M_1 - c_i^k U_1 - a_i^k M_2),
\]

\[
B_i = \frac{d_i^k m_i}{c_i^k} \quad i = 1, 2.
\]

**Proof.** From the first and second equations of system (5), we have

\[
x_i(k + 1) \leq x_i(k) \exp\left[ r_i(k) \left( 1 - \frac{d_i^k}{r_i^k} x_i(k) \right) \right], \quad i = 1, 2.
\]

Hence, from Lemma 3, we can obtain

\[
\limsup_{k \to \infty} x_i(k) \leq \frac{1}{a_i^k} \exp(r_i^k - 1) \text{ def } M_i. \tag{22}
\]

According to Lemma 5 and the above inequality, we have

\[
\limsup_{k \to \infty} \sum_{s=0}^{\infty} L_i(s) x_i(k - s) = \limsup_{k \to \infty} \sum_{s=0}^{\infty} L_i(k - s) x_i(s) \leq \limsup_{k \to \infty} x_i(k) \leq M_i. \tag{23}
\]

For any \( \varepsilon > 0 \), there exists a positive integer \( N_1 \) such that

\[
\sum_{s=0}^{\infty} L_i(s) x_i(k - s) \leq M_i + \varepsilon, \quad \text{for all } n > N_1. \tag{24}
\]

By the third and fourth equations of system (5), we have
\[ u_i(k + 1) \leq (1 - \varepsilon_i^t)u_i(k) + d_i^u(M_i + \varepsilon). \tag{25} \]

Hence, by applying Lemmas 1 and 2 to (25), we obtain
\[ \limsup_{k \to \infty} u_i(k) \leq \frac{d_i^u(M_i + \varepsilon)}{\varepsilon_i}. \tag{26} \]

Setting \( \varepsilon \to 0 \), it follows that
\[ \limsup_{k \to \infty} u_i(k) \leq \frac{d_i^u M_i}{\varepsilon_i} \equiv U_i. \tag{27} \]

Condition (18) implies that, for enough small positive constant \( \varepsilon_1 \), the following inequalities hold:
\[ r_1^i - b_1^u(M_2 + \varepsilon_1) - \gamma^u(M_1 + \varepsilon_1)(M_2 + \varepsilon_1)^2 - c_1^u(U_1 + \varepsilon_1) > 0, \]
\[ r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1) > 0. \tag{28} \]

For above \( \varepsilon_1 \), it follows from (22) and (27) that there exists a positive integer \( N_2 \geq N_1 \) such that
\[ \sum_{s=0}^{\infty} H_i(s)u_i(k - s) \leq U_i + \varepsilon_1, \]
\[ \sum_{s=0}^{\infty} K_{ji}(s)x_j(k - s) \leq M_j + \varepsilon_1, \tag{29} \]
\[ x_i(k) \leq M_i + \varepsilon_1 \quad \text{for all } n > N_2. \]

Thus, for all \( n > N_2 \), from (28), (29), and the first two equations of system (5), we have
\[ x_1(k + 1) \geq x_1(k) \exp\left\{ (r_1^i - b_1^u(M_2 + \varepsilon_1) - \gamma^u(M_1 + \varepsilon_1) \right\} \]
\[ \cdot (M_2 + \varepsilon_1)^2 - c_1^u(U_1 + \varepsilon_1) (1 - D_1 x_1(k)) \}, \]
\[ x_2(k + 1) \geq x_2(k) \exp\left\{ (r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1) \right\} \]
\[ \cdot (1 - D_2 x_2(k)) \}, \tag{30} \]

where
\[ D_1 = \frac{a_1^u}{r_1^i - b_1^u(M_2 + \varepsilon_1) - \gamma^u(M_1 + \varepsilon_1)(M_2 + \varepsilon_1)^2 - c_1^u(U_1 + \varepsilon_1)}, \]
\[ D_2 = \frac{a_2^u}{r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1)}. \tag{31} \]

Notice that
\[ \frac{a_1^u}{a_1^i} \geq 1, \]
\[ \exp(r_1^i - 1) \frac{r_1^i - b_1^u(M_2 + \varepsilon_1) - \gamma^u(M_1 + \varepsilon_1)(M_2 + \varepsilon_1)^2 - c_1^u(U_1 + \varepsilon_1)}{r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1)} > 1, \]
\[ \exp(r_2^i - 1) \frac{r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1)}{r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1)} > 1. \tag{32} \]

then
\[ D_1 \cdot M_1 = \frac{a_1^u}{r_1^i - b_1^u(M_2 + \varepsilon_1) - \gamma^u(M_1 + \varepsilon_1)(M_2 + \varepsilon_1)^2 - c_1^u(U_1 + \varepsilon_1)} \cdot \frac{1}{a_1^i} \exp(r_1^i - 1) > 1, \]
\[ D_2 \cdot M_2 = \frac{a_2^u}{r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1)} \cdot \frac{1}{a_2^i} \exp(r_2^i - 1) > 1. \tag{33} \]

Hence, according to Lemma 4, we have
\[ \liminf_{k \to \infty} x_1(k) \geq \frac{1}{D_1} \cdot \exp\left( r_1^i - b_1^u(M_2 + \varepsilon_1) - \gamma^u(M_1 + \varepsilon_1)(M_2 + \varepsilon_1)^2 - c_1^u(U_1 + \varepsilon_1) - a_1^u M_1 \right), \]
\[ \liminf_{k \to \infty} x_2(k) \geq \frac{1}{D_2} \cdot \exp\left( r_2^i - b_2^u(M_1 + \varepsilon_1) - c_2^u(U_2 + \varepsilon_1) - a_2^u M_2 \right). \tag{34} \]

Setting \( \varepsilon_1 \to 0 \), it follows that
\[ \liminf_{k \to \infty} x_1(k) \geq \frac{1}{D_1} \cdot \exp\left( r_1^i - b_1^u M_2 - \gamma^u M_1 M_2^2 - c_1^u U_1 - a_1^u M_1 \right) \equiv m_1, \tag{35} \]
\[ \liminf_{k \to \infty} x_2(k) \geq \frac{1}{D_2} \cdot \exp\left( r_2^i - b_2^u M_1 - c_2^u U_2 - a_2^u M_2 \right) \equiv m_2. \tag{36} \]
where
\[
D_1 = \left( r_1^u - b_1^u M_2 - c_1^u (U_1) \right)
\]
\[
D_2 = \left( r_2^u - b_2^u M_1 \right)
\]

(37)

According to Lemma 5, from (35) and (36), we have, for any \( \varepsilon > 0 \) small enough (without loss of generality, assume that \( \varepsilon < (1/2)\min \{ m_i \} \), \( i = 1, 2 \), there exists \( N_3 > N_2 \), such that
\[
\sum_{s=0}^{\infty} L_i (s) x_i (k-s) \geq m_i - \varepsilon, \quad \text{for all } n \geq N_3.
\]

(38)

For \( n \geq N_3 \), from (38) and the third and fourth equations of system (5), we have
\[
\eta_i (k) \geq \frac{d_i^1 (m_i - \varepsilon)}{\varepsilon^i}.
\]

(39)

Hence, by applying Lemmas 1 and 2 to (39), we obtain
\[
\limsup_{k \to \infty} \eta_i (k) \geq \frac{d_i^1 (m_i - \varepsilon)}{\varepsilon^i}.
\]

(40)

Setting \( \varepsilon \to 0 \), it follows that
\[
\limsup_{k \to \infty} \eta_i (k) \geq \frac{d_i^1 m_i}{\varepsilon^i} = B_i.
\]

(41)

This ends the proof of Theorem 3.1. \( \square \)

4. Extinction

Concerned with the extinction property of system (5), we could establish the following results.

Theorem 2. Assume that
\[
\frac{r_1^u}{r_2^u} < \min \left\{ \frac{d_1^1 e_1^i + c_1^1 d_1^i}{b_1^u e_1^i}, \frac{b_1^u}{b_2^u e_1^i} \frac{b_1^u}{b_2^u e_1^i} \right\} \quad \text{(H1)},
\]

(42)

holds. Let \( (x_1 (k), x_2 (k), u_1 (k), u_2 (k)) \) be any positive solution of system (5), then
\[
\lim_{k \to \infty} x_1 (k) = 0, \quad \lim_{k \to \infty} u_1 (k) = 0.
\]

(43)

Theorem 3. Assume that
\[
\frac{r_1^u}{r_2^u} > \max \left\{ \frac{1}{M_1}, \frac{1}{M_2} \right\} \left( (1 + M_1) \frac{a_1^1 e_1^i + c_1^1 d_1^i}{b_1^u e_1^i}, \frac{b_1^u}{b_2^u e_1^i} + c_1^1 d_1^i \right) \quad \text{(H2)},
\]

\[
\eta \leq \frac{1}{M_1 M_2^2} \left( \frac{r_1^u}{r_2^u} - (1 + M_1) \frac{a_1^1 e_1^i + c_1^1 d_1^i}{b_2^u e_1^i}, \frac{b_1^u}{b_2^u e_1^i} + c_1^1 d_1^i \right),
\]

(44)

Consider the following discrete Lyapunov functional

hold. Let \( (x_1 (k), x_2 (k), u_1 (k), u_2 (k)) \) be any positive solution of system (5), then
\[
\lim_{k \to \infty} x_2 (k) = 0, \quad \lim_{k \to \infty} u_2 (k) = 0.
\]

(45)

Proof of Theorem 2. Condition \((H_1)\) is equivalent to
\[
\frac{c_1^i}{c_2^i} > \frac{r_1^u}{r_2^u} \frac{b_1^u}{d_1^i} - \frac{a_1^u}{d_1^i},
\]

(46)

From (46), one could choose positive constants \( \alpha, \beta, \eta_1, \eta_2 \), and \( \eta_3 \) and enough small positive \( \varepsilon \) such that
\[
\frac{r_1^u}{r_2^u} < \frac{\beta}{\alpha}, \quad \frac{c_1^i}{c_2^i} > \frac{\beta b_1^u - a d_1^i}{a d_1^i} > \frac{r_1^u b_1^u}{r_2^u d_2^i} \frac{a_1^u}{d_1^i},
\]

(47)

That is,
\[
\alpha r_1^u - \beta r_2^u < -\lambda_1 < 0,
\]

\[
\eta_1 e_1^i - a c_1^1 < 0,
\]

\[
\beta e_2^u - \eta_1 d_1^i < 0,
\]

\[
\beta b_2^u - a d_1^i - \eta_2 d_1^i < 0,
\]

\[
\eta_2 d_2^u + a d_1^i - \frac{\alpha t}{1 + M_2 + \varepsilon} < 0.
\]

(48)

Let \( (x_1 (k), x_2 (k), u_1 (k), u_2 (k)) \) be a positive solution of system (5). For above \( \varepsilon \), from Theorem 1, there exists an enough large \( T_1 \), such that
\[
x_i (t) < M_i + \varepsilon,
\]

\[
u_i (t) < U_i + \varepsilon,
\]

(49)

Consider the following discrete Lyapunov functional
\[
V_1(k) = x_1^\alpha(k)x_2^\beta(k)\exp\left\{ \eta_1 u_2(k) - \eta_1 u_1(k) - \frac{\alpha \sum_{s=0}^{\infty} K_{12}(s) \sum_{q=k-s}^{k-1} b_1(q+s) x_2(q)}{1 + M_2} - \frac{\alpha \sum_{s=0}^{\infty} H_1(s) \sum_{q=k-s}^{k-1} c_1(q+s) u_1(q)}{1 + M_2} \\
+ \frac{\beta \sum_{s=0}^{\infty} K_{21}(s) \sum_{q=k-s}^{k-1} b_2(q+s) x_1(q)}{1 + M_1} + \frac{\beta \sum_{s=0}^{\infty} K_{12}(s) \sum_{q=k-s}^{k-1} b_2(q+s) x_1(q)}{1 + M_1} \right\}.
\]

By calculating, we obtain

\[
\frac{V_1(k+1)}{V_1(k)} = \exp\left\{ a \left( r_1(k) - a_1(k)x_1(k) - \frac{b_1(k) \sum_{s=0}^{\infty} K_{12}(s) x_2(k-s)}{1 + \sum_{s=0}^{\infty} K_{12}(s) x_2(k-s)} - y(k) x_1(k) x_2(k) - c_1(k) \sum_{s=0}^{\infty} H_1(s) u_1(k-s) \right) \\
- \beta \left( r_2(k) - \frac{b_2(k) \sum_{s=0}^{\infty} K_{21}(s) x_1(k-s)}{1 + \sum_{s=0}^{\infty} K_{21}(s) x_1(k-s)} - a_2(k) x_2(k) - c_2(k) \sum_{s=0}^{\infty} H_2(s) u_2(k-s) \right) \\
- \eta_1 \left( -e_1(k) u_1(k) + d_1(k) \sum_{s=0}^{\infty} L_1(s) x_1(k-s) \right) + \eta_2 \left( -e_2(k) u_2(k) + d_2(k) \sum_{s=0}^{\infty} L_2(s) x_2(k-s) \right) \\
- \frac{\alpha}{1 + M_2} \sum_{s=0}^{\infty} K_{12}(s) \left( b_1(k+x) x_2(k) - b_1(k) x_2(k-s) \right) - \alpha \sum_{s=0}^{\infty} H_1(s) \left( c_1(k+s) u_1(k) - c_1(k) u_1(k-s) \right) \\
+ \frac{\beta}{1 + M_1} \sum_{s=0}^{\infty} K_{21}(s) \left( b_2(k+s) x_1(k) - b_2(k) x_1(k-s) \right) + \beta \sum_{s=0}^{\infty} H_2(s) \left( c_2(k+s) u_2(k) - c_2(k) u_2(k-s) \right) \\
- \eta_1 \sum_{s=0}^{\infty} L_1(s) d_1(k+x) x_1(k-s) + \eta_2 \sum_{s=0}^{\infty} L_2(s) d_2(k+x) x_2(k-s) \right\} \\
\leq \exp\left\{ ar_1(k) - \beta r_2(k) \right\} + \left( -aa_1(k) + \beta \sum_{s=0}^{\infty} K_{21}(s) b_2(k+s) - \eta_1 \sum_{s=0}^{\infty} L_1(s) d_1(k+s) \right) x_1(k) \\
+ \left( \beta a_2(k) + \eta_2 \sum_{s=0}^{\infty} L_2(s) d_2(k+s) - \frac{\alpha \sum_{s=0}^{\infty} K_{12}(s) b_1(k+s)}{1 + (M_2 + \epsilon)} \right) x_2(k) + \left( \eta_1 e_1(k) - \eta_1 \sum_{s=0}^{\infty} H_1(s) c_1(k+s) \right) u_1(k) \\
+ \left( \beta \sum_{s=0}^{\infty} H_2(s) c_2(k+s) - \eta_2 e_2(k) \right) u_2(k) \right\} \\
\leq \exp\left\{ (ar_1^u - \beta r_2^u) + (\beta e_2^u - aa_1^u) - \eta_1 d_1^u \right\} x_1(k) + \left( \eta_1 e_1^u + \beta a_2^u - \frac{\alpha b_1^u}{1 + (M_2 + \epsilon)} \right) x_2(k) \\
+ \left( \eta_1 e_1^u - ac_1^u \right) u_1(k) + \left( \beta e_2^u - \eta_2 e_2^u \right) u_2(k) \right\}.
\]
From inequalities (48) and (51), we can obtain

\[ V_1(k + 1) \leq V_1(k) \exp(-\lambda_1). \quad (52) \]

Therefore,

\[ V_1(k) \leq V_1(0) \exp(-k\lambda_1). \quad (53) \]

Hence,

\[ V_1(0) = x_1^a(0)x_2^{-\beta}(0)\exp\left\{ \eta_2u_2(0) - \eta_1u_1(0) - \frac{\alpha \sum_{s=0}^{\infty} K_{12}(s) \sum_{q=-s}^{s-1} b_1(q+s)x_2(q)}{1 + M_2} - \sum_{s=0}^{\infty} H_1(s) \sum_{q=-s}^{s-1} c_1(q+s)u_1(q) + \sum_{s=0}^{\infty} H_2(s) \sum_{q=-s}^{s-1} c_2(q+s)u_2(q) + \eta_2 \sum_{s=0}^{\infty} L_2(s) \sum_{q=-s}^{s-1} d_2(q+s)x_2(q) \right\} \]

\[ < \beta \sum_{s=0}^{\infty} K_{21}(s) \sum_{q=-s}^{s-1} b_2(q+s)x_1(q) \]

\[ + \beta \sum_{s=0}^{\infty} H_2(s) \sum_{q=-s}^{s-1} c_2(q+s)u_2(q) + \eta_2 \sum_{s=0}^{\infty} L_2(s) \sum_{q=-s}^{s-1} d_2(q+s)x_2(q) \]

\[ - \eta_1 \sum_{s=0}^{\infty} L_1(s) \sum_{q=-s}^{s-1} d_1(q+s)x_1(q) \}

\[ < \infty. \]

On the contrary, we also have

\[ V_1(k) \geq x_1^a(k)x_2^{-\beta}(k)\exp\left\{ -\eta_1u_1(k) - \frac{\alpha \sum_{s=0}^{\infty} K_{12}(s) \sum_{q=-s}^{s-1} b_1(q+s)x_2(q)}{1 + M_2} - \sum_{s=0}^{\infty} H_1(s) \sum_{q=-s}^{s-1} c_1(q+s)u_1(q) \}

\[ - \sum_{s=0}^{\infty} L_1(s) \sum_{q=-s}^{s-1} d_1(q+s)x_1(q) \}

\[ \geq x_1^a(k)M^{-\beta}\exp\left\{ -\eta_1M - \frac{ab_1^\mu M \sum_{s=0}^{\infty} K_{12}(s)}{1 + M_2} - \sum_{s=0}^{\infty} H_1(s)M + \eta_1d_1^\mu \sum_{s=0}^{\infty} L_1(s) \}

\[ = x_1^a(k)M^{-\beta}\exp\left\{ -\eta_1(M + d_1^\mu MY_1) - \alpha \left( \frac{b_1^\mu M \Theta_{12}}{1 + M_2} + c_1^\mu \lambda_1 \right) \} \}

\[ \Delta = (V_1(0))^{1/\alpha} M^{\beta/s}\exp\left\{ \frac{\eta_1}{\alpha} (M + d_1^\mu MY_1) \right\}

\[ + \left( \frac{b_1^\mu M \Theta_{12}}{1 + M_2} + c_1^\mu \lambda_1 \right) \}

\[ < \infty. \]

Combining inequalities (53), (55), and (56), we have

\[ x_1(k) \leq \Delta \exp\left\{ -\frac{\lambda_1}{\alpha} k \right\}, \quad (57) \]

where

\[ x_i(k) < M, \]

\[ u_i(k) < M, \quad \text{for all } k \in \mathbb{Z}, i = 1, 2. \]

Hence, we obtain that
\[ \lim_{k \to +\infty} x_1(k) = 0. \]  
\[ (59) \]

Similar to the corresponding proof of Theorem 1 by Chen et al. [7], we can easily obtain that \( \lim_{k \to +\infty} u_1(k) = 0. \) This ends the proof of Theorem 2.

**Proof of Theorem 3.** Condition \( (H_4) \) is equivalent to

\[
\begin{align*}
\frac{c^u_1}{c^l_1} &< \frac{r^l_1 b^l_2}{(1 + M_1) r^m_2 d^m_1} \frac{a^u_1}{d^m_1}, \\
\frac{c^l_2}{c^u_2} &> \frac{r^l_2 b^l_1}{r^m_1 d^m_2} \frac{a^l_1}{d^m_2}. \\
\end{align*}
\]
\[ (60) \]

From \( (H_4) \) and \( (60) \), there exist positive constants \( \alpha, \beta, \delta_1, \) and \( \delta_2 \) and enough small positive \( \epsilon \) such that

\[
\begin{align*}
\frac{r^l_1}{r^l_2} &> \frac{\beta}{\alpha}, \\
\frac{c^u_1}{c^l_1} &< \frac{\beta b^l_1 - (1 + M_1 + \epsilon) a^u_1}{(1 + M_1 + \epsilon) a^u_1} < \frac{r^l_1 b^l_2}{(1 + M_1 + \epsilon) r^m_2 d^m_1} - \frac{a^m_1}{d^m_1}, \\
\frac{c^l_2}{c^u_2} &> \frac{ab^u_1 - \beta a^l_1}{\beta d^m_2} > \frac{r^l_2 b^l_1}{r^m_1 d^m_2} - \frac{a^l_1}{d^m_2}, \\
&\frac{1 + M_1 + \epsilon}{b^l_2 c^l_1} \frac{\beta}{\alpha} < \frac{r^l_1 - \gamma^m \left( (M_1 + \epsilon) (M_2 + \epsilon) \right)^2}{r^l_2}, \\
&\frac{b^u_2 c^u_1}{a^u_1 c^u_1 + c^u_2 d^u_2} \frac{\beta}{\alpha} < \frac{r^l_1 - \gamma^m \left( (M_1 + \epsilon) (M_2 + \epsilon) \right)^2}{r^l_2}.
\end{align*}
\]
\[ (61) \]

That is,

\[
\begin{align*}
ac^u_1 - \delta_1 c^l_1 &< 0, \\
\delta_2 c^u_2 - \beta c^l_2 &< 0, \\
\alpha a^u_1 - \frac{\beta b^l_2}{1 + M_1 + \epsilon} + \delta_1 d^u_1 &< 0, \\
ab^u_1 - \beta a^l_1 - \delta_2 d^u_2 &< 0, \\
-ar^l_1 + \beta r^u_2 + \alpha \gamma^m (M_1 + \epsilon) (M_2 + \epsilon)^2 &< -\xi_1 < 0.
\end{align*}
\]

Let \( (x_1(k), x_2(k), u_1(k), u_2(k))^T \) be a positive solution of system \( (5) \). For above \( \epsilon \), from Theorem 1, there exists an enough large \( T_2 \), such that

\[ x_i(k) < M_1 + \epsilon, u_i(k) < U_i + \epsilon, \quad t \geq T_2, \quad i = 1, 2. \]
\[ (63) \]

Define the following Lyapunov functional as follows:

\[
\begin{align*}
V_2(k) &= x_1^a(k) x_2^b(k) \exp \left\{ \delta_1 u_1(k) - \delta_2 u_2(k) \right\} \\
&\quad + a \sum_{s=0}^{\infty} K_{12}(s) \sum_{q=k-s}^{k-1} b_1(q + s) x_2(q) \\
&\quad + a \sum_{s=0}^{\infty} H_1(s) \sum_{q=k-s}^{k-1} c_1(q + s) u_1(q) \\
&\quad - \frac{\beta \sum_{s=0}^{\infty} K_{21}(s) \sum_{q=k-s}^{k-1} b_2(q + s) x_1(q)}{1 + M_1} \\
&\quad - \beta \sum_{s=0}^{\infty} H_2(s) \sum_{q=k-s}^{k-1} c_2(q + s) u_2(q) \\
&\quad - \delta_2 \sum_{s=0}^{\infty} L_2(s) \sum_{q=k-s}^{k-1} d_2(q + s) x_2(q) \\
&\quad + \delta_1 \sum_{s=0}^{\infty} L_1(s) \sum_{q=k-s}^{k-1} d_1(q + s) x_1(q).
\end{align*}
\]
\[ (64) \]
Theorem 5. Assume that \((H_i)\) holds; also,
\[
\frac{\epsilon_1^2}{\epsilon_2^2} < \frac{A_{21} \rho_1}{d_1^2},
\]
holds; then, for any positive solution \((x_1(k), x_2(k), u_1(k), u_2(k))\) of system (5) and any positive solution \((x_1^*(k), u_2^*(k))\) of system (66), we have
\[
\lim_{k \to +\infty} x_1(k) = 0, \\
\lim_{k \to +\infty} u_1(k) = 0, \\
\lim_{k \to +\infty} (x_2(k) - x_2^*(k)) = 0, \\
\lim_{k \to +\infty} (u_2(k) - u_2^*(k)) = 0,
\]
where \(A_{21} = \min \{a_1^2, (2/M_2) - a_2^2\} \).

Proof of Theorem 4. By conditions (68), we can choose positive constants \(\rho_1 \) and \(\rho_2 \) such that
\[
\frac{\epsilon_2^2}{\epsilon_1^2} > \frac{\rho_1}{\rho_2} > \frac{d_2^2}{A_{22}}.
\]

Thus, there exists enough small positive constant \(\omega\) and \(\epsilon\) such that
\[
\rho_1 A_{22}^\epsilon - \rho_2 a_2^\omega > \omega, \\
\rho_2 \epsilon_1^\omega - \rho_1 a_2^\omega > \omega,
\]
where \(A_{22}^\epsilon = \min \{a_2^\epsilon, (2/(M_2 + \epsilon)) - a_2^\omega\} \).

Now, we define a Lyapunov functional as follows:
\[
G_1(k) = \rho_1 \left[ \ln x_2(k) - \ln x_2^*(k) \right] \\
+ \sum_{s=0}^{\infty} K_{21}(s) \sum_{q=n-s}^{n-1} b_2(q + s) x_1(q) \\
+ \sum_{s=0}^{\infty} H_2(s) \sum_{q=n-s}^{n-1} c_2(q + s) [u_2(q) - u_2^*(q)] \\
+ \rho_2 \left[ u_2(k) - u_2^*(k) \right] \\
+ \sum_{s=0}^{\infty} L_2(s) \sum_{q=n-s}^{n-1} d_2(q + s) [x_2(q) - x_2^*(q)]
\]

One could easily see that \(G_1(k) \geq 0 \) for all \(k \in Z^+ \). Also, for any fixed \(k^* \in Z^+ \), from (54), one could see that
\[
G_1(k^*) = \rho_1 \left[ \ln x_2(k^*) - \ln x_2^*(k^*) \right] + \rho_1 M_{22}^\epsilon \sum_{s=0}^{\infty} K_{21}(s) s \\
+ \rho_1 c_2^\epsilon \sup_{q \in Z^+, q < s} |u_2(q) - u_2^*(q)| \sum_{s=0}^{\infty} H_2(s) s \\
+ \rho_2 \left[ u_2(k^*) - u_2^*(k^*) \right] \\
+ \rho_2 d_2^\omega \sup_{q \in Z^+, q < k^*} |x_2(q) - x_2^*(q)| \sum_{s=0}^{\infty} L_2(s) s \\
< +\infty.
\]
It follows from the second equation of system (5) and the mean value theorem that

\[
\Delta G_1(k) \leq \rho_1 \left[ -\frac{1}{\varphi_2(k)} \left( a_\nu(k) - a_\mu(k) \right) \left( x_2(k) - x_2^*(k) \right) + b_2^\nu x_1(k) + c_2^\nu [u_2(k) - u_2^*(k)] \right] \\
+ \rho_2 \left[ -e_1^\nu [u_2(k) - u_2^*(k)] + d_1^\nu [x_2(k) - x_2^*(k)] \right].
\]

where \( \rho = \rho_1 b_2^\nu \).

Summating both sides of the above inequality from \( k^* \) to \( k \), we have

\[
\sum_{p=k^*}^{k} \left( G_1(p+1) - G_1(p) \right) \leq -\omega \sum_{p=k^*}^{k} \left( |x_2(p) - x_2^*(p)| + |u_2(p) - u_2^*(p)| \right) \\
+ \rho \sum_{p=k^*}^{k} x_1(p).
\]

Hence,

\[
G_1(k+1) + \omega \sum_{p=k^*}^{k} \left( |x_2(p) - x_2^*(p)| + |u_2(p) - u_2^*(p)| \right) \\
\leq G_1(k^*) + \rho \sum_{p=k^*}^{k} x_1(p).
\]

Then, from (75) and (57), we have

\[
\sum_{p=k^*}^{k} \left( |x_2(p) - x_2^*(p)| + |u_2(p) - u_2^*(p)| \right) \\
\leq G_1(k^*) + \rho \sum_{p=k^*}^{k} x_1(p).
\]

Therefore,

\[
\sum_{p=k^*}^{\infty} \left( |x_2(p) - x_2^*(p)| + |u_2(p) - u_2^*(p)| \right) < +\infty,
\]

which means that

\[
\lim_{k \to \infty} \left( |x_2(k) - x_2^*(k)| + |u_2(k) - u_2^*(k)| \right) = 0.
\]

Consequently,

\[
\lim_{k \to \infty} (x_2(k) - x_2^*(k)) = 0, \\
\lim_{k \to \infty} (u_2(k) - u_2^*(k)) = 0.
\]

This completes the proof of Theorem 4. \( \square \)

**Proof of Theorem 5.** The proof of Theorem 5 is similar to that of Theorem 4, and we omit the detail here. \( \square \)

## 5. Examples

In this section, we shall give three examples to illustrate the feasibility of main results.

**Example 1.** Consider the following equations:
\( x_1 (k + 1) = x_1 (k) \exp \left\{ - (5.2 + 0.2 \sin (k)) x_1 (k) - \frac{0.5 \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)}{1 + \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)} - 3x_1 (k) x_2^2 (k) - 3 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-2i} u_1 (k - s) \right\}, \)

\( x_2 (k + 1) = x_2 (k) \exp \left\{ 0.95 - 0.05 \cos (k) - \frac{0.2 \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)}{1 + \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)} - (2.5 + 0.5 \sin (k)) x_2 (k) - 0.8 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-2i} u_2 (k - s) \right\}, \)

\( u_1 (k + 1) = u_1 (k) (1 - 0.5) + 0.25 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-2i} x_1 (k - s), \)

\( u_2 (k + 1) = u_2 (k) (1 - 0.8) + 0.2 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-2i} x_2 (k - s). \)

(83)

One could easily see that conditions (C1) and (C2) are satisfied. Also, by calculating, one has

\[ M_1 = \frac{1}{d_1} \exp (r_1 - 1) = 0.2, \]

\[ M_2 = \frac{1}{d_2} \exp (r_2 - 1) = 0.5, \]

\[ U_1 = \frac{d_2}{c_1} M_1 = 0.1, \]

\[ U_2 = \frac{d_2}{c_2} M_2 = \frac{1}{8}, \]

\[ r_1 - b_1 M_2 - c_1 U_1 = 0.45 > 0, \]

\[ r_2 - b_2 M_1 - c_1 U_2 = 0.4 > 0, \]

\[ \gamma' = 3 < \frac{r_1 - b_1 M_2 - c_1 U_1}{M_1 M_2^2} = 22.5. \]

Clearly, condition (18) is satisfied, and so from Theorem 1, we have

\[ m_1 \leq \lim inf_{k \to \infty} x_1 (k) < \lim sup_{k \to \infty} x_1 (k) \leq M_1, \]

\[ B_1 \leq \lim inf_{k \to \infty} u_1 (k) < \lim sup_{k \to \infty} u_1 (k) \leq U_1, \]

where \( (x_1 (k), x_2 (k), u_1 (k), u_2 (k)) \) is any positive solution of system (83).

Figure 1 shows the dynamic behaviors of system (83), which strongly supports the above assertions.

Example 2. Consider the following equations:

\[ x_1 (k + 1) = x_1 (k) \exp \left\{ 1 - (3.2 + 0.2 \sin (k)) x_1 (k) - \frac{5 \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)}{1 + \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)} - 0.0005x (k) x_2^2 (k) - 0.3 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-2i} u_1 (k - s) \right\}, \]

\[ x_2 (k + 1) = x_2 (k) \exp \left\{ 0.95 - 0.05 \cos (k) - \frac{1.5 \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)}{1 + \sum_{i=0}^{\infty} ((e - 1)/e)e^{-2} x_1 (k - s)} - 0.4 x_2 (k) - 0.1 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-2i} u_2 (k - s) \right\}, \]

\[ u_1 (k + 1) = u_1 (k) (1 - 0.9) + 5 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-3i} x_1 (k - s), \]

\[ u_2 (k + 1) = u_2 (k) (1 - 0.6) + 2 \sum_{i=0}^{\infty} \frac{e^i - 1}{e^i} e^{-3i} x_2 (k - s). \]

(86)

One could easily see that conditions (C1) and (C2) are satisfied. Also, by calculating, one has
Figure 1: Numeric simulations of the solutions \((x_1(k), x_2(k), u_1(k), u_2(k))\) of system (83), with the initial conditions \((x_1(s), x_2(s), u_1(s), u_2(s)) = (0.15, 0.23, 0.31, 0.56), (0.32, 0.4, 0.43, 0.7), (0.5, 0.67, 0.6, 0.84), s = \ldots, -n, \ldots, -1, 0\), respectively.

\[
\frac{r''}{r_2} = \frac{10}{9} \approx 1.11, \\
M_2 = \frac{1}{a_2} \exp(r'' - 1) = 2.5, \\
d_2 t' + c_2 d_2 = 3.11, \\
1 + M_2 \frac{b_2 e_2}{a_2^2 e_2 + c_2^2 d_2} \approx 2.72, \\
A_{22} = \min \left\{ \frac{d_2}{M_2}, \frac{2}{a_2^2} - \frac{a_2^2}{d_2^2} \right\} = 0.4, \\
0.1 = c_2^2 \frac{d_2}{d_2^2} = 0.12.
\]

Clearly, condition \((H_1)\) and (68) are satisfied, and so from Theorem 2 and 4 we know that species \(x_1\) will be driven to extinction, while species \(x_2\) is global attractive.

Figure 2 shows the dynamic behaviors of system (86), which strongly support our results.

Example 3. Consider the following equations:

\[
x_1(k + 1) = x_1(k) \exp \left\{ 1 - (2.5 + 0.5 \sin(k))x_1(k) - \frac{5}{1 + \sum_{i=0}^{\infty}} (e - 1)e^{-e}x_2(k - s) \right\} - 0.0005x(k)x_2^2(k) - 0.1 \frac{e^2 - 1}{e^2} e^{-2s}u_1(k - s), \\
x_2(k + 1) = x_2(k) \exp \left\{ 0.65 - 0.05 \cos(k) - \frac{5}{1 + \sum_{i=0}^{\infty}} (e - 1)e^{-e}x_2(k - s) - 3x_2(k) - 0.3 \frac{e^2 - 1}{e^2} e^{-2s}u_2(k - s) \right\}, \\
u_1(k + 1) = u_1(k)(1 - 0.9) + 5 \sum_{j=0}^{\infty} \frac{e^3 - 1}{e^2} e^{-3s}x_1(k - s), \\
u_2(k + 1) = u_2(k)(1 - 0.9) + 2 \sum_{j=0}^{\infty} \frac{e^3 - 1}{e^2} e^{-3s}x_2(k - s).
\]
which strongly supports our results.

Figure 2: Numeric simulations of the solutions \( (x_1(k), x_2(k), u_1(k), u_2(k)) \) of system (86), with the initial conditions \( (x_1(s), x_2(s), u_1(s), u_2(s)) = (0.3, 0.2, 2.7, 0.4), (0.7, 1.2, 2.1, 6), (1.1, 0.7, 1.5, 2.3), s = \ldots, -n, -n + 1, \ldots, -1, 0 \), respectively.

One could easily see that conditions \((C_1)\) and \((C_2)\) are satisfied. Also, by calculating, one has
\[
M_1 = \frac{1}{a_1} \exp(r_1^u - 1) = 0.5, \quad M_2 = \frac{1}{a_2} \exp(r_2^u - 1) = 0.25,
\]
\[
(1 + M_1) \frac{a_1^u e_1^u + c_1^u d_1^u}{b_1 e_1} = 1.07,
\]
\[
\frac{b_2^u e_2^u + c_2^u d_2^u}{a_2 e_2} = 1.36,
\]
\[
\frac{r_1}{r_2} = 1.43 > \max\{1.07, 1.36\},
\]
\[
\frac{1}{M_1 M_2^2} \left( r_1^u - (1 + M_1) r_2^u \frac{a_1^u e_1^u + c_1^u d_1^u}{b_1^2 e_1} \right) = 8,
\]
\[
\frac{1}{M_1 M_2^2} \left( r_1^u - r_2^u \frac{b_2^u e_2^u + c_2^u d_2^u}{a_2^2 e_2} \right) = 1.6,
\]
\[
A_{11} = \min \left\{ a_1^u, \frac{2}{M_1} - a_1^u \right\} = 1,
\]
\[
0.1 = e_1 < A_{11} e_1^u = 0.18.
\]

(89)

Figure 3 shows the dynamics behaviors of system (88), which strongly supports our results.

6. Discussion

(1) From the conditions of Theorems 2–5, we can easily find that the feedback control variables and toxic substances play a crucial role on the extinction of system (5). We find that, by choosing suitable feedback control variable, one of the species will be extinct or permanent, that is, feedback control variable, which represents the biological control or some harvesting procedures, is an unstable factor of the system. From the conditions of Theorem 2, we also find that, despite the second species could produce toxicity, if the toxic rate is very low such that inequality \((H_2)\) holds; then, the second species is still driven to extinction; in other words, that lower rate of toxic production has no influence on the extinction property of system (5).

(2) From the conditions of Theorem 1, we find that the feedback control variables and toxic substances play an important role on the permanence of the system; only the rate of toxic production and feedback control variables are small enough such that inequality \((18)\) holds, and the toxic substances and feedback controls have no effect on the permanence of the system.

(3) Yu [7] obtained a set of sufficient conditions that guarantees the extinction of system (3), as direct corollaries of Theorems 2–5; one could also obtain other conditions for the extinction of system (3), which supplement and complements the results of Yu [7].

(4) Now, concerned with the extinction property of system (2), with some minor revisions to the proof of Theorems 2 and 3, we could obtain the following results.

Corollary 1. Assume that
\[
\frac{r_1^u}{r_2^u} < \min \left\{ \frac{a_1^u}{b_1^u (1 + M_2)^2} \right\}
\]
holds, and let \((x_1(k), x_2(k))\) be any positive solution of system (2), then
\[
\lim_{k \to \infty} x_1(k) = 0.
\]
then Theorems 4 and 5, by simple computation, we have system (2) is driven to extinction. As direct corollaries of rest species under the assumption that one of the species in

\[ x(t) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases} \]

\[ y(t) = \begin{cases} 0 & \text{if } y > 0, \\ 1 & \text{otherwise} \end{cases} \]

\[ z(t) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{otherwise} \end{cases} \]

\[ w(t) = \begin{cases} 0 & \text{if } w > 0, \\ 1 & \text{otherwise} \end{cases} \]

\[ u(t) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{otherwise} \end{cases} \]

\[ v(t) = \begin{cases} 0 & \text{if } v > 0, \\ 1 & \text{otherwise} \end{cases} \]

\[ s(t) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{otherwise} \end{cases} \]

\[ t(t) = \begin{cases} 0 & \text{if } t > 0, \\ 1 & \text{otherwise} \end{cases} \]

\[ \text{corollary } 1 \text{ holds; let } (x_1(t), x_2(t), u_1(t), u_2(t)) \text{ be any positive solution of the system } (2), \text{ then} \]

\[ \lim_{n \to \infty} x_2(t) = 0. \quad (94) \]

It is interesting to investigate the stability property of the rest species under the assumption that one of the species in system (2) is driven to extinction. As direct corollaries of Theorems 4 and 5, by simple computation, we have

Corollary 2. Assume that

\[ \frac{r_1}{r_2} > \max \left\{ \left(1 + M_1\right)\frac{a_1^\mu}{b_1^\mu}, \frac{b_2^{\mu}}{a_2^{\mu}} \right\}, \quad (92) \]

\[ \gamma < \min \frac{1}{M_1M_2} \left\{ \left(1 + M_1\right)r_2a_1^{\mu}b_2^{\mu}r_1 - r_2^{\mu}a_1^{\mu}b_2^{\mu} \right\}, \quad (93) \]

holds; let \((x_1(t), x_2(t))\) be any positive solution of system (2), then

\[ \lim_{n \to \infty} x_2(t) = 0. \quad (94) \]

Corollary 3. Assume that (90) holds; also,

\[ \frac{a_1^{\mu}}{a_2^{\mu}} \exp \left\{ r_2^{\mu} - 1 \right\} < 2 \quad (95) \]

holds; then, for any positive solution \((x_1(t), x_2(t))\) of system (2), we have

\[ \lim_{n \to \infty} x_1(t) = 0, \quad (96) \]

\[ \lim_{n \to \infty} (x_2(t) - x_2^\ast(t)) = 0, \]

where \(x_2^\ast(t)\) is any positive solution of the system

\[ x_2(t + 1) = x_2(t) \exp \left(r_2(t) - a_2(t)x_2(t) \right) \]

Corollary 4. Assume that (92) and (93) hold; also,

\[ \frac{a_1^{\mu}}{a_1^{\mu}} \exp \left\{ r_1^{\mu} - 1 \right\} < 2 \quad (97) \]

holds; then, for any positive solution \((x_1(t), x_2(t))\) of system (2), we have

\[ \lim_{n \to \infty} x_1(t) = 0, \quad (96) \]

\[ \lim_{n \to \infty} (x_2(t) - x_2^\ast(t)) = 0, \]

where \(x_2^\ast(t)\) is any positive solution of the system

\[ x_2(t + 1) = x_2(t) \exp \left(r_2(t) - a_2(t)x_2(t) \right) \]

\[ \lim_{n \to \infty} x_2(t) = 0, \]

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\[ \lim_{n \to \infty} x_2(t) = 0, \]


