Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 7056254, 16 pages
https://doi.org/10.1155/2020/7056254

Research Article

An Exponential Spline Difference Scheme for Solving a Class of Boundary Value Problems of Second-Order Ordinary Differential Equations

Dunqian Cao

School of Mathematics and Physics, Guangxi University for Nationalities, Nanning 530006, China

Correspondence should be addressed to Dunqian Cao; caodunqian@gxun.edu.cn

Received 8 May 2020; Revised 2 July 2020; Accepted 3 July 2020; Published 28 August 2020

Academic Editor: Juan L. G. Guirao

In this paper, we mainly study an exponential spline function space, construct a basis with local supports, and present the relationship between the function value and the first and the second derivative at the nodes. Using these relations, we construct an exponential spline-based difference scheme for solving a class of boundary value problems of second-order ordinary differential equations (ODEs) and analyze the error and the convergence of this method. The results show that the algorithm is high accurate and conditionally convergent, and an accuracy of $(1/240)h^6$ was achieved with smooth functions.

1. Introduction

In physics, chemistry, biology, sociology, and many other disciplines, there are tremendous problems that can be described by differential equations (DEs), but it is difficult to get their explicit expressions. So, people began to seek the numerical solutions of these problems, which can also be applied to scientific research and engineering practice if their accuracy satisfies the needs. Especially the advent of computers makes it possible to quickly carry out a large number of calculations, which also makes the numerical solution method of DEs become one of the most important branches of computational mathematics. Due to its high smoothness, low power, and easy calculation, the spline function has been widely used in computer graphics, data interpolation and fitting, shape control, and numerical solutions of DEs. There are two main schemes in numerical solutions of DEs using spline functions: the spline finite element method and the spline difference method. The first has a wide range of application and can be applied to many types of equations, but it requires a large amount of calculation. While the second is simple process with a small amount of calculation and high accuracy, but it can only be applied to specific types of equations.

In this paper, we mainly focus a class of second-order ordinary differential equations (ODEs):

$$u'' + q(x)u' + p(x)u = g(x), \quad x \in [a, b], \tag{1}$$

which meets one of the following boundary conditions.

(1) First boundary condition:

$$\begin{aligned} u(a) &= \mu_0, \\ u(b) &= \mu_1. \end{aligned} \tag{2}$$

(2) Second boundary condition:

$$\begin{aligned} u^{(1)}(a) &= \mu_0, \\ u^{(1)}(b) &= \mu_1. \end{aligned} \tag{3}$$

(3) Third boundary condition:

$$\begin{aligned} u(a) &= \mu_0, \\ u^{(1)}(b) &= \mu_1. \end{aligned} \tag{4}$$

(4) Fourth boundary condition:

$$u^{(1)}(a) = \mu_0, \quad u(b) = \mu_1. \tag{5}$$

where μ_0 and μ_1 are constants and $p(x)$, $q(x)$, and $g(x)$ are continuous functions in the interval of $[a, b]$.

Many scholars have been studying such two-point boundary value problems. Albasiny and Hoskins [1] and Raghavarao et al. [2] used cubic polynomial splines to solve such problems. Blue [3] used quintic polysplines to solve such problems. Caglar et al. [4] used cubic B-splines in their solving scheme. Chawla and Shiva Kumar[5] extended the problem to semi-infinite regions.

In recent years, with the deepening of research, people have begun to use nonpolynomial splines to solve such problems. Zahra [6], Rao and Kumar [7], Tirmizi et al. [8], Ramadan et al. [9], Surla and Stojanović [10], Jha [11, 12], and Kadalbajoo and Patidar [13] have carried out a lot of research in this area and achieved very high computational accuracy.

However, there are still many theoretical problems to be broken in the study of nonpolynomial splines. Due to the diversity of nonpolynomial splines, it is crucial to choose the basis and parameters in solving the problem. However, there is still no reference in this regard. In this paper, a selected set of spline basis functions was used to deduce the relationship between the derivative and the function value and then to obtain the second-order difference scheme for solving second-order ODEs, which provides a method for solving such problems.

2. Exponential Spline Function Space

Exponential spline refers to a type of spline in which the nonpolynomial factors of spline basis functions contain only exponential functions. The exponential spline in this sense is not very specific; it can contain many forms of exponential spline, which can produce substantially different splines, and is inconvenient to study. Therefore, the exponential spline refers to that with a specific form in the rest of this work.

Next, we define an exponential spline function space. Let

$$s_i(x) = a_i + b_i(x - x_i) + c_i e^{\tau_1(x - x_i)} + d_i e^{\tau_2(x - x_i)}, \quad i = 1, 2, \cdots, n, \tag{6}$$

where a_i, b_i, c_i, and d_i are coefficients and τ_1 and τ_2 are parameters with $\tau_1 \neq \tau_2$.

Definition 1. The following function space

$$\mathscr{E}_3^r(\Delta_n) = \{s\colon s(x) = s_i(x),\ x \in I_i, i = 1, 2, \cdots, ns(x) \in C^r[a, b], r < 3\}, \tag{7}$$

is called the cubic r-order exponential spline function space.

Obviously, the function $s(x)$ in $\mathscr{E}_3^2(\Delta_n)$ must meet

$$s_i^{(r)}(x_i) = s_{i+1}^{(r)}(x_i), \quad r = 0, 1, 2;\ i = 1, 2, \cdots, n-1. \tag{8}$$

The dimension of $\mathscr{E}_3^2(\Delta_n)$ is $n + 3$. Then, we find a set of basic functions with local supports for $\mathscr{E}_3^2(\Delta_n)$.

Assume $s(x) \in \mathscr{E}_3^2(\Delta_n)$, for given j, $2 \le j \le n-2$; let

$$\begin{cases} s_{j-1}^{(r)}(x_{j-2}) = 0, & r = 0, 1, 2, \\ s_{j+2}^{(r)}(x_{j+2}) = 0, & r = 0, 1, 2, \\ s_{j+k}^{(r)}(x_{j+k}) = s_{j+k+1}^{(r)}(x_{j+k}), & k = -1, 0, 1,\ r = 0, 1, 2, \\ s_j(x_j) = 1. \end{cases} \tag{9}$$

We can obtain

$$\begin{aligned} a_{i-1} &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|}, & b_{i-1} &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|}, & c_{i-1} &= \frac{|\mathbf{A}_3|}{|\mathbf{A}|}, & d_{i-1} &= \frac{|\mathbf{A}_4|}{|\mathbf{A}|}, \\ a_i &= \frac{|\mathbf{A}_5|}{|\mathbf{A}|}, & b_i &= \frac{|\mathbf{A}_6|}{|\mathbf{A}|}, & c_i &= \frac{|\mathbf{A}_7|}{|\mathbf{A}|}, & d_i &= \frac{|\mathbf{A}_8|}{|\mathbf{A}|}, \\ a_{i+1} &= \frac{|\mathbf{A}_9|}{|\mathbf{A}|}, & b_{i+1} &= \frac{|\mathbf{A}_{10}|}{|\mathbf{A}|}, & c_{i+1} &= \frac{|\mathbf{A}_{11}|}{|\mathbf{A}|}, & d_{i+1} &= \frac{|\mathbf{A}_{12}|}{|\mathbf{A}|}, \\ a_{i+2} &= \frac{|\mathbf{A}_{13}|}{|\mathbf{A}|}, & b_{i+2} &= \frac{|\mathbf{A}_{14}|}{|\mathbf{A}|}, & c_{i+2} &= \frac{|\mathbf{A}_{15}|}{|\mathbf{A}|}, & d_{i+2} &= \frac{|\mathbf{A}_{16}|}{|\mathbf{A}|}, \end{aligned} \tag{10}$$

where

$$\mathbf{A} = [\,\mathbf{A}_1' \,\mathbf{A}_2'\,],$$

$$\mathbf{A}_1' = \begin{bmatrix} 1 & -h_{i-1} & e^{(-\theta_{i-1})} & e^{(-\eta_{i-1})} & 0 & 0 & 0 & 0 \\ 0 & 1 & \tau_1 e^{(-\theta_{i-1})} & \tau_2 e^{(-\eta_{i-1})} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_1^2 e^{(-\theta_{i-1})} & \tau_2^2 e^{(-\eta_{i-1})} & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 & h_i & -e^{(-\theta_i)} & -e^{(-\eta_i)} \\ 0 & 1 & \tau_1 & \tau_2 & 0 & -1 & -\tau_1 e^{(-\theta_i)} & -\tau_2 e^{(-\eta_i)} \\ 0 & 0 & \tau_1^2 & \tau_2^2 & 0 & 0 & -\tau_1^2 e^{(-\theta_i)} & -\tau_2^2 e^{(-\eta_i)} \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tau_1 & \tau_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_1^2 & \tau_2^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$\mathbf{A}_2' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & h_{i+1} & -e^{(-\theta_{i+1})} & -e^{(-\eta_{i+1})} & 0 & 0 & 0 & 0 \\ 0 & -1 & -\tau_1 e^{(-\theta_{i+1})} & -\tau_2 e^{(-\eta_{i+1})} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau_1^2 e^{(-\theta_{i+1})} & -\tau_2^2 e^{(-\eta_{i+1})} & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 & h_{i+2} & -e^{(-\theta_{i+2})} & -e^{(-\eta_{i+2})} \\ 0 & 1 & \tau_1 & \tau_2 & 0 & -1 & -\tau_1 e^{(-\theta_{i+2})} & -\tau_2 e^{(-\eta_{i+2})} \\ 0 & 0 & \tau_1^2 & \tau_2^2 & 0 & 0 & -\tau_1^2 e^{(-\theta_{i+2})} & -\tau_2^2 e^{(-\eta_{i+2})} \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tau_1 & \tau_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_1^2 & \tau_2^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

$\mathbf{A}_k\,(k = 1, 2, \ldots, 16)$ is the matrix obtained by replacing the k column of $\mathbf{A}$ with $[0, \ldots, 0, 1]^T$, $\theta_i = \tau_1 h_i$, and $\eta_i = \tau_2 h_i$.

For even splitting, i.e., $h_i = h, i = 1, 2, \ldots, n$, and $\tau_2 = -\tau_1 = \tau \neq 0$, the results obtained are

$$
\begin{aligned}
a_{j-1} &= -\frac{\omega\theta}{\sigma},\\
b_{j-1} &= -\frac{\omega\theta}{h\sigma},\\
c_{j-1} &= \frac{1}{2}\frac{\omega(-\cosh(\theta)+\sinh(\theta))}{\sigma},\\
d_{j-1} &= \frac{1}{2}\frac{\omega(\cosh(\theta)+\sinh(\theta))}{\sigma},\\
a_j &= \frac{\theta(-4\sinh(3\theta)+\sinh(4\theta)-4\sinh(\theta)+6\sinh(2\theta))}{\sigma},\\
b_j &= \frac{\theta(-3\sinh(3\theta)+2\sinh(2\theta)+\sinh(4\theta)+\sinh(\theta))}{h\sigma},\\
c_j &= -\frac{1}{2\sigma}(-5+4\cosh(2\theta)-2\sinh(2\theta)-4\cosh(3\theta)+3\sinh(3\theta)+\cosh(4\theta)-\sinh(4\theta)+4\cosh(\theta)-\sinh(\theta)),\\
d_j &= -\frac{1}{2\sigma}(\cosh(4\theta)+\sinh(4\theta)-5-4\cosh(3\theta)-3\sinh(3\theta)+4\cosh(\theta)+\sinh(\theta)+4\cosh(2\theta)+2\sinh(2\theta)),\\
a_{j+1} &= -\frac{\omega\theta}{\sigma},\\
b_{j+1} &= -\frac{\theta(-3\sinh(3\theta)+2\sinh(2\theta)+\sinh(4\theta)+\sinh(\theta))}{h\sigma},\\
c_{j+1} &= \frac{1}{4\sigma}(-5+4\cosh(2\theta)+10\sinh(2\theta)-4\cosh(3\theta)+\cosh(4\theta)-\sinh(4\theta)+4\cosh(\theta)-16\sinh(\theta)),\\
d_{j+1} &= \frac{1}{4\sigma}(\cosh(4\theta)+\sinh(4\theta)-5-4\cosh(3\theta)+4\cosh(\theta)+16\sinh(\theta)+4\cosh(2\theta)-10\sinh(2\theta)),\\
a_{j+2} &= 0,\\
b_{j+2} &= \frac{\omega\theta}{h\sigma},\\
c_{j+2} &= \frac{\omega}{2\sigma},\\
d_{j+2} &= -\frac{\omega}{2\sigma},
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\omega &= 5\sinh(\theta)-4\sinh(2\theta)+\sinh(3\theta),\\
\sigma &= 5+\theta\sinh(4\theta)-4\theta\sinh(\theta)+6\theta\sinh(2\theta)-4\theta\sinh(3\theta)\\
&\quad +4\cosh(3\theta)-4\cosh(\theta)-\cosh(4\theta)-4\cosh(2\theta),\\
\theta &= \tau h.
\end{aligned}
\tag{13}
$$

Define a function

$$
B_j(x) = \begin{cases} s_{j-1}(x), & x \in I_{j-1},\\ s_j(x), & x \in I_j,\\ s_{j+1}(x), & x \in I_{j+1},\\ s_{j+2}(x), & x \in I_{j+2},\\ 0, & \text{otherwise}, \end{cases}
\quad j = 2, 3, \ldots, n-2,
\tag{14}
$$

where the coefficients of $s_{j-1}(x), s_j(x), s_{j+1}(x)$, and $s_{j+2}(x)$ are given by the solution of function (9). Besides, for

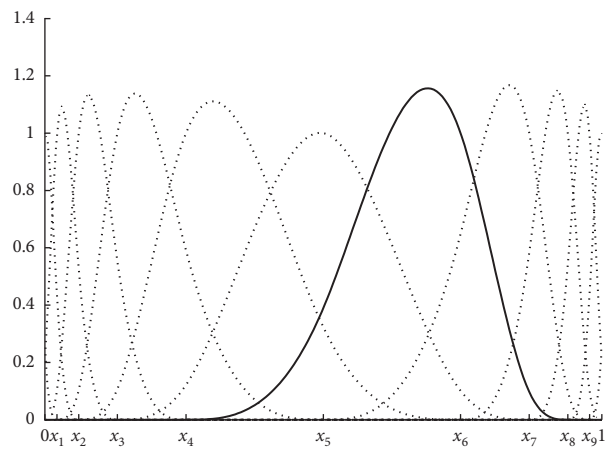

FIGURE 1: Basis function set $\{B_j(x)\}_{j=-1}^{n+1}$ of the exponential spline function space.

$j = -1, 0, 1, n-1, n, n+1$, an interval expansion will be conducted, i.e., $(x_0, x_1, \ldots, x_n)$ will be extended to be

$$(x_{-3}, x_{-2}, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}). \tag{15}$$

Thus, j in (11) can take the values of $-1, 0, 1, \ldots, n, n+1$. Of course, the function domain is still $[x_0, x_n]$.

For the space basis of $\mathscr{E}_3^2(\Delta_n)$, we have the following proposition.

Proposition 1. *When $\tau_1 \neq \tau_2$, the function set of $\{B_j(x)\}_{j=-1}^{n+1}$ is a set of basis of $\mathscr{E}_3^2(\Delta_n)$.*

For special cases, we can prove the following.

Theorem 1. *When $\tau_2 = -\tau_1 = \tau \neq 0$ and $h_i = h, i = -2, -1, \ldots, n+3$, the function set of $\{B_j(x)\}_{j=-1}^{n+1}$ is a set of basis of $\mathscr{E}_3^2(\Delta_n)$.*

Proof. Obviously, $B_j(x) \in \mathscr{E}_3^2(\Delta_n)$, $j = -1, 0, 1, \ldots, n, n+1$, and the number of functions equals the dimension of $\mathscr{E}_3^2(\Delta_n)$. Therefore, we need to show that $\{B_j(x)\}_{j=-1}^{n+1}$ is linearly independent.

For $x = x_i, i = -1, 0, \ldots, n, n+1$, let $\mathbf{B} = (\beta_{ij}) = (B_j(x_i))_{(n+3)\times(n+3)}$, if matrix $\mathbf{B}$ is invertible, then $\{B_j(x)\}_{j=-1}^{n+1}$ is linearly independent. It is easy to know that $\mathbf{B}$ is a tridiagonal matrix, and

$$\begin{aligned} b_{i,i-1} &= a_{i-1} + c_{i-1} + d_{i-1}, \\ b_{i,i} &= a_i + c_i + d_i = 1, \\ b_{i,i+1} &= a_{i+1} + c_{i+1} + d_{i+1}, \end{aligned} \tag{16}$$

where $i = -1, 0, \ldots, n+1$. Because

$$\begin{aligned} |b_{i,i-1}| + |b_{i,i+1}| &= |a_{i-1} + c_{i-1} + d_{i-1}| + |a_{i+1} + c_{i+1} + d_{i+1}| \\ &= \left|\frac{8}{\sigma}\left((\cosh(\theta) - 1)^2 (1 - \cosh(\theta))^2 + \theta \sinh(\theta)\right)\right| \\ &= \left|\frac{1 - \cosh(\theta)^2 + \theta \sinh(\theta)}{1 - \cosh(\theta)^2 + \theta \sinh(\theta)\cosh(\theta)}\right| \\ &\leq \frac{1}{2} < 1 = |b_{ii}|. \end{aligned} \tag{17}$$

$\mathbf{B}$ is a strictly tridiagonal matrix. Thus, $\mathbf{B}$ is invertible. This proves that $\{B_j(x)\}_{j=-1}^{n+1}$ is linearly independent.

When $h_i = h, i = -2, -1, \ldots, n+3$, $\{B_j(x)\}_{j=-1}^{n+1}$ has the following properties. □

Proposition 2. *For any ,$x \in [a, b]$*

$$\sum_{j=-1}^{n+1} B_j(x) = C, \tag{18}$$

where C is not related to x, and

$$C = \frac{\tau_1\tau_2 h(\tau_2 - \tau_1)\left(e^{(-\tau_1 h)} + e^{(-\tau_2 h)} - e^{(-h(\tau_1+\tau_2))} - 1\right)}{(\tau_2^2 - \tau_1^2)e^{-h(\tau_1+\tau_2)} + (\tau_1^2 + \tau_2^2 + \tau_1\tau_2^2 h - \tau_1^2\tau_2 h)e^{(-\tau_1 h)} - (\tau_1^2 + \tau_2^2 - \tau_1\tau_2^2 h + \tau_1^2\tau_2 h)e^{(-\tau_2 h)} + \tau_1^2 - \tau_2^2}. \tag{19}$$

The basis function $B_j(x)$ has a local supporting set of $[x_{i-2}, x_{i+2}]$. Figure 1 shows all functions in the function set of $\{B_j(x)\}_{j=-1}^{n+1}$ at the same coordinate.

3. Properties of Exponential Spline Functions

The relationship between the function value and the first derivative and that between the function value and the second derivative are commonly used in numerically solving DEs. Next, we will derive some properties of the functions in $\mathscr{E}_3^2(\Delta_n)$, and these relationships will be used in numerically calculating DEs.

Let $s(x) \in \mathscr{E}_3^2(\Delta_n)$; then,

$$s(x) = s_i(x), \quad x \in I_i,\ i = 1, 2, \ldots, n. \tag{20}$$

Denote

$$\begin{aligned} S_i &= s(x_i), \\ D_i &= s'(x_i), \\ M_i &= s'', \quad i = 1, 2, \ldots, n, \end{aligned} \tag{21}$$

since $s_i(x_i) = s_{i+1}(x_i), s_i''(x_i) = s_{i+1}''(x_i)$, the factors of a_i, b_i, c_i, and d_i can be written as

$$\begin{aligned} a_i &= \frac{1}{\tau_1^2\tau_2^2\rho_i}\left(\left(\tau_1^2 - \tau_2^2\right)M_{i-1} + \left(\tau_2^2 e^{-\tau_2 h_i} - \tau_1^2 e^{-\tau_1 h_i}\right)M_i + \tau_1^2\tau_2^2\rho_i S_i\right), \\ b_i &= \frac{1}{h_i\tau_1^2\tau_2^2\rho_i}\left(\left(\tau_2^2 e^{-\tau_1 h_i} - \tau_1^2 e^{-\tau_2 h_i} + \tau_1^2 - \tau_2^2\right)M_{i-1}\right. \\ &\quad + \left(\tau_1^2 e^{-\tau_1 h_i - \tau_2 h_i} - \tau_2^2 e^{-\tau_1 h_i - \tau_2 h_i} - \tau_1^2 e^{-\tau_1 h_i} + \tau_2^2 e^{-\tau_2 h_i}\right)M_i \\ &\quad \left. + \tau_1^2\tau_2^2\rho_i S_i + \tau_1^2\tau_2^2 e^{-\tau_2 h_i} S_{i-1} - \tau_1^2\tau_2^2 e^{-\tau_1 h_i} S_{i-1}\right), \\ c_i &= \frac{1}{\tau_1^2\rho_i}\left(M_{i-1} - e^{-\tau_2 h_i} M_i\right), \\ d_i &= -\frac{1}{\tau_2^2\rho_i}\left(M_{i-1} - e^{-\tau_1 h_i} M_i\right), \end{aligned} \tag{22}$$

where $\rho_i = e^{-\tau_1 h_i} - e^{-\tau_2 h_i}, i = 1, 2, \ldots, n$ (the same applies hereinafter). Using the continuous condition of the first derivative

$$s_i'(x_i) = s_{i+1}'(x_i), \quad i = 1, 2, \ldots, n-1, \tag{23}$$

we obtain

$$\alpha_{1i} M_{i-1} + \alpha_{2i} M_i + \alpha_{3i} M_{i+1} - \frac{1}{h_i} S_{i-1} + \frac{h_{i+1} + h_i}{h_i h_{i+1}} S_i - \frac{1}{h_{i+1}} S_{i+1} = 0, \tag{24}$$

where $i = 1, 2, \ldots, n-1$,

$$\begin{aligned} \alpha_{1i} &= -\frac{\left(h_i\tau_1\tau_2^2 - h_i\tau_1^2\tau_2 - \tau_1^2 e^{(-\tau_2 h_i)} + \tau_2^2 e^{(-\tau_1 h_i)} + \tau_1^2 - \tau_2^2\right)}{\tau_1^2\tau_2^2\rho_i h_i}, \\ \alpha_{2i} &= -\frac{1}{\tau_1^2\tau_2^2\rho_i\rho_{i+1} h_i h_{i+1}}\left(\left(h_{i+1}\tau_1^2 - h_{i+1}\tau_2^2\right)e^{(-\tau_1 h_i - \tau_2 h_i - \tau_1 h_{i+1})} + \left(h_{i+1}\tau_2^2 - h_{i+1}\tau_1^2\right)e^{(-\tau_1 h_i - \tau_2 h_i - \tau_2 h_{i+1})}\right. \\ &\quad + \left(-h_i\tau_1^2 + h_i\tau_2^2\right)e^{(-\tau_1 h_i)} + \left(h_i\tau_1^2 - h_i\tau_2^2\right)e^{(-\tau_2 h_i)} + \left(h_{i+1}\tau_1^2 + h_i\tau_1^2\right)e^{(-\tau_2 h_{i+1} - \tau_1 h_i)} \\ &\quad + \left(h_i\tau_2^2 + h_{i+1}\tau_2^2\right)e^{(-\tau_1 h_{i+1} - \tau_2 h_i)} + \left(-h_{i+1}\tau_2^2 - h_i\tau_1^2 + h_i h_{i+1}\tau_1\tau_2^2 - h_i h_{i+1}\tau_1^2\tau_2\right)e^{(-\tau_2 (h_{i+1} + h_i))} \\ &\quad \left. + \left(-h_{i+1}\tau_1^2 - h_i\tau_2^2 - h_i h_{i+1}\tau_1\tau_2^2 + h_i h_{i+1}\tau_1^2\tau_2\right)e^{(-\tau_1 (h_{i+1} + h_i))}\right), \\ \alpha_{3i} &= \frac{\tau_2^2 e^{-\tau_2 h_{i+1}} - \tau_1^2 e^{-\tau_1 h_{i+1}} + \left(h_{i+1}\tau_1^2\tau_2 - h_{i+1}\tau_1\tau_2^2 + \tau_1^2 - \tau_2^2\right)e^{-h_{i+1}(\tau_1 + \tau_2)}}{\tau_1^2\tau_2^2\rho_{i+1} h_{i+1}}. \end{aligned} \tag{25}$$

If $h_i = h > 0, i = 1, 2, \ldots, n$ and $\tau_2 = -\tau_1 = \tau$, then the corresponding coefficients become

$$\begin{aligned}
\alpha_{1i} &= \frac{\left(e^{2\tau h} - 2\tau h e^{\tau h} - 1\right)}{h\tau^2\left(e^{2\tau h} - 1\right)},\\
\alpha_{2i} &= \frac{2\left(-e^{2\tau h} + \tau h e^{2\tau h} + 1 + \tau h\right)}{h\tau^2\left(e^{2\tau h} - 1\right)},\\
\alpha_{3i} &= \frac{\left(e^{2\tau h} - 2\tau h e^{\tau h} - 1\right)}{h\tau^2\left(e^{2\tau h} - 1\right)},\\
\alpha_{4i} &= -\frac{1}{h},\\
\alpha_{5i} &= \frac{2}{h},\\
\alpha_{6i} &= -\frac{1}{h}.
\end{aligned} \tag{26}$$

Let $\tau \longrightarrow 0$; then, (24) is equivalent to

$$h^2\left(M_{i-1} + 4M_i + M_{i+1}\right) - 6S_{i-1} + 12S_i - 6S_{i+1} = 0. \tag{27}$$

In a similar way, we can obtain the relationship between the function value and the first derivative:

$$\beta_{1i}D_{i-1} + \beta_{2i}D_i + \beta_{3i}D_{i+1} + \beta_{4i}S_{i-1} + \beta_{5i}S_i + \beta_{6i}S_{i+1} = 0, \tag{28}$$

where

$$\begin{aligned}
\beta_{1i} = \frac{1}{K_iK_{i+1}}\Big[&\left(\tau_1 - \tau_2\right)^2\left(-\tau_1 + \tau_1\tau_2h_i - \tau_2\right)e^{\left(-h_{i+1}\left(\tau_1+\tau_2\right)\right)} - \tau_2^2\left(\tau_1 - \tau_2\right)e^{\left(-\tau_1h_{i+1} - \tau_2h_{i+1} - \tau_1h_i\right)}\\
&+ \tau_1^2\left(\tau_1 - \tau_2\right)e^{\left(-\tau_1h_{i+1} - \tau_2h_{i+1} - \tau_2h_i\right)} - \tau_2^2\left(-\tau_1 + \tau_1\tau_2h_{i+1} + \tau_2\right)e^{\left(-\tau_1\left(h_{i+1}+h_i\right)\right)}\\
&- \tau_1^2\left(\tau_1\tau_2h_{i+1} + \tau_1 - \tau_2\right)e^{\left(-\tau_2\left(h_{i+1}+h_i\right)\right)} + \tau_2^2\left(\tau_2 - \tau_1\right)e^{\left(-\tau_1h_i\right)} - \tau_1^2\left(\tau_2 - \tau_1\right)e^{\left(-\tau_2h_i\right)}\\
&+ \left(\tau_1 - \tau_2\right)\left(-\tau_1 + \tau_1\tau_2h_i - \tau_2\right)\left(-\tau_1 + \tau_1\tau_2h_{i+1} + \tau_2\right)e^{\left(-\tau_1h_{i+1}\right)}\\
&- \left(\tau_1 - \tau_2\right)\left(-\tau_1 + \tau_1\tau_2h_i - \tau_2\right)\left(\tau_1\tau_2h_{i+1} + \tau_1 - \tau_2\right)e^{\left(-\tau_2h_{i+1}\right)}\\
&+ \tau_1^2\left(-\tau_1 + \tau_1\tau_2h_{i+1} + \tau_2\right)e^{\left(-\tau_1h_{i+1} - \tau_2h_i\right)} + \tau_2^2\left(\tau_1\tau_2h_{i+1} + \tau_1 - \tau_2\right)e^{\left(-\tau_1h_i - \tau_2h_{i+1}\right)}\\
&+ \left(\tau_1 - \tau_2\right)^2\left(-\tau_1 + \tau_1\tau_2h_i - \tau_2\right)\Big],
\end{aligned} \tag{29}$$

$$\begin{aligned}
\beta_{2i} = \frac{1}{K_iK_{i+1}}\Big[&-\tau_1^2\left(\tau_1 - \tau_2\right)\left(-1 + \tau_2h_{i+1}\right)e^{\left(-\tau_1h_{i+1} - \tau_1h_i - \tau_2h_i\right)} + \tau_2^2\left(\tau_1h_{i+1} - 1\right)\left(\tau_1 - \tau_2\right)e^{\left(-\tau_2h_{i+1} - \tau_1h_i - \tau_2h_i\right)}\\
&- \tau_1^2\left(\tau_1 - \tau_2\right)\left(-1 + \tau_2h_i\right)e^{\left(-\tau_1h_{i+1} - \tau_2h_{i+1} - \tau_1h_i\right)} + \tau_2^2\left(\tau_1 - \tau_2\right)\left(\tau_1h_i - 1\right)e^{\left(-\tau_1h_{i+1} - \tau_2h_{i+1} - \tau_2h_i\right)} -\\
&\left(-\tau_2^3 - \tau_1^2\tau_2 + \tau_1^3\tau_2^2Zh_ih_{i+1} - \tau_1^3\tau_2h_i - \tau_1\tau_2^3h_{i+1} - \tau_1\tau_2^3h_i + \tau_1^2\tau_2^2h_{i+1} - \tau_1^2\tau_2^3h_ih_{i+1}\right.\\
&\left.-\tau_1^3\tau_2h_{i+1} + \tau_1^2\tau_2^2h_i + \tau_1^3 + \tau_1\tau_2^2\right)e^{\left(-\tau_1\left(h_i+h_{i+1}\right)\right)} - \left(\tau_1 + \tau_2\right)\left(\tau_1 - \tau_2\right)^2e^{\left(-\left(\tau_1+\tau_2\right)\left(h_i+h_{i+1}\right)\right)}\\
&+\left(-\tau_1^2\tau_2 + \tau_1\tau_2^2 - \tau_1^2\tau_2^3h_ih_{i+1} + \tau_1^3 + \tau_1^3\tau_2^2h_ih_{i+1} + \tau_1\tau_2^3h_i - \tau_2^3\right.\\
&\left.-\tau_1^2\tau_2^2h_i + \tau_1^3\tau_2h_i - \tau_1^2\tau_2^2h_{i+1} + \tau_1^3\tau_2h_{i+1} + \tau_1\tau_2^3h_{i+1}\right)e^{\left(-\tau_2\left(h_i+h_{i+1}\right)\right)}\\
&- \tau_2^2\left(\tau_2 - \tau_1\right)\left(\tau_1h_i + 1\right)e^{\left(-\tau_1h_i\right)} - \tau_1^2\left(\tau_2h_i + 1\right)\left(\tau_1 - \tau_2\right)e^{\left(-\tau_2h_i\right)}\\
&+ \tau_2^2\left(\tau_1h_{i+1} + 1\right)\left(\tau_1 - \tau_2\right)e^{\left(-\tau_1h_{i+1}\right)} + \tau_1^2\left(\tau_2 - \tau_1\right)\left(\tau_2h_{i+1} + 1\right)e^{\left(-\tau_2h_{i+1}\right)}\\
&-\tau_1^2\tau_2^2\left(h_i + h_{i+1}\right)e^{\left(-\tau_1h_{i+1} - \tau_2h_i\right)} - \tau_1^2\tau_2^2\left(h_i + h_{i+1}\right)e^{\left(-\tau_1h_i - \tau_2h_{i+1}\right)} + \left(\tau_1 + \tau_2\right)\left(\tau_1 - \tau_2\right)^2\Big],
\end{aligned} \tag{30}$$

$$\begin{aligned}\beta_{3,i} = \frac{1}{K_iK_{i+1}}\Big[&(\tau_1-\tau_2)^2(\tau_1\tau_2h_{i+1}+\tau_1+\tau_2)e^{(-h_{i+1}(\tau_1+\tau_2))}+\tau_1^2(\tau_2-\tau_1)e^{(-\tau_1h_{i+1}-\tau_1h_i-\tau_2h_i)}\\&-\tau_2^2(\tau_2-\tau_1)e^{(-\tau_2h_{i+1}-\tau_1h_i-\tau_2h_i)}-(\tau_2-\tau_1)(\tau_2+\tau_1\tau_2h_i-\tau_1)(\tau_1\tau_2h_{i+1}+\tau_1+\tau_2)e^{(-\tau_1h_{i+1}-\tau_2h_{i+1}-\tau_1h_i)}\\&-(\tau_1-\tau_2)(\tau_1\tau_2h_i+\tau_1-\tau_2)(\tau_1\tau_2h_{i+1}+\tau_1+\tau_2)e^{(-\tau_1h_{i+1}-\tau_2h_{i+1}-\tau_2h_i)}\\&-\tau_1^2(\tau_2+\tau_1\tau_2h_i-\tau_1)e^{(-\tau_1(h_{i+1}+h_i))}+(\tau_1-\tau_2)^2(\tau_1\tau_2h_{i+1}+\tau_1+\tau_2)e^{(-(\tau_1+\tau_2)(h_{i+1}+h_i))}\\&-\tau_2^2(\tau_1\tau_2h_i+\tau_1-\tau_2)e^{(-\tau_2(h_{i+1}+h_i))}-\tau_1^2(\tau_1-\tau_2)e^{(-\tau_1h_{i+1})}+\tau_2^2(\tau_1-\tau_2)e^{(-\tau_2h_{i+1})}\\&+\tau_1^2(\tau_1\tau_2h_i+\tau_1-\tau_2)e^{(-\tau_1h_{i+1}-\tau_2h_i)}+\tau_2^2(\tau_2+\tau_1\tau_2h_i-\tau_1)e^{(-\tau_1h_i-\tau_2h_{i+1})}\Big],\end{aligned} \tag{31}$$

$$\begin{aligned}\beta_{4,i} = \frac{\tau_1\tau_2}{K_iK_{i+1}}\Big[&(\tau_1-\tau_2)^2\mathrm{e}^{(-h_{i+1}(\tau_1+\tau_2))}+\tau_2(\tau_1-\tau_2)\mathrm{e}^{(-\tau_1h_{i+1}-\tau_2h_{i+1}-\tau_1h_i)}\\&-\tau_1(\tau_1-\tau_2)\mathrm{e}^{(-\tau_1h_{i+1}-\tau_2h_{i+1}-\tau_2h_i)}+\tau_2(\tau_1\tau_2h_{i+1}+\tau_2-\tau_1)\mathrm{e}^{(-\tau_1(h_{i+1}+h_i))}\\&+\tau_1(\tau_1+\tau_1\tau_2h_{i+1}-\tau_2)\mathrm{e}^{(-\tau_2(h_{i+1}+h_i))}+\tau_2(\tau_1-\tau_2)\mathrm{e}^{(-\tau_1h_i)}\\&-\tau_1(\tau_1-\tau_2)\mathrm{e}^{(-\tau_2h_i)}+(\tau_1-\tau_2)(\tau_1\tau_2h_{i+1}+\tau_2-\tau_1)\mathrm{e}^{(-\tau_1h_{i+1})}\\&-(\tau_1-\tau_2)(\tau_1+\tau_1\tau_2h_{i+1}-\tau_2)\mathrm{e}^{(-\tau_2h_{i+1})}-\tau_1(\tau_1\tau_2h_{i+1}+\tau_2-\tau_1)\mathrm{e}^{(-\tau_1h_{i+1}-\tau_2h_i)}\\&-\tau_2(\tau_1+\tau_1\tau_2h_{i+1}-\tau_2)\mathrm{e}^{(-\tau_1h_i-\tau_2h_{i+1})}+(\tau_1-\tau_2)^2\Big],\end{aligned} \tag{32}$$

$$\begin{aligned}\beta_{6,i} = \frac{\tau_1\tau_2}{K_iK_{i+1}}\Big[&-(\tau_1-\tau_2)^2e^{(-h_{i+1}(\tau_1+\tau_2))}+\tau_1(\tau_1-\tau_2)e^{(-\tau_1h_{i+1}-\tau_1h_i-\tau_2h_i)}\\&-\tau_2(\tau_1-\tau_2)e^{(-\tau_2h_{i+1}-\tau_1h_i-\tau_2h_i)}-(\tau_1-\tau_2)(-\tau_1+\tau_1\tau_2h_i+\tau_2)e^{(-\tau_1h_{i+1}-\tau_2h_{i+1}-\tau_1h_i)}\\&+(\tau_1-\tau_2)(-\tau_2+\tau_1\tau_2h_i+\tau_1)e^{(-\tau_1h_{i+1}-\tau_2h_{i+1}-\tau_2h_i)}+\tau_1(-\tau_1+\tau_1\tau_2h_i+\tau_2)e^{(-\tau_1(h_{i+1}+h_i))}\\&-(\tau_1-\tau_2)^2e^{(-(\tau_1+\tau_2)(h_{i+1}+h_i))}+\tau_2(-\tau_2+\tau_1\tau_2h_i+\tau_1)e^{(-\tau_2(h_{i+1}+h_i))}+\tau_1(\tau_1-\tau_2)e^{(-\tau_1h_{i+1})}\\&-\tau_2(\tau_1-\tau_2)e^{(-\tau_2h_{i+1})}-\tau_1(-\tau_2+\tau_1\tau_2h_i+\tau_1)e^{(-\tau_1h_{i+1}-\tau_2h_i)}-\tau_2(-\tau_1+\tau_1\tau_2h_i+\tau_2)e^{(-\tau_1h_i-\tau_2h_{i+1})}\Big],\end{aligned} \tag{33}$$

$$K_i = (-\tau_1+\tau_1\tau_2h_i+\tau_2)e^{(-\tau_1h_i)}+(-\tau_1-\tau_1\tau_2h_i+\tau_2)e^{(-\tau_2h_i)}+(\tau_1-\tau_2)e^{(-h_i(\tau_1+\tau_2))}+\tau_1-\tau_2. \tag{34}$$

If $\tau_2 = -\tau_1 = \tau$ and $h_i = h, i = 1, 2, \ldots, n$, then

$$\begin{aligned}\beta_{1i} &= \frac{(-2\tau he^{(\tau h)}+e^{(2\tau h)}-1)\tau}{4e^{(\tau h)}-2-2e^{(2\tau h)}+\tau he^{(2\tau h)}-\tau h},\\ \beta_{2i} &= \frac{2(\tau he^{(2\tau h)}-e^{(2\tau h)}+\tau h+1)\tau}{4e^{(\tau h)}-2-2e^{(2\tau h)}+\tau he^{(2\tau h)}-\tau h},\\ \beta_{3i} &= \frac{(-2\tau e^{(\tau h)}h+e^{(2\tau h)}-1)\tau}{4e^{(\tau h)}-2-2e^{(2\tau h)}+\tau he^{(2\tau h)}-\tau h},\\ \beta_{4i} &= \frac{(-1+e^{(\tau h)})\tau^2}{-2e^{(\tau h)}+\tau e^{(\tau h)}h+2+\tau h},\\ \beta_{5i} &= 0,\\ \beta_{6i} &= -\frac{(-1+e^{(\tau h)})\tau^2}{-2e^{(\tau h)}+\tau e^{(\tau h)}h+2+\tau h}.\end{aligned} \tag{35}$$

Let $\tau \longrightarrow 0$; then, (28) is equivalent to

$$h(D_{i-1}+4D_i+D_{i+1})+3S_{i-1}-3S_{i+1} = 0. \tag{36}$$

This is consistent with the cubic 2nd-order polynomial spline function relationship.

Besides, using

$$\begin{aligned}s_i(x_i) &= S_i,\\ s_i'(x_i) &= D_i,\\ s_i''(x_i) &= M_i,\\ s_i''(x_{i-1}) &= M_{i-1},\end{aligned} \tag{37}$$

we can obtain

$$\begin{aligned}a_i &= \frac{(\tau_1^2-\tau_2^2)M_{i-1}+\left(\tau_2^2e^{(-\tau_2h_i)}-\tau_1^2e^{(-\tau_1h_i)}\right)M_i+\tau_1^2\tau_2^2\rho_iS_i}{\tau_1^2\tau_2^2\rho_i},\\ b_i &= \frac{(\tau_1-\tau_2)M_{i-1}+\left(\tau_2e^{(-\tau_2h_i)}-\tau_1e^{(-\tau_1h_i)}\right)M_i+\tau_1\tau_2\rho_iD_i}{\tau_1\tau_2\rho_i},\\ c_i &= -\frac{-M_{i-1}+e^{(-\tau_2h_i)}M_i}{\tau_1^2\rho_i},\\ d_i &= \frac{-M_{i-1}+e^{(-\tau_1h_i)}M_i}{\tau_2^2\rho_i}.\end{aligned} \tag{38}$$

Then, using continuous condition of $s_i(x_i) = s_{i+1}(x_i)$, $s_i'(x_i) = s_{i+1}'(x_i)$, we obtain

$$
\begin{aligned}
&-\frac{\left(h_{i+1}\tau_1\tau_2^2 - h_{i+1}\tau_1^2\tau_2 + \tau_2^2 e^{-\tau_1 h_{i+1}} - \tau_1^2 e^{-\tau 2 h_{i+1}} + \tau_1^2 - \tau_2^2\right)}{\tau_1^2\tau_2^2\rho_{i+1}} M_i \\
&+\frac{\left(\tau_1^2 e^{-\tau_1 h_{i+1}} - \tau_2^2 e^{-\tau_2 h_{i+1}} + h_{i+1}\tau_1\tau_2^2 e^{-\tau_2 h_{i+1}} - h_{i+1}\tau_1^2\tau_2 e^{-\tau_1 h_{i+1}} + \tau_2^2 e^{-h_{i+1}\left(\tau_1+\tau_2\right)} - \tau_1^2 e^{-h_{i+1}\left(\tau_1+\tau_2\right)}\right)}{\tau_1^2\tau_2^2\rho_{i+1}} M_{i+1} \\
&+ S_i - S_{i+1} + h_{i+1} D_{i+1} = 0,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&-\frac{\left(\tau_2 e^{\left(-\tau_2 h_{i+1}\right)} - \tau_1 e^{-\tau_1 h_{i+1}} \tau_1 e^{-h_{i+1}\left(\tau_1+\tau 2\right)} - \tau_2 e^{-h_{i+1}\left(\tau_1+\tau_2\right)}\right)}{\tau_1\tau_2\rho_{i+1}} M_{i+1} \\
&-\frac{\left(\tau_2 e^{-\tau_1 h_{i+1}} - \tau_1 e^{-\tau_2 h_{i+1}} + \tau_1 - \tau_2\right)}{\tau_1\tau_2\rho_{i+1}} M_i + D_i - D_{i+1} = 0.
\end{aligned}
\tag{40}
$$

Solving (39) yields

$$
\begin{aligned}
D_i = &-\frac{\left(\tau_1^2 e^{-\tau_2 h_i} - \tau_2^2 e^{-\tau_1 h_i} - h_i\tau_1\tau_2^2 + h_i\tau_1^2\tau_2 - \tau_1^2 + \tau_2^2\right)}{h_i\tau_1^2\tau_2^2\rho_i} M_{i-1} \\
&+\frac{\left(\left(-h_i\tau_1\tau_2^2 + \tau_2^2\right)e^{-\tau_2 h_i} + \left(-\tau_1^2 + h_i\tau_1^2\tau_2\right)e^{-\tau_1 h_i} + \left(\tau_1^2 - \tau_2^2\right)e^{-h_i\left(\tau_2+\tau_1\right)}\right)}{h_i\tau_1^2\tau_2^2\rho_i} M_i \\
&-\frac{S_{i-1}}{h_i} + \frac{S_i}{h_i}.
\end{aligned}
\tag{41}
$$

Meanwhile, by eliminating D_{i+1} with (39) and (40) and rearranging, we obtain

$$
\begin{aligned}
D_i = &\frac{\left(\left(-\tau_1^2 - h_{i+1}\tau_1^2\tau_2\right)e^{-\tau_2 h_{i+1}} + \left(\tau_2^2 + h_{i+1}\tau_1\tau_2^2\right)e^{-\tau_1 h_{i+1}} - \tau_2^2 + \tau_1^2\right)}{h_{i+1}\tau_1^2\tau_2^2\rho_{i+1}} M_i \\
&-\frac{\left(-\tau_1^2 + h_{i+1}\tau_1\tau_2^2 - h_{i+1}\tau_1^2\tau_2 + \tau_2^2\right)e^{-h_{i+1}\left(\tau_1+\tau_2\right)} + \tau_1^2 e^{-\tau_1 h_{i+1}} - \tau_2^2 e^{-\tau_2 h_{i+1}}}{h_{i+1}\tau_1^2\tau_2^2\rho_{i+1}} M_{i+1},
\end{aligned}
\tag{42}
$$

$$
-\frac{S_i}{h_{i+1}} + \frac{S_{i+1}}{h_{i+1}}.
\tag{43}
$$

We can also use $s_{i+1}(x_{i+1}) = S_{i+1}, s_{i+1}(x_i) = S_i, s'_{i+1}(x_{i+1}) = D_{i+1}$, and $s'_{i+1}(x_i) = D_i$ to obtain

$$
\begin{aligned}
a_i &= \frac{1}{\varrho_i}\Big((\tau_1 h_i - \tau_2 h_i + \rho_i)D_{i-1} + \Big(h_i\tau_2 e^{-\tau_2 h_i} - \tau_1 e^{-\tau_1 h_i} h_i - \rho_i\Big)D_i \\
&\quad + \Big(\tau_2 e^{-\tau_2 h_i} + \tau_1 - \tau_2 - \tau_1 e^{-\tau_1 h_i}\Big)S_{i-1} \\
&\quad + \Big((-\tau_1 - \tau_1 h_i \tau_2)e^{-\tau_2 h_i} + (\tau_1 h_i \tau_2 + \tau_2)e^{-\tau_1 h_i} + (\tau_1 - \tau_2)e^{-h_i(\tau_1+\tau_2)}\Big)S_i\Big), \\
b_i &= \frac{1}{\varrho_i}\Big(\Big(\tau_1 - \tau_2 - \tau_1 e^{-\tau_2 h_i} + \tau_2 e^{-\tau_1 h_i}\Big)D_{i-1} + \Big(\tau_2 e^{-\tau_2 h_i} - \tau_1 e^{-\tau_1 h_i} + (\tau_1 - \tau_2)e^{-h_i(\tau_1+\tau_2)}\Big)D_i \\
&\quad - \tau_1\tau_2\rho_i S_{i-1} + \tau_1\tau_2\rho_i S_i\Big), \\
c_i &= \frac{1}{\varrho_i}\Big(\Big(e^{-\tau_2 h_i} - 1 + \tau_2 h_i\Big)D_{i-1} + \Big(-e^{-\tau_2 h_i} + 1 - h_i\tau_2 e^{-\tau_2 h_i}\Big)D_i \\
&\quad + \Big(-\tau_2 e^{-\tau_2 h_i} + \tau_2\Big)S_{i-1} + \Big(-\tau_2 + \tau_2 e^{-\tau_2 h_i}\Big)S_i\Big), \\
d_i &= \frac{1}{\varrho_i}\Big(\Big(-\tau_1 h_i - e^{(-\tau_1 h_i)} + 1\Big)D_{i-1} + \Big(\tau_1 e^{-\tau_1 h_i} h_i + e^{-\tau_1 h_i} - 1\Big)D_i \\
&\quad + \Big(-\tau_1 + \tau_1 e^{-\tau_1 h_i}\Big)S_{i-1} + \Big(-\tau_1 e^{-\tau_1 h_i} + \tau_1\Big)S_i\Big),
\end{aligned}
\tag{44}
$$

where $\varrho_i = (-\tau_1 h_i \tau_2 + \tau_2 - \tau_1)e^{-\tau_2 h_i} + (\tau_1 h_i \tau_2 + \tau_2 - \tau_1)e^{-\tau_1 h_i} + (\tau_1 - \tau_2)e^{-h_i(\tau_1+\tau_2)} + \tau_1 - \tau_2$. Fast Hermite interpolation can achieved by using this set of relations.

4. Exponential Spline Difference Method

The spline difference method uses the relationship between the spline function and its derivative to construct the differential expression to numerically solve DEs, by which the numerical solution at nodes can be obtained, and that within the subintervals can also be calculated by using the spline function expressions. It is the advantage of this method compared with the general difference schemes. In fact, it can be said that the approximate analytical solution using the splines is obtained.

4.1. Differential Expression. The following presents a spline difference method for solving (1) which satisfies one of the boundary conditions (2)–(5) for the boundary value problems of ODEs. Due to the limitation of this method, we only consider the case of $q(x) = 0$ in this section. For convenience of description, we first consider the boundary condition (2). From (1), we can obtain

$$u'' = g(x) - p(x)u, \quad x \in [a, b]. \tag{45}$$

By discretization the above equation, we obtain

$$u''_i = g_i - p_i u_i, \quad i = 0, 1, 2, \ldots, n, \tag{46}$$

where $g_i = g(x_i)$ and $p_i = p(x_i)$.

Substitute S with u and M with u'' in (24), and we obtain

$$t_i = (\alpha_{1i}u''_{i-1} + \alpha_{2i}u''_i + \alpha_{3i}u''_{i-1}) - \frac{u_{i-1}}{h_i} + \frac{(h_i + h_{i+1})u_i}{h_i h_{i+1}} - \frac{u_{i+1}}{h_{i+1}}, \tag{47}$$

where $i = 1, \ldots, n-1$, $u_i = u(x_i)$ and $u''_i = u''(x_i)$. t_i is the local truncation error at x_i. By substituting (46) into (47) and rearranging, we obtain

$$
\begin{aligned}
&\left(-\frac{1}{h_i} - \alpha_{1i}p_{i-1}\right)u_{i-1} + \left(\frac{h_i + h_{i+1}}{h_i h_{i+1}} - \alpha_{2i}p_i\right)u_i \\
&\quad + \left(-\frac{1}{h_{i+1}} - \alpha_{3i}p_{i+1}\right)u_{i+1} + \alpha_{1i}g_{i-1} + \alpha_{2i}g_i + \alpha_{3i}g_{i+1} = t_i,
\end{aligned}
\tag{48}
$$

where $i = 1, 2, \ldots, n-1$. Thus, we get $n-1$ equations about $u_0, u_1, \ldots, u_n$.

The following equation can also be derived from the boundary conditions (2):

$$
\begin{aligned}
u_0 &= \mu_0, \\
u_n &= \mu_1.
\end{aligned}
\tag{49}
$$

So, there will be $n+1$ equations in (48) and (49) in total, and it matrix form can be written as

$$\mathbf{AU} = \mathbf{F} + \mathbf{T}, \tag{50}$$

where $\mathbf{A} = \mathbf{B} - \mathbf{W}$, with

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & & & & & \\ -\frac{1}{h_1} & \frac{h_1+h_2}{h_1h_2} & -\frac{1}{h_2} & & & & & \\ & -\frac{1}{h_2} & \frac{h_2+h_3}{h_2h_3} & -\frac{1}{h_3} & & & & \\ & & & \ddots & & & & \\ & & & -\frac{1}{h_{n-2}} & \frac{h_{n-2}+h_{n-1}}{h_{n-2}h_{n-1}} & -\frac{1}{h_{n-1}} & & \\ & & & & -\frac{1}{h_{n-1}} & \frac{h_{n-1}+h_n}{h_{n-1}h_n} & -\frac{1}{h_n} \\ & & & & 0 & 0 & 1 \end{bmatrix}, \tag{51}$$

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & & & & \\ \alpha_{11}p_0 & \alpha_{21}p_1 & \alpha_{31}p_2 & & & & \\ & \alpha_{12}p_1 & \alpha_{22}p_2 & \alpha_{32}p_3 & & & \\ & & & \ddots & & & \\ & & & \alpha_{1,n-2}p_{n-3} & \alpha_{2,n-2}p_{n-2} & \alpha_{3,n-2}p_{n-1} & \\ & & & & \alpha_{1,n-1}p_{n-2} & \alpha_{2,n-1}p_{n-1} & \alpha_{3,n-1}p_n \\ & & & & 0 & 0 & 0 \end{bmatrix}, \tag{52}$$

$$\begin{aligned} \mathbf{F} &= \left[\mu_0, \{-(\alpha_{1i}g_{i-1} + \alpha_{2i}g_i + \alpha_{3i}g_{i+1})\}_{i=1}^{n-1}, \mu_1\right]^T, \\ \mathbf{T} &= \left[\{t_i\}_{i=0}^{n}\right]^T, \\ \mathbf{U} &= \left[\{u_i\}_{i=0}^{n}\right]^T. \end{aligned} \tag{53}$$

If $\mathbf{A}$ is nonsingular, then equation (50) has a unique solution. Solving (50) yields the approximate values of u_i in $u(x)$ at the splitting points $x_i, i = 0, 1, \ldots, n$. Since $\mathbf{A}$ is a tridiagonal matrix, the catch-up method can be used to reduce the calculation in practice.

To obtain the spline function expression, we can calculate u_i'' from (2) after finding u_i and substitute them into (22) to obtain a_i, b_i, c_i, and d_i so that the approximate solution of the spline over the entire interval will be found.

The case of other boundary conditions will be discussed below. For the second boundary conditions, from (41), (42), and (3), we obtain

$$\alpha_{20}M_0 + \alpha_{30}M_1 + \frac{S_0}{h_1} - \frac{S_1}{h_1} = -\mu_0, \tag{54}$$

$$\alpha_{1n}M_{n-1} + \alpha_{2n}M_n - \frac{S_{n-1}}{h_n} + \frac{S_n}{h_n} = \mu_1, \tag{55}$$

where

$$\begin{aligned} \alpha_{20} &= \frac{\left((\tau_1^2 + h_1\tau_1^2\tau_2)e^{-\tau_2 h_1} + (-\tau_2^2 - \tau_1 h_1 \tau_2^2)e^{-\tau_1 h_1} - \tau_1^2 + \tau_2^2\right)}{h_1\tau_1^2\tau_2^2\rho_1}, \\ \alpha_{30} &= -\frac{\left(\tau_2^2 e^{-\tau_2 h_1} - \tau_1^2 e^{-\tau_1 h_1} + (\tau_1^2 - \tau_1 h_1\tau_2^2 + h_1\tau_1^2\tau_2 - \tau_2^2)e^{-h_1(\tau_2+\tau_1)}\right)}{h_1\tau_1^2\tau_2^2\rho_1}, \\ \alpha_{1n} &= -\frac{\left(\tau_1^2 e^{-\tau_2 h_n} - \tau_2^2 e^{-\tau_1 h_n} - h_n\tau_1\tau_2^2 + h_n\tau_1^2\tau_2 - \tau_1^2 + \tau_2^2\right)}{h_n\tau_1^2\tau_2^2\rho_n}, \\ \alpha_{2n} &= \frac{\left((-h_n\tau_1\tau_2^2 + \tau_2^2)e^{-\tau_2 h_n} + (-\tau_1^2 + h_n\tau_1^2\tau_2)e^{-\tau_1 h_n} + (\tau_1^2 - \tau_2^2)e^{-h_n(\tau_2+\tau_1)}\right)}{h_n\tau_1^2\tau_2^2\rho_n}. \end{aligned} \tag{56}$$

Solving this type of boundary problem only needs to modify the first line and the last row of $\mathbf{B}$, $\mathbf{W}$, and $\mathbf{F}$ in (47), namely,

$$\begin{aligned} \mathbf{B}_0 &= \begin{bmatrix} \frac{1}{h_1} & -\frac{1}{h_1} & 0 & \cdots & 0 \end{bmatrix}, \\ \mathbf{W}_0 &= [\, \alpha_{20}p_0 \;\; \alpha_{30}p_1 \;\; 0 \;\; \cdots \;\; 0 \,], \\ \mathbf{B}_n &= \begin{bmatrix} 0 & \cdots & 0 & -\frac{1}{h_n} & \frac{1}{h_n} \end{bmatrix}, \\ \mathbf{W}_n &= [\, 0 \;\; \cdots \;\; 0 \;\; \alpha_{n1}p_{n-1} \;\; \alpha_{n2}p_n \,], \\ \mathbf{F}_0 &= -\mu_0 - (\alpha_{20}g_0 + \alpha_{30}g_1), \\ \mathbf{F}_n &= \mu_1 - (\alpha_{1n}g_{n-1} + \alpha_{2n}g_n). \end{aligned} \tag{57}$$

And the third and the fourth boundary conditions should also be modified accordingly.

4.2. Error Estimation. Suppose $u(x)$ is sufficiently smooth in $[a, b]$, by Taylor expanding $u_{i-1}, u_{i+1}, u_{i-1}'', u_{i+1}''$ in difference equation (47) at x_i, we obtain

$$t_i = \sum_{j=2}^{6} \xi_j u_i^{(j)} + O\left(h_i^7\right), \tag{58}$$

where

$$\xi_2 = \alpha_{1i} + \alpha_{2i} + \alpha_{3i} - \frac{1}{2}(h_i + h_{i+1}),$$
$$\xi_3 = -\alpha_{1i}h_i + \alpha_{3i}h_{i+1} + \frac{1}{6}\left(h_i^2 - h_{i+1}^2\right),$$
$$\xi_4 = \frac{1}{2}\alpha_{1i}h_i^2 + \frac{1}{2}\alpha_{3i}h_{i+1}^2 - \frac{1}{24}\left(h_i^3 + h_{i+1}^3\right), \tag{59}$$
$$\xi_5 = -\frac{1}{6}\alpha_{1i}h_i^3 + \frac{1}{6}\alpha_{3i}h_{i+1}^3 + \frac{1}{120}\left(h_i^4 + h_{i+1}^4\right),$$
$$\xi_6 = \frac{1}{24}\alpha_{1i}h_i^4 + \frac{1}{24}\alpha_{3i}h_{i+1}^4 - \frac{1}{720}\left(h_i^5 + h_{i+1}^5\right).$$

Let $\tau_1 \longrightarrow 0$ and $\tau_2 \longrightarrow 0$, we find $\xi_2 = \xi_3 = 0$ and $\xi_4 = (1/24)(h_{i+1}^3 + h_i^3)$. When

$$\alpha_{1i} = \frac{1}{12}\frac{h_i^2 + h_i h_{i+1} - h_{i+1}^2}{h_i},$$
$$\alpha_{2i} = \frac{1}{12}\frac{h_i^3 + 4h_i^2 h_{i+1} + 4h_i h_{i+1}^2 + h_{i+1}^3}{h_i h_{i+1}}, \tag{60}$$
$$\alpha_{3i} = -\frac{1}{12}\frac{h_i^2 - h_i h_{i+1} - h_{i+1}^2}{h_{i+1}},$$

we obtain $\xi_2 = \xi_3 = \xi_4 = 0$,

$$\xi_5 = \frac{1}{360}(-h_i + h_{i+1})(2h_i + h_{i+1})(h_i + 2h_{i+1})(h_i + h_{i+1}). \tag{61}$$

$\xi_5 = 0$ for an even splitting, i.e., $h_i = h, i = 1, 2, \ldots, n$, and $\xi_6 = (1/240)h^6$. This indicates that the even splitting has higher accuracy if $u(x)$ is sufficiently smooth.

4.3. Convergence Analysis. We mainly discuss the convergence of equation (47) and the differential expression of (49) in the sense of $\|\mathbf{E}\|_\infty$.

Lemma 1 (see [14]). *If the n-order matrix* **B** *satisfies one of the following two conditions:*

(1) **B** *is a strictly diagonally dominant matrix*

(2) **B** *is a second half strong diagonally dominant matrix*

Then, **B** *is nonsingular and* $\rho(\mathbf{I} - \mathbf{D}^{-1}\mathbf{B}) < 1$, *where* $\mathbf{D} = diag(\mathbf{B})$ *and* **I** *is the identity matrix.*

By (14),

$$\mathbf{AS} = \mathbf{F}. \tag{62}$$

If **A** is reversible, combining (50), we obtain

$$\mathbf{E} = \mathbf{A}^{-1}\mathbf{T} = (\mathbf{B} - \mathbf{W})^{-1}\mathbf{T} = \left(\mathbf{I} - \mathbf{B}^{-1}\mathbf{W}\right)^{-1}\mathbf{B}^{-1}\mathbf{T}, \tag{63}$$

where $\mathbf{E} = (e_i) = \mathbf{U} - \mathbf{S}$, and **B**, **W**, and **T** are given by (51)–(53). The reversibility of **B** can be proved by 1. From the boundary condition (2), we can get $e_0 = e_n = 0$. Thus, discussing the convergence of $\|\mathbf{E}\|_\infty$ is consistent with that of $\|(e_i)_{i=1}^{n-1}\|_\infty$. So, in the following convergence discussion, the first and the last row and the first and the last column will be removed from the original matrices of **A**, **B**, and **W** to obtain the $n-1$-order ones, keeping the subscript value unchanged.

If

$$\left\|\mathbf{B}^{-1}\right\|_\infty \|\mathbf{W}\|_\infty < 1, \tag{64}$$

then

$$\|\mathbf{E}\|_\infty \le \frac{\left\|\mathbf{B}^{-1}\right\|_\infty \|\mathbf{T}\|_\infty}{1 - \left\|\mathbf{B}^{-1}\right\|_\infty \|\mathbf{W}\|_\infty} \le \left\|\mathbf{B}^{-1}\right\|_\infty \|\mathbf{T}\|_\infty. \tag{65}$$

To calculate $\mathbf{B}^{-1}$, the following lemma is needed:

Lemma 2 (see [15]). *Let the square matrix* **A** *be an n-order tridiagonal one, with the following expression:*

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}, \tag{66}$$

and $b_i c_i \ne 0, i = 1, 2, \ldots, n-1$, $a_i \ne 0, i = 1, 2, \ldots, n$; *then, the expression of* $\mathbf{A}^{-1} = (a_{ij}^{(-1)})$ *can be written as*

$$a_{ij}^{(-1)} = \begin{cases} (-1)^{i+j}\left(\prod_{k=i}^{j-1} b_k\right)\dfrac{\det(\mathbf{A}[1,\cdots,i-1])\det(\mathbf{A}[j+1,\cdots,n])}{\det(\mathbf{A})}, & i < j, \\ (-1)^{i+j}\left(\prod_{k=j}^{i-1} \dfrac{1}{c_k}\right)\dfrac{\det(\mathbf{A}[1,\cdots,i-1])\det(\mathbf{A}[j+1,\cdots,n])}{\det(\mathbf{A})}, & i \ge j, \end{cases} \tag{67}$$

where $\mathbf{A}[i_1, i_2, \ldots, i_k]$ *denotes the matrix consisting of the elements of* $i_1, i_2, \ldots, i_k$ *rows crossed with the columns in* **A**. *Specifically,* $\mathbf{A}[1, \ldots, i-1] = 1$ *for* $i = 1$.

Using (2), $\mathbf{B}^{-1} = (b_{ij}^{(-1)})$ *in (65) can be written as (with* $\sum_{k=1}^{n} h_k = b - a$*)*

Table 1: The maximum error $\|E\|_\infty$ in solving (72) with the exponential spline difference method.

	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
Ref. [7]	$1.84e-4$	$4.61e-5$	$1.15e-5$	$2.88e-6$	$7.21e-7$
Our method ($\tau = [n^{1/4}, -n^{1/4}]$)	$8.11e-6$	$2.03e-6$	$5.09e-7$	$1.28e-7$	$4.25e-8$

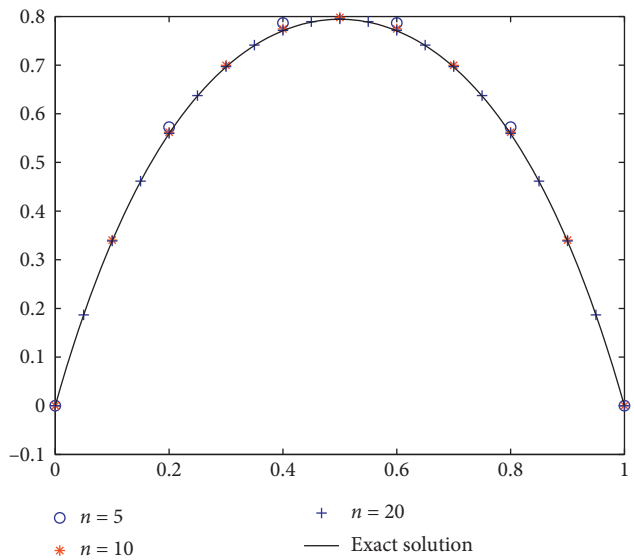

Figure 2: Results of the boundary value problem (72) for $\varepsilon = 0.1$, and $n = 5, 10$, and 20, respectively.

Table 2: The maximum error $\|E\|_\infty$ and root mean square error for problem (74).

	$n = 10$	$n = 20$	$n = 40$	$n = 80$
Ref. [11]	$1.02e-3$	$2.96e-5$	$3.69e-6$	$5.11e-8$
Our method ($\tau = [10.11, -0.012]$)	$8.57e-7$	$8.18e-7$	$8.03e-7$	$7.97e-7$

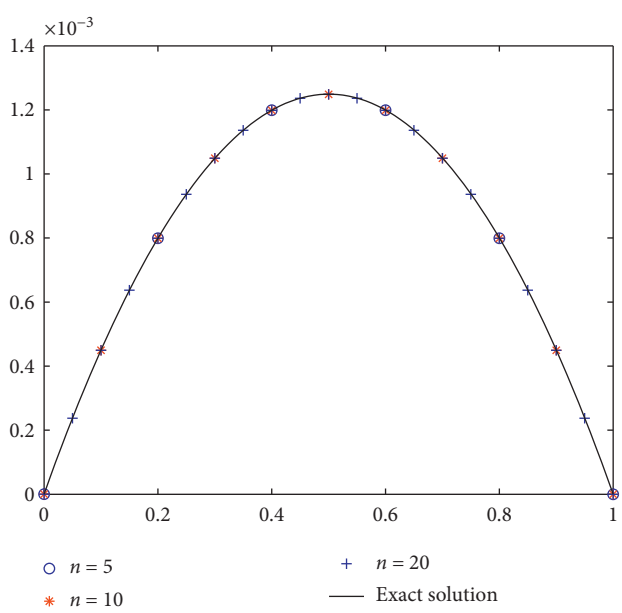

Figure 3: Results of the boundary value problem (74) for $\varepsilon = 0.01$, and $n = 5, 10$, and 20, respectively.

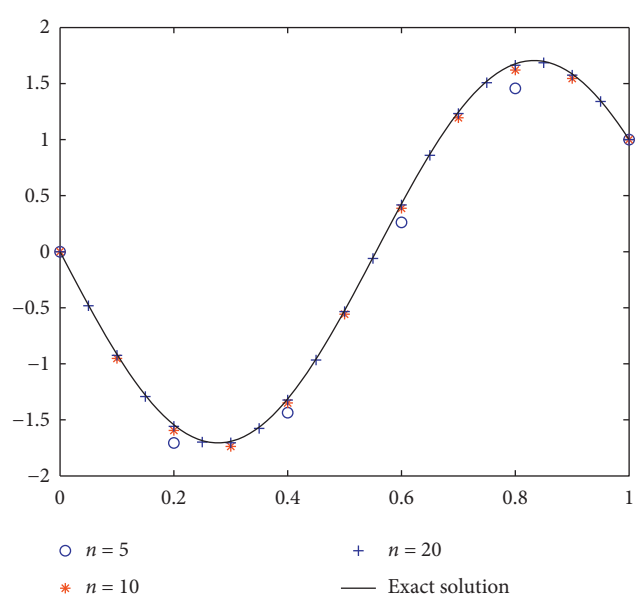

Figure 4: Results of the boundary value problem (76) for $\varepsilon = 1/32$, and $n = 5, 10$, and 20, respectively.

Table 3: The maximum error $\|E\|_\infty$ and root mean square error for problem (76).

ε	Method	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	Ref. [16]	$2.6e-4$	$6.5e-5$	$1.6e-5$	$4.1e-6$	$1.0e-6$	$2.6e-7$	$6.4e-8$
	Our method ($\tau = [1.18, -12.75]$)	$2.34e-4$	$6.37e-5$	$1.65e-5$	$4.20e-6$	$1.06e-6$	$2.72e-7$	$5.21e-8$
1/4	Ref. [16]	$1.9e-3$	$4.7e-4$	$1.2e-4$	$2.9e-5$	$7.4e-6$	$1.8e-6$	$4.6e-7$
	Our method ($\tau = [1.91, -11.77]$)	$3.04e-3$	$8.19e-4$	$2.11e-4$	$5.38e-5$	$1.35e-5$	$3.36e-6$	$1.94e-7$
1/8	Ref. [16]	$7.3e-2$	$1.9e-2$	$4.7e-3$	$1.2e-3$	$2.9e-4$	$7.3e-5$	$1.8e-5$
	Our method ()	$8.23e-2$	$2.39e-2$	$6.13e-3$	$1.54e-3$	$3.93e-4$	$9.66e-5$	$6.74e-7$
1/16	Ref. [16]	$2.9e-2$	$7.1e-3$	$1.8e-3$	$4.4e-4$	$1.1e-4$	$2.8e-5$	$6.9e-6$
	Our method ($\tau = [13, -1]$)	$1.61e-2$	$4.03e-3$	$1.04e-3$	$2.67e-4$	$6.80e-5$	$1.66e-5$	$3.74e-6$
1/32	Ref. [16]	—	$3.8e$—2	$9.8e$—3	$2.5e$—3	$6.2e$—4	$1.5e$—4	$3.9e$-5
	Our method ($\tau = [1, -13]$)	$7.52e-2$	$1.99e-2$	$5.06e-3$	$1.30e-3$	$3.25e-4$	$7.93e-5$	$1.12e-5$
1/64	Ref. [16]	—	$3.4e-2$	$8.7e-3$	$2.2e-3$	$5.5e-4$	$1.4e-4$	$3.4e-5$
	Our method ($\tau = [12.51, -1.48]$)	$8.54e-2$	$2.33e-2$	$6.39e-3$	$1.63e-3$	$4.10e-4$	$1.03e-4$	$1.85e-5$
1/128	Ref. [16]	—	—	$2.8e-2$	$7.2e-3$	$1.8e-3$	$4.5e-4$	$1.1e-4$
	Our method ($\tau = [11.53, -2.46]$)	—	$6.46e-2$	$1.69e-2$	$4.33e-3$	$1.09e-3$	$2.73e-4$	$5.70e-5$
1/256	Ref. [16]	—	—	—	—	$6.3e-2$	$1.6e-2$	$3.9e-3$
	Our method ($\tau = [13, -1]$)	—	—	—	—	$3.21e-2$	$8.09e-3$	$9.13e-4$
1/512	Ref. [16]	—	—	—	—	$4.2e-2$	$1.0e-2$	$2.5e-3$
	Our method ($\tau = [11.53, -2.46]$)	—	—	—	—	$2.12e-2$	$5.35e-3$	$6.92e-4$
1/1024	Ref. [16]	—	—	—	—	—	$3.4e-2$	$8.5e-3$
	Our method ($\tau = [3.93, -10]$)	—	—	—	—	—	$1.92e-2$	$2.55e-3$

$$b_{ij}^{(-1)} = \begin{cases} \dfrac{1}{b-a}\displaystyle\sum_{k=1}^{i} h_k \sum_{k=j+1}^{n} h_k, & i < j, \\ \dfrac{1}{b-a}\displaystyle\sum_{k=1}^{j} h_k \sum_{k=i+1}^{n} h_k, & i \geq j, \end{cases} \tag{68}$$

then $0 < b_{ij}^{(-1)} \leq (b-a)/4$. *Thus,*

$$\left\|\mathbf{B}^{-1}\right\|_\infty \leq \frac{1}{4} n(b-a) \leq \frac{(b-a)^2}{4h_{\min}}. \tag{69}$$

According to (27), we find $|t_i| = O(h_i^2)$. Thus,

$$\left\|\mathbf{B}^{-1}\right\|_\infty \|\mathbf{T}\|_\infty \leq \frac{(b-a)^2}{4} \frac{h_{\max}^2}{h_{\min}}. \tag{70}$$

So, if $h_{\max} = O(h_{\min})$, then $\|\mathbf{B}^{-1}\|_\infty \|\mathbf{W}\|_\infty \longrightarrow 0$ when $h_{\max} \longrightarrow 0$.

So far, we have proved that

Theorem 2. *For given* τ_1, τ_2, *and* Δ: $0 = x_0 < x_1 < \cdots < x_n = 1$, $h_i = x_i - x_{i-1}, i = 1, 2, \ldots, n$, $h_{\max} = \max_{1 \leq i \leq n}\{h_i\}$, $h_{\min} = \min_{1 \leq i \leq n}\{h_i\}$, *if* $\|\mathbf{B}^{-1}\|_\infty \|\mathbf{W}\|_\infty < 1$, *then*

$$\|\mathbf{E}\|_\infty = O\left(\frac{h_{\max}^2}{h_{\min}}\right), \tag{71}$$

that is, the differential expression of (47) is convergent if $h_{\max} = O(h_{\min})$.

5. Example

Example 1. Solve the following singular boundary value problem (see [7]):

$$\begin{cases} -\varepsilon u'' + p(x)u = g(x), & 0 \leq x \leq 1, \\ u(0) = u(1) = 0. \end{cases} \tag{72}$$

where $p(x) = 1 + x(1-x)$ and $g(x) = 1 + x(1-x) + [2\sqrt{\varepsilon} - x \quad (1-x)^2]e^{-x/\sqrt{\varepsilon}} + [2\sqrt{\varepsilon} - x^2(1-x)]e^{-(1-x)/\sqrt{\varepsilon}}$, and its analytical solution is

$$u(x) = 1 + (x-1)e^{-x/\sqrt{\varepsilon}} - xe^{-(1-x)/\sqrt{\varepsilon}}. \tag{73}$$

Table 1 lists the results of the exponential spline difference method for $\varepsilon = 0.1$ and that of the method proposed in [7].

Figure 2 shows the results for $\varepsilon = 0.1$, and $n = 5$ and 10, respectively. To show the difference between the exact

solution and the numerical one, we deliberately use fewer split points and simply connect two adjacent solutions with straight lines. A smoother exponential spline function which is closer to the exact solution can be constructed using these obtained numerical solutions.

Example 2. Consider the convection-dominated equation (see [11]):

$$\begin{cases} u'' - \varepsilon u' - \varepsilon u = -\varepsilon, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{74}$$

The analytical solution is given by

$$u(x) = 1 + \frac{\left(e^{(1-\sqrt{(1+4)/\lambda})/2/\lambda} - 1\right)e^{((1+\sqrt{(1+4)/\lambda})/2/\lambda)x} + \left(1 - e^{(1+\sqrt{(1+4)/\lambda})/2/\lambda}\right)e^{((1-\sqrt{(1+4)/\lambda})/2/\lambda)x}}{((1+\sqrt{(1+4)/\lambda})/2/\lambda) - (1-\sqrt{(1+4)/\lambda})/2/\lambda}. \tag{75}$$

The computational results are shown in Table 2 for $\varepsilon = 0.01$ and various values of n (Figure 3).

Example 3. Consider the convection-dominated equation (see [16–19]) Figure 4:

$$\begin{cases} \varepsilon u'' + u = 0, & 0 < x < 1, \\ u(0) = 0, u(1) = 1, \end{cases} \tag{76}$$

$$u(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}, \varepsilon \neq (n\pi)^{-2}. \tag{77}$$

The computational results are shown in Table 3 for $\varepsilon = 1/2$ and various values of n.

6. Conclusions

Exponential spline, which is a generalization of polynomial spline, is an ideal function approximation tool due to its excellent curve fitting ability. High accuracy can be achieved in solving second-order ODEs using the exponential spline scheme. The spline difference method is an ideal scheme because it can give not only the numerical results but also the spline function expressions by reusing these numerical results at the same time, whereas it is only suitable for solving certain types of equations and does not have generality. And the selection of the appropriate parameters is also needed for this method, but there is no better guideline for the selecting.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Education Department of Guangxi Province in 2015 (KY2015YB081).

References

[1] E. L. Albasiny and W. D. Hoskins, "Cubic spline solution to two-point boundary value problems," *Computer Journal*, vol. 12, no. 2, pp. 151–153, 1969.

[2] C. V. Raghavarao, Y. V. S. S. Sanyasiraju, and S. Suresh, "A note on application of cubic splines to two point boundary value problems," *Computers & Mathematics with Applications*, vol. 27, no. 11, pp. 45–48, 1994.

[3] J. L. Blue, "Spline function methods for nonlinear boundary-value problems," *Communications of the ACM*, vol. 12, no. 6, pp. 327–330, 1969.

[4] H. Caglar, N. Caglar, and K. Elfaituri, "B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems," *Applied Mathematics and Computation*, vol. 175, no. 1, pp. 72–79, 2006.

[5] M. M. Chawla and P. N. Shivakumar, "A finite difference method for a class of two-point boundary value problems over a semi-infinite range," *Journal of Computational and Applied Mathematics*, vol. 15, no. 2, pp. 251–256, 1986.

[6] W. K. Zahra, "Exponential spline solutions for a class of two point boundary value problems over a semi-infinite range," *Numerical Algorithms*, vol. 52, no. 4, pp. 561–573, 2009.

[7] S. C. S. Rao and M. Kumar, "Exponential B-spline collocation method for self-adjoint singularly perturbed boundary value problems," *Applied Numerical Mathematics*, vol. 58, no. 10, pp. 1572–1581, 2008.

[8] I. A. Tirmizi, F. Fazal-i-Haq, and S. Siraj-ul-Islam, "Non-polynomial spline solution of singularly perturbed boundary-value problems," *Applied Mathematics and Computation*, vol. 196, no. 1, pp. 6–16, 2008.

[9] M. A. Ramadan, I. F. Lashien, and W. K. Zahra, "Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 476–484, 2007.

[10] K. Surla and M. Stojanović, "Solving singularly perturbed boundary-value problems by spline in tension," *Journal of Computational and Applied Mathematics*, vol. 24, no. 3, pp. 355–363, 1988.

[11] N. Jha and R. K. Mohanty, "Quintic hyperbolic non-polynomial spline and finite difference method for nonlinear second order differential equations and its application," *Journal of the Egyptian Mathematical Society*, vol. 22, no. 1, pp. 115–122, 2014.

[12] N. Jha, "High order accurate quintic nonpolynomial spline finite difference approximations for the numerical solution of non-linear two point boundary value problems," *International*

Journal of Modeling, Simulation, and Scientific Computing, vol. 5, no. 1, p. 1350018, 2014.

[13] M. K. Kadalbajoo and K. C. Patidar, "Numerical solution of singularly perturbed two-point boundary value problems by spline in tension," *Applied Mathematics and Computation*, vol. 131, no. 2-3, pp. 299–320, 2002.

[14] Z. You, *Non-singular M-Matrix*, pp. 89–108, Huazhong Institute of Technology Press, Wuhan, China, 1981.

[15] C. Yang, C. Xu, and Y. Gao, "Explicit inverses of nonsingular tridiagonal matrices," *Journal of Engineering Mathematics*, vol. 21, no. 1, pp. 48–52, 2004.

[16] M. K. Kadalbajoo and K. C. Patidar, "Numerical solution of singularly perturbed two point boundary value problems by spline in compression," *International Journal of Computer Mathematics*, vol. 77, no. 2, pp. 263–28310, 2001.

[17] S.-T. Liu and Y. Xu, "Galerkin methods based on hermite splines for singular perturbation problems," *SIAM Journal on Numerical Analysis*, vol. 43, no. 6, pp. 2607–2623, 2006.

[18] H. Liu, Y. Pan, S. Li, and Y. Chen, "Adaptive fuzzy backstepping control of fractional-order nonlinear systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 8, pp. 2209–2217, 2017.

[19] H. Liu, H. Wang, J. Cao, A. Alsaedi, and T. Hayat, "Composite learning adaptive sliding mode control of fractional-order nonlinear systems with actuator faults," *Journal of the Franklin Institute*, vol. 356, no. 16, pp. 9580–9599, 2019.