In this paper, we mainly study an exponential spline function space, construct a basis with local supports, and present the relationship between the function value and the first and the second derivative at the nodes. Using these relations, we construct an exponential spline-based difference scheme for solving a class of boundary value problems of second-order ordinary differential equations (ODEs) and analyze the error and the convergence of this method. The results show that the algorithm is high accurate and conditionally convergent, and an accuracy of \((1/240)h^6\) was achieved with smooth functions.

1. Introduction

In physics, chemistry, biology, sociology, and many other disciplines, there are tremendous problems that can be described by differential equations (DEs), but it is difficult to get their explicit expressions. So, people began to seek the numerical solutions of these problems, which can also be applied to scientific research and engineering practice if their accuracy satisfies the needs. Especially the advent of computers makes it possible to quickly carry out a large number of calculations, which also makes the numerical solution method of DEs become one of the most important branches of computational mathematics. Due to its high smoothness, low power, and easy calculation, the spline function has been widely used in computer graphics, data interpolation and fitting, shape control, and numerical solutions of DEs. There are two main schemes in numerical solutions of DEs using spline functions: the spline finite element method and the spline difference method. The first has a wide range of application and can be applied to many types of equations, but it requires a large amount of calculation. While the second is simple process with a small amount of calculation and high accuracy, but it can only be applied to specific types of equations.

In this paper, we mainly focus a class of second-order ordinary differential equations (ODEs):

\[ u'' + q(x)u' + p(x)u = g(x), \quad x \in [a, b], \]

which meets one of the following boundary conditions.

(1) First boundary condition:

\begin{align}
  u(a) &= \mu_0, \\
  u(b) &= \mu_1.
\end{align}

(2) Second boundary condition:

\begin{align}
  u^{(1)}(a) &= \mu_0, \\
  u^{(1)}(b) &= \mu_1.
\end{align}

(3) Third boundary condition:

\begin{align}
  u(a) &= \mu_0, \\
  u^{(1)}(b) &= \mu_1.
\end{align}

In recent years, with the deepening of research, people have begun to use nonpolynomial splines to solve such problems. Zahra [6], Rao and Kumar [7], Tirmizi et al. [8], Ramadan et al. [9], Surla and Stojanović [10], Jha [11,12], and Kadalbajoo and Patidar [13] have carried out a lot of research in this area and achieved very high computational accuracy. However, there are still many theoretical problems to be broken in the study of nonpolynomial splines. Due to the diversity of nonpolynomial splines, it is crucial to choose the basis and parameters in solving the problem. However, there is still no reference in this regard. In this paper, a selected set of spline basis functions was used to deduce the relationship between the derivative and the function value and then to obtain the second-order difference scheme for solving second-order ODEs, which provides a method for solving such problems.

2. Exponential Spline Function Space

Exponential spline refers to a type of spline in which the nonpolynomial factors of spline basis functions contain only exponential functions. The exponential spline in this sense is not very specific; it can contain many forms of exponential spline, which can produce substantially different splines, and is inconvenient to study. Therefore, the exponential spline refers to that with a specific form in the rest of this work.

Next, we define an exponential spline function space. Let

\[ s_i(x) = a_i + b_i (x - x_i) + c_i e^{\tau_1 (x - x_i)} + d_i e^{\tau_2 (x - x_i)}, \]

where \( a_i, b_i, c_i, \) and \( d_i \) are coefficients and \( \tau_1 \) and \( \tau_2 \) are parameters with \( \tau_1 \neq \tau_2 \).

**Definition 1.** The following function space

\[ \mathcal{E}^r_d(\Delta_n) = \{ s: s(x) = s_i(x), x \in I_i, i = 1, 2, \ldots, ns(x) \in C^r[a,b], r < 3 \}, \]

is called the cubic \( r \)-order exponential spline function space. Obviously, the function \( s(x) \) in \( \mathcal{E}^r_d(\Delta_n) \) must meet

\[ s_i^{(r)}(x_i) = s_i^{(r)}(x_i), \quad r = 0, 1, 2; \quad i = 1, 2, \ldots, n - 1. \]

The dimension of \( \mathcal{E}^r_d(\Delta_n) \) is \( n + 3 \). Then, we find a set of basic functions with local supports for \( \mathcal{E}^r_d(\Delta_n) \).

Assume \( s(x) \in \mathcal{E}^r_d(\Delta_n) \), for given \( j, 2 \leq j \leq n - 2 \); let

\[
\begin{align*}
    s_{j-1}^{(r)}(x_{j-2}) &= 0, & r &= 0, 1, 2, \\
    s_{j+2}^{(r)}(x_{j+2}) &= 0, & r &= 0, 1, 2, \\
    s_{j+k}^{(r)}(x_{j+k}) &= s_{j+k+1}^{(r)}(x_{j+k}), & k &= -1, 0, 1, r = 0, 1, 2, \\
    s_j(x_j) &= 1. 
\end{align*}
\]

We can obtain

\[
\begin{align*}
    a_{i-1} &= \frac{[A_i]}{[A]}, & b_{i-1} &= \frac{[A_2]}{[A]}, & c_{i-1} &= \frac{[A_3]}{[A]}, & d_{i-1} &= \frac{[A_4]}{[A]}, \\
    a_i &= \frac{[A_5]}{[A]}, & b_i &= \frac{[A_6]}{[A]}, & c_i &= \frac{[A_7]}{[A]}, & d_i &= \frac{[A_8]}{[A]}, \\
    a_{i+1} &= \frac{[A_9]}{[A]}, & b_{i+1} &= \frac{[A_{10}]}{[A]}, & c_{i+1} &= \frac{[A_{11}]}{[A]}, & d_{i+1} &= \frac{[A_{12}]}{[A]}, \\
    a_{i+2} &= \frac{[A_{13}]}{[A]}, & b_{i+2} &= \frac{[A_{14}]}{[A]}, & c_{i+2} &= \frac{[A_{15}]}{[A]}, & d_{i+2} &= \frac{[A_{16}]}{[A]}, \\
\end{align*}
\]

where
\[
A = [A_1', A_2'],
\]

\[
A_1' =
\begin{bmatrix}
1 & -h_{i-1} & e^{-\theta_{i-1}} & e^{-\eta_{i-1}} & 0 & 0 & 0 & 0 \\
0 & 1 & \tau_1 e^{-\theta_{i-1}} & \tau_2 e^{-\eta_{i-1}} & 0 & 0 & 0 & 0 \\
0 & 0 & \tau_1^2 e^{-\theta_{i-1}} & \tau_2^2 e^{-\eta_{i-1}} & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & -1 & h_i & -e^{-\theta_i} & -e^{-\eta_i} \\
0 & 1 & \tau_1 & \tau_2 & 0 & -1 & -\tau_1 e^{-\theta_i} & -\tau_2 e^{-\eta_i} \\
0 & 0 & \tau_1^2 & \tau_2^2 & 0 & 0 & -\tau_1^2 e^{-\theta_i} & -\tau_2^2 e^{-\eta_i} \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \tau_1 & \tau_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
A_2' =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & h_{i+1} & -e^{-\theta_{i+1}} & -e^{-\eta_{i+1}} & 0 & 0 & 0 & 0 \\
0 & 1 & -\tau_1 e^{-\theta_{i+1}} & -\tau_2 e^{-\eta_{i+1}} & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau_1^2 e^{-\theta_{i+1}} & -\tau_2^2 e^{-\eta_{i+1}} & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & -1 & h_{i+2} & -e^{-\theta_{i+2}} & -e^{-\eta_{i+2}} \\
0 & 1 & \tau_1 & \tau_2 & 0 & -1 & -\tau_1 e^{-\theta_{i+2}} & -\tau_2 e^{-\eta_{i+2}} \\
0 & 0 & \tau_1^2 & \tau_2^2 & 0 & 0 & -\tau_1^2 e^{-\theta_{i+2}} & -\tau_2^2 e^{-\eta_{i+2}} \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \tau_1 & \tau_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[A_k(k = 1, 2, \ldots, 16)\] is the matrix obtained by replacing the \(k\) column of \(A\) with \([0, \ldots, 0, 1]^T\), \(\theta_i = \tau_1 h_i\), and \(\eta_i = \tau_2 h_i\).

For even splitting, i.e., \(h_i = h, i = 1, 2, \ldots, n\), and \(\tau_2 = -\tau_1 = \tau \neq 0\), the results obtained are 

\[\tau_1 = \tau_2 h_i.\]
\[
a_{j-1} = \frac{\omega \theta}{\sigma}, \\
b_{j-1} = \frac{\omega \theta}{h \sigma}, \\
c_{j-1} = \frac{1}{2} \frac{\omega (-\cosh(\theta) + \sinh(\theta))}{\sigma}, \\
d_{j-1} = \frac{1}{2} \frac{\omega (\cosh(\theta) + \sinh(\theta))}{\sigma}, \\
a_j = \frac{\theta (-4 \sinh(3\theta) + \sinh(4\theta) - 4 \sinh(\theta) + 6 \sinh(2\theta))}{\sigma}, \\
b_j = \frac{\theta (-3 \sinh(3\theta) + 2 \sinh(2\theta) + \sinh(4\theta) + \sinh(\theta))}{h \sigma}, \\
c_j = \frac{1}{2\sigma} (-5 + 4 \cosh(2\theta) - 2 \sinh(2\theta) - 4 \cosh(3\theta) + 3 \sinh(3\theta) + \cosh(4\theta) - \sinh(4\theta) + 4 \cosh(\theta) - \sinh(\theta)), \\
d_j = \frac{1}{2\sigma} (\cosh(4\theta) + \sinh(4\theta) - 5 - 4 \cosh(3\theta) - 3 \sinh(3\theta) + 4 \cosh(\theta) + \sinh(\theta) + 4 \cosh(2\theta) + 2 \sinh(2\theta)), \\
a_{j+1} = \frac{-\omega \theta}{\sigma}, \\
b_{j+1} = \frac{-\theta (-3 \sinh(3\theta) + 2 \sinh(2\theta) + \sinh(4\theta) + \sinh(\theta))}{h \sigma}, \\
c_{j+1} = \frac{1}{4\sigma} (-5 + 4 \cosh(2\theta) + 10 \sinh(2\theta) - 4 \cosh(3\theta) + \cosh(4\theta) - \sinh(4\theta) + 4 \cosh(\theta) - 16 \sinh(\theta)), \\
d_{j+1} = \frac{1}{4\sigma} (\cosh(4\theta) + \sinh(4\theta) - 5 - 4 \cosh(3\theta) + 4 \cosh(\theta) + 16 \sinh(\theta) + 4 \cosh(2\theta) - 10 \sinh(2\theta)), \\
a_{j+2} = 0, \\
b_{j+2} = \frac{\omega \theta}{h \sigma}, \\
c_{j+2} = \frac{\omega}{2\sigma}, \\
d_{j+2} = \frac{-\omega}{2\sigma},
\]

where

\[
\omega = 5 \sinh(\theta) - 4 \sinh(2\theta) + \sinh(3\theta), \\
\sigma = 5 + \theta \sinh(4\theta) - 4 \theta \sinh(\theta) + 6 \theta \sinh(2\theta) - 4 \theta \sinh(3\theta) \\
\quad + 4 \cosh(3\theta) - 4 \cosh(\theta) - \cosh(4\theta) - 4 \cosh(2\theta), \\
\theta = \tau h.
\]

Define a function

\[
B_j(x) = \begin{cases} 
    s_{j-1}(x), & x \in I_{j-1}, \\
    s_j(x), & x \in I_j, \\
    s_{j+1}(x), & x \in I_{j+1}, \\
    s_{j+2}(x), & x \in I_{j+2}, \\
    0, & \text{otherwise}, 
\end{cases} 
\]

where the coefficients of \( s_{j-1}(x), s_j(x), s_{j+1}(x), \) and \( s_{j+2}(x) \) are given by the solution of function (9). Besides, for
j = −1, 0, 1, n − 1, n, n + 1, an interval expansion will be conducted, i.e., \( (x_0, x_1, \ldots, x_n) \) will be extended to be 
\( (x_{-1}, x_{-2}, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}) \). (15)

Thus, \( j \) in (11) can take the values of \( −1, 0, 1, \ldots, n, n + 1 \). Of course, the function domain is still \( [x_0, x_n] \).

For the space basis of \( \mathcal{E}^2_3(\Delta_n) \), we have the following proposition.

**Proposition 1.** When \( \tau_1 \neq \tau_2 \), the function set of \( \left\{ B_j(x) \right\}_{j=1}^{n+1} \) is a set of basis of \( \mathcal{E}^2_3(\Delta_n) \).

For special cases, we can prove the following.

**Theorem 1.** When \( \tau_2 = -\tau_1 = \tau \neq 0 \) and \( h_i = h, i = -2, -1, \ldots, n, n + 1 \), the function set of \( \left\{ B_j(x) \right\}_{j=1}^{n+1} \) is a set of basis of \( \mathcal{E}^2_3(\Delta_n) \).

**Proof.** Obviously, \( B_j(x) \in \mathcal{E}^2_3(\Delta_n) \), \( j = -1, 0, 1, \ldots, n, n + 1 \), and the number of functions equals the dimension of \( \mathcal{E}^2_3(\Delta_n) \). Therefore, we need to show that \( \left\{ B_j(x) \right\}_{j=1}^{n+1} \) is linearly independent.

For \( x = x_i, i = -1, 0, \ldots, n, n + 1 \), let \( B = (\beta_{ij}) = (B_j(x_i))_{(n+3) \times (n+3)} \) if matrix \( B \) is invertible, the basis \( \left\{ B_j(x) \right\}_{j=1}^{n+1} \) is linearly independent. It is easy to know that \( B \) is a tridiagonal matrix, and

\[
\begin{align*}
b_{i,j-1} &= a_{i-1} + c_{i-1} + d_{i-1}, \\
b_{i,i} &= a_i + c_i + d_i = 1, \\
b_{i+1,j} &= a_{i+1} + c_{i+1} + d_{i+1},
\end{align*}
\]

where \( i = -1, 0, \ldots, n + 1 \). Because
\[
\begin{align*}
|b_{i-1,j}| + |b_{i+1,j}| &= |a_{i-1} + c_{i-1} + d_{i-1}| + |a_{i+1} + c_{i+1} + d_{i+1}| \\
&= \frac{8}{\sigma}(\cosh(\theta) - 1)^2(1 - \cosh(\theta))^2 + \theta \sinh(\theta)) \\
&= \frac{1 - \cosh(\theta)^2 + \theta \sinh(\theta)\cosh(\theta)}{1 - \cosh(\theta)^2 + \theta \sinh(\theta)\cosh(\theta)} \\
&\leq \frac{1}{2} < |b_i|,
\end{align*}
\]

(17)

\( B \) is a strictly tridiagonal matrix. Thus, \( B \) is invertible.

This proves that \( \left\{ B_j(x) \right\}_{j=1}^{n+1} \) is linearly independent.

When \( h_i = h, i = -2, -1, \ldots, n, n + 3, \left\{ B_j(x) \right\}_{j=1}^{n+1} \) has the following properties.

**Proposition 2.** For any \( x \in [a, b] \)
\[
\sum_{j=1}^{n+1} B_j(x) = C,
\]

(18)

where \( C \) is not related to \( x \), and

\[
C = \frac{\tau_1 \tau_2 h (\tau_2 - \tau_1) (e^{(-\tau_1 h)} + e^{(-\tau_2 h)} - e^{(-h(\tau_1 + \tau_2))} - 1)}{(r_2^2 - r_1^2) e^{-h(\tau_1 + \tau_2)} + (r_1^2 + r_2^2 + r_1 r_2 h - r_1^2 r_2 h) e^{(-\tau_1 h)} - (r_1^2 + r_2^2 - r_1 r_2 h + r_1^2 r_2 h) e^{(-\tau_2 h)} + r_1^2 - r_2^2}.
\]

(19)
The basis function $B_j(x)$ has a local supporting set of $[x_{i-2}, x_{i+2}]$. Figure 1 shows all functions in the function set of $\{B_j(x)\}_{j=1}^{n+1}$ at the same coordinate.

3. Properties of Exponential Spline Functions

The relationship between the function value and the first derivative and that between the function value and the second derivative are commonly used in numerically solving DEs. Next, we will derive some properties of the functions in $\mathcal{C}^2_1(\Delta_n)$, and these relationships will be used in numerically calculating DEs.

Let $s(x) \in \mathcal{C}^2_1(\Delta_n)$, then,

$$s(x) = s_i(x), \quad x \in I_i, \ i = 1, 2, \ldots, n.$$  \hspace{1cm} (20)

Denote

$$S_i = s(x_i), \quad D_i = s'(x_i), \quad M_i = s''(x_i), \ i = 1, 2, \ldots, n,$$

since $s_i(x_i) = s_{i-1}(x_i), s''_i(x_i) = s''_{i-1}(x_i)$, the factors of $a_i, b_i, c_i,$ and $d_i$ can be written as

$$a_i = \frac{1}{r_1^2 r_2 \rho_i} \left( (r_1^2 - r_2) M_{i-1} + (r_2^2 e^{-r_1 h_i} - r_1^2 e^{-r_2 h_i}) M_i + r_1^2 r_2 \rho_i S_i \right),$$

$$b_i = \frac{1}{h_i r_1^2 r_2 \rho_i} \left( (r_2^2 e^{-r_1 h_i} - r_1^2 e^{-r_2 h_i} + r_1^2 - r_2^2) M_{i-1} \right.$$

$$\left. + (r_1^2 e^{-r_1 h_i - r_2 h_i} - r_2^2 e^{-r_1 h_i - r_2 h_i} + r_1^2 e^{-r_2 h_i} + r_2^2 e^{-r_1 h_i}) M_i \right) + r_1^2 r_2 \rho_i S_i + \tau_2^2 e^{-r_1 h_i} M_{i-1} - r_1^2 e^{-r_2 h_i} S_{i-1},$$

$$c_i = \frac{1}{r_2 \rho_i} \left( M_{i-1} - e^{-r_1 h_i} M_i \right),$$

$$d_i = -\frac{1}{r_2 \rho_i} \left( M_{i-1} - e^{-r_2 h_i} M_i \right),$$

where $\rho_i = e^{-r_1 h_i} - e^{-r_2 h_i}, i = 1, 2, \ldots, n$ (the same applies hereinafter). Using the continuous condition of the first derivative

$$s'_i(x_i) = s'_{i+1}(x_i), \quad i = 1, 2, \ldots, n-1,$$  \hspace{1cm} (23)

we obtain

$$a_{ii} = -\frac{h_i r_1^2 r_2 - h_i r_1^2 r_2 - r_1^2 e^{(-r_1 h_i)} + r_2^2 e^{(-r_1 h_i)} + r_1^2 - r_2^2}{r_1^2 r_2 \rho_i h_i},$$

$$a_{ii} = -\frac{1}{r_1^2 r_2 \rho_i h_i} \left( h_{i+1} r_1^2 - h_{i+1} r_2^2 \right) e^{(-r_1 h_i - r_2 h_i - r_1 h_{i+1})} + \left( h_{i+1} r_1^2 - h_{i+1} r_1^2 \right) e^{(-r_1 h_i - r_2 h_i - r_1 h_{i+1})}$$

$$+ (h_1 r_1^2 + h_1 r_1^2) e^{(-r_1 h_i)} + \left( h_{i+1} r_1^2 + h_{i+1} r_1^2 \right) e^{(-r_1 h_i - r_1 h_{i+1})}$$

$$+ (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i)} + (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i - r_1 h_{i+1})}$$

$$+ (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i)} + (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i - r_1 h_{i+1})}$$

$$+ (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i)} + (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i - r_1 h_{i+1})}$$

$$+ (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i)} + (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i - r_1 h_{i+1})}$$

$$\left. + (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i)} + (h_{i+1} r_1^2 + h_{i+1} r_1^2) e^{(-r_1 h_i - r_1 h_{i+1})} \right),$$

$$a_{ii} = \frac{r_2^2 e^{-r_1 h_i} - r_1^2 e^{-r_1 h_i} + (h_{i+1} r_1^2 + h_{i+1} r_1^2 + r_1^2 - r_2^2) e^{-h_i (r_1 + r_2)}}{r_1^2 r_2 \rho_i h_i},$$

If $h_i = h > 0, i = 1, 2, \ldots, n$ and $r_2 = -r_1 = r$, then the corresponding coefficients become
\[\alpha_{i1} = \frac{(e^{2x} - 2\tau e^{x} - 1)}{h^2(e^{2x} - 1)},\]

\[\alpha_{i2} = \frac{2(-e^{2x} + \tau e^{2x} + 1 + \tau h)}{h^2(e^{2x} - 1)},\]

\[\alpha_{i3} = \frac{(e^{2x} - 2\tau h e^{x} - 1)}{h^2(e^{2x} - 1)},\]

\[\alpha_{i4} = -\frac{1}{h},\]

\[\alpha_{i5} = \frac{2}{h},\]

\[\alpha_{i6} = -\frac{1}{h},\]

\[\beta_{i1} = \frac{1}{K_i K_{i+1}} \left[ (\tau_1 - \tau_2)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] (\tau_1 - \tau_2)^2 (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \]

\[\beta_{i2} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-1 + \tau_1 \tau_2 h_i + \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

Let \( \tau \to 0 \); then, (24) is equivalent to

\[h^2(M_{i+1} + 4M_i + M_{i+1}) - 6S_{i+1} + 12S_i - 6S_{i+1} = 0. \]  

In a similar way, we can obtain the relationship between the function value and the first derivative:

\[\beta_{i1}D_{i+1} + \beta_{i2}D_i + \beta_{i3}D_{i+1} + \beta_{i4}S_{i+1} + \beta_{i5}S_i + \beta_{i6}S_{i+1} = 0, \]

where

\[\beta_{i1} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] (\tau_1 - \tau_2)^2 (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \]

\[\beta_{i2} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-1 + \tau_1 \tau_2 h_i + \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i3} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i4} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i5} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i6} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i7} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i8} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i9} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i10} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]

\[\beta_{i11} = \frac{1}{K_i K_{i+1}} \left[ \left( \tau_1 - \tau_2 \right)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{(-h_i(\tau_1 + \tau_2))} - \tau_2^2 (\tau_1 h_i + 1) (\tau_1 - \tau_2) e^{(-h_{i+1}(\tau_1 + \tau_2))} \right] \]
\[
\beta_{3i} = \frac{1}{K_i K_{i+1}} \left[ (\tau_1 - \tau_2)^2 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} + \tau_2^2 (\tau_2 - \tau_1) e^{(-h_{i+1} (\tau_2 - \tau_1))} + \tau_1^2 (\tau_1 - \tau_2) e^{(-h_{i+1} (\tau_1 - \tau_2))} + \tau_1^2 (\tau_2 + \tau_1) e^{(-h_{i+1} (\tau_2 + \tau_1))} \right] \\
- \tau_2^2 (\tau_2 - \tau_1) e^{(-h_{i+1} (\tau_2 - \tau_1))} + \tau_2 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_2 + \tau_1))} + \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} \\
- (\tau_1 + \tau_2) (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} - \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} + \tau_2 (\tau_2 + \tau_1) e^{(-h_{i+1} (\tau_2 + \tau_1))} + \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} \\
- (\tau_1^2 (\tau_1 - \tau_2) e^{(-h_{i+1} (\tau_2 - \tau_1))} + \tau_2 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_2 + \tau_1))} + \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} + \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} \\
+ \tau_2 (\tau_2 + \tau_1) e^{(-h_{i+1} (\tau_2 + \tau_1))} + \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} \\
\right] \\
(31)
\]

\[
\beta_{4i} = \frac{\tau_1 \tau_2}{K_i K_{i+1}} \left[ (\tau_1 - \tau_2)^2 e^{(-h_{i+1} (\tau_1 + \tau_2))} + \tau_2 (\tau_1 - \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} \right] \\
\]

\[
\beta_{5i} = \frac{\tau_1 \tau_2}{K_i K_{i+1}} \left[ (\tau_1 - \tau_2)^2 e^{(-h_{i+1} (\tau_1 + \tau_2))} + \tau_1 (\tau_1 + \tau_2) e^{(-h_{i+1} (\tau_1 + \tau_2))} \right] \\
\]

\[
K_i = (-\tau_1 + \tau_1 \tau_2 h_i + \tau_2) e^{(-h_i)} + (-\tau_1 - \tau_1 \tau_2 h_i + \tau_2) e^{(-h_i (\tau_i + \tau_2))} + \tau_1 - \tau_2. \quad (34)
\]

If \( \tau_2 = -\tau_1 = \tau \) and \( h_i = h, i = 1, 2, \ldots, n \), then

\[
\beta_{1i} = \frac{(-2r \tau e^{(\tau h)} + e^{(2\tau h)} - 1) \tau}{4 e^{(\tau h)} - 2 - 2 e^{(2\tau h)} + \tau (e^{(2\tau h)} - \tau h)} \\
\beta_{2i} = \frac{2 (\tau h e^{(2\tau h)} - e^{(2\tau h)} + \tau h + 1) \tau}{4 e^{(\tau h)} - 2 - 2 e^{(2\tau h)} + \tau (e^{(2\tau h)} - \tau h)} \\
\beta_{3i} = \frac{(-2r \tau e^{(\tau h)} + e^{(2\tau h)} - 1) \tau}{4 e^{(\tau h)} - 2 - 2 e^{(2\tau h)} + \tau (e^{(2\tau h)} - \tau h)} \\
\beta_{4i} = \frac{2 (\tau h e^{(2\tau h)} - e^{(2\tau h)} + \tau h + 1) \tau}{4 e^{(\tau h)} - 2 - 2 e^{(2\tau h)} + \tau (e^{(2\tau h)} - \tau h)} \\
\beta_{5i} = 0, \\
\beta_{6i} = \frac{(-1 + e^{(\tau h)}) \tau^2}{-2 e^{(\tau h)} + \tau e^{(\tau h)} h + 2 + \tau h} \\
\]

Let \( \tau \to 0 \); then, (28) is equivalent to

\[
h(D_{i+1} + 4D_i + D_{i-1}) + 3S_{i-1} - 3S_{i+1} = 0. \quad (36)
\]

This is consistent with the cubic 2nd-order polynomial spline function relationship.

Besides, using

\[
s_i(x_i) = S_i, \quad \dot{s}_i(x_i) = D_i, \quad S'_{i+1}(x_i) = M_{i+1}, \quad \dot{S}_{i+1}(x_i) = M_{i+1}, \quad (37)
\]

we can obtain

\[
a_i = \frac{(\tau_1^2 - \tau_2^2) M_{i+1} + (\tau_2^2 - \tau_1^2) e^{(-\tau_i h_i)} M_i + \tau_1^2 \tau_2 \tau_1^2 \rho_i S_i}{\tau_1 \tau_2 \rho_i}, \\
b_i = \frac{(\tau_1 - \tau_2) M_{i+1} + (\tau_2 e^{(-\tau_i h_i)} - \tau_1 e^{(-\tau_i h_i)}) M_i + \tau_1 \tau_2 \rho_i D_i}{\tau_1 \tau_2 \rho_i}, \\
c_i = -\frac{M_{i+1} + e^{(-\tau_i h_i)} M_i}{\tau_1 \tau_2 \rho_i}, \\
d_i = \frac{M_{i+1} + e^{(-\tau_i h_i)} M_i}{\tau_1 \tau_2 \rho_i}. \quad (38)
\]

Then, using continuous condition of \( s_i(x_i) = s_{i+1}(x_i), \dot{s}_i(x_i) = s'_{i+1}(x_i) \), we obtain
\[- (h_{i+1} r_1^2 - h_i r_1 r_2 + r_1^2 e^{-\tau_i h_i} + r_2^2 e^{-\tau_i h_i} + r_1^2 - r_2^2) M_i \]
\[+ \left( \frac{r_1^2 e^{-\tau_i h_i} - r_2^2 e^{-\tau_i h_i} + h_{i+1}(r_1^2 e^{-\tau_i h_i} + r_2^2 e^{-\tau_i h_i} (\tau + \tau_i))}{r_1^2 \tau_2 \rho_{i+1}} \right) \]
\[+ S_i - S_{i+1} + h_{i+1} D_{i+1} = 0, \]
\[\text{(39)}\]

Solving (39) yields

\[D_i = \left( \frac{- (r_1^2 e^{-\tau_i h_i} - r_2^2 e^{-\tau_i h_i} - h_i r_1 r_2 + h_i r_1 r_2 - r_1^2 + r_2^2) M_{i-1}}{h_i r_1 r_2 \rho_i} \right) \]
\[+ \left( \frac{(-h_i r_1 r_2^2 + h_i r_1 r_2^2) e^{-\tau_i h_i} + (-r_1^2 + h_i r_1 r_2^2) e^{-\tau_i h_i} + (r_1^2 - r_2^2) e^{-\tau_i h_i (\tau + \tau_i)})}{h_i r_1^2 \tau_2 \rho_i} \right) \]
\[- \frac{S_{i-1}}{h_i} + \frac{S_i}{h_i} \]
\[\text{(41)}\]

Meanwhile, by eliminating \(D_{i+1}\) with (39) and (40) and rearranging, we obtain

\[D_i = \left( \frac{(-r_1^2 + h_{i+1} r_1^2 r_2 e^{-\tau_i h_i} + (r_1^2 + h_{i+1} r_1^2 r_2^2) e^{-\tau_i h_i} - r_2^2 + r_1^2) M_i}{h_{i+1} r_1^2 r_2 \rho_{i+1}} \right) \]
\[- \frac{(-r_1^2 + h_{i+1} r_1^2 r_1^2 - h_{i+1} r_1^2 r_2^2 + r_2^2) e^{-\tau_i h_i} (\tau + \tau_i) + (r_1^2 - r_2^2) e^{-\tau_i h_i (\tau + \tau_i)})}{h_{i+1} r_1^2 r_2 \rho_{i+1}} \]
\[+ \frac{S_i}{h_{i+1}} + \frac{S_{i+1}}{h_{i+1}} \]
\[\text{(42)}\]

We can also use \(s_{i+1}(x_{i+1}) = S_{i+1}, s_i(x_i) = S_i, s_{i+1}(x_{i+1}) = D_{i+1}\), and \(s_{i+1}(x_i) = D_i\) to obtain
\[ a_i = \frac{1}{\tau_i} \left( (\tau_i h_i - \tau_2 h_i + \rho_i) D_i - 1 + \left(h_i, 2e^{-\tau_i h_i} - \tau_i e^{-\tau_i h_i} h_i - \rho_i \right) D_i \right) \\
+ \left( \tau_i e^{-\tau_i h_i} + \tau_1 - \tau_2 e^{-\tau_i h_i} \right) S_{i-1} \\
+ \left( -\tau_i - \tau_i h_i + \tau_2 e^{-\tau_i h_i} \right) S_{i-1} \right), \]

\[ b_i = \frac{1}{\tau_i} \left( (\tau_i - \tau_2 - \tau_i e^{-\tau_i h_i} + 2e^{-\tau_i h_i}) D_i - 1 + \left(\tau_i e^{-\tau_i h_i} - \tau_1 e^{-\tau_i h_i} + (\tau_1 - \tau_2) e^{-\tau_i h_i} \right) D_i \right) \\
- \tau_1 \tau_2 \rho_i S_{i-1} + \tau_1 \tau_2 \rho_i S_i, \]

\[ c_i = \frac{1}{\tau_i} \left( (e^{-\tau_i h_i} - 1 + \tau_2 h_i) D_i - 1 + (e^{-\tau_i h_i} + 1 - h_i, 2e^{-\tau_i h_i}) D_i \right) \\
+ \left( -\tau_2 e^{-\tau_i h_i} + \tau_3 S_{i-1} + (\tau_2 + \tau_3 e^{-\tau_i h_i}) S_i \right), \]

\[ d_i = \frac{1}{\tau_i} \left( (\tau_i h_i - e^{-(\tau_i h_i)} + 1) D_i - 1 + \left(\tau_i e^{-\tau_i h_i} h_i + e^{-\tau_i h_i} - 1 \right) D_i \right) \\
+ \left( -\tau_1 + \tau_1 e^{-\tau_i h_i} + \tau_1 \right) S_{i-1} + \left( -\tau_1 e^{-\tau_i h_i} + \tau_1 \right) S_i, \]

where \[ \tau_i = (-\tau_1 h_i, 2 + \tau_2 - \tau_1) e^{-\tau_i h_i} + (\tau_1 h_i, 2 + \tau_2 - \tau_1) e^{-\tau_i h_i} + (\tau_1 h_i, 2 + \tau_2 - \tau_1) e^{-\tau_i h_i} + \tau_1 - \tau_2. \]

Fast Hermite interpolation can be achieved by using this set of relations.

### 4. Exponential Spline Difference Method

The spline difference method uses the relationship between the spline function and its derivative to construct the differential expression to numerically solve ODEs, by which the numerical solution at nodes can be obtained, and that within the subintervals can also be calculated by using the spline function expressions. It is the advantage of this method compared with the general difference schemes. In fact, it can be said that the approximate analytical solution using the splines is obtained.

\[ t_i = (a_{1i} u''_{i-1} + a_{2i} u''_{i} + a_{3i} u''_{i+1}) - u_{i-1} \frac{h_i}{h_i + h_{i+1}} + \frac{(h_i + h_{i+1}) u_i}{h_i h_{i+1}} - u_{i+1} \frac{h_{i+1}}{h_i h_{i+1}}, \]

where \( i = 1, \ldots, n - 1, u_i = u(x_i) \) and \( u_i'' = u''(x_i) \). \( t_i \) is the local truncation error at \( x_i \). By substituting (46) into (47) and rearranging, we obtain

\[ \left( \frac{1}{h_i} - a_{1i} \rho_i \right) u_{i-1} + \left( h_i + h_{i+1} \right) u_i + \frac{h_i + h_{i+1}}{h_i h_{i+1}} - a_{2i} \rho_i \right) u_i \\
+ \left( \frac{1}{h_{i+1}} - a_{3i} \rho_{i+1} \right) u_{i+1} + a_{1i} g_{i-1} + a_{2i} g_i + a_{3i} g_{i+1} = t_i, \]

where \( i = 1, 2, \ldots, n - 1 \). Thus, we get \( n - 1 \) equations about \( u_0, u_1, \ldots, u_n \).

#### 4.1. Differential Expression

The following presents a spline difference method for solving (1) which satisfies one of the boundary conditions (2)–(5) for the boundary value problems of ODEs. Due to the limitation of this method, we only consider the case of \( q(x) = 0 \) in this section. For convenience of description, we first consider the boundary condition (2). From (1), we can obtain

\[ u'' = g(x) - p(x)u, \quad x \in [a, b]. \]

By discretization the above equation, we obtain

\[ u_i'' = g_i - p_i u_i, \quad i = 0, 1, 2, \ldots, n, \]

where \( g_i = g(x_i) \) and \( p_i = p(x_i) \).

Substitute \( S \) with \( u \) and \( M \) with \( u'' \) in (24), and we obtain

\[ \text{The following equation can also be derived from the boundary conditions (2):} \]

\[ u_0 = \mu_0, \]

\[ u_n = \mu_1. \]

So, there will be \( n + 1 \) equations in (48) and (49) in total, and it matrix form can be written as

\[ \mathbf{A} \mathbf{u} = \mathbf{F} + \mathbf{T}, \]

where \( \mathbf{A} = \mathbf{B} - \mathbf{W} \), with
For these second boundary conditions, from (41), (42), equation (47) at $x_i$, we obtain

\[ U_i = \frac{S_0}{h_i} - \frac{S_1}{h_i} = -\mu_0, \]

(54)

\[ U_i = \frac{S_{i-1}}{h_{i-1}} + \frac{S_n}{h_n} = \mu_1, \]

(55)

where

\[
\begin{align*}
\alpha_{20} &= \frac{((r_1^2 + h_1 r_1^2 r_2) e^{-r_1 h_1} + (-r_2^2 - r_1 h_1 r_2^2) e^{-r_2 h_2} - r_1^2 + r_2^2)}{h_1 r_1^2 r_2^2}, \\
\alpha_{30} &= \frac{(r_2^2 e^{-r_2 h_2} - r_1^2 e^{-r_1 h_1} + (r_1^2 - r_2 h_2 r_1^2 + h_1 r_1^2 r_2^2 - r_2^2) e^{-r_2 h_2} (r_2 + r_1))}{h_1 r_1^2 r_2^2}, \\
\alpha_{1n} &= \frac{(r_1^2 e^{-r_1 h_1} - r_2^2 e^{-r_2 h_2} - h_n r_1 r_2^2 + h_n r_1^2 r_2 - r_1^2 + r_2^2)}{h_n r_1^2 r_2^2}, \\
\alpha_{2n} &= \frac{(-h_n r_1 r_2^2 + r_2^2) e^{-r_2 h_2} + (-r_1^2 + h_n r_1^2 r_2) e^{-r_1 h_1} + (r_1^2 - r_2^2) e^{-h_2} (r_2 + r_1))}{h_n r_1^2 r_2^2}.
\end{align*}
\]  

Solving this type of boundary problem only needs to modify the first line and the last row of $B$, $W$, and $F$ in (47), namely,

\[
B_0 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{h_1} & \frac{1}{h_1} & 0 \end{bmatrix},
\]

\[
W_0 = \begin{bmatrix} \alpha_{20} p_0 & \alpha_{30} p_1 & 0 \ldots 0 \end{bmatrix},
\]

\[
B_n = \begin{bmatrix} \ldots & 0 & 0 \end{bmatrix},
\]

\[
W_n = \begin{bmatrix} \ldots & 0 & \alpha_{n1} p_{n-1} & \alpha_{n2} p_n \end{bmatrix},
\]

\[
F_0 = -\mu_0 - (\alpha_{20} g_0 + \alpha_{30} g_1),
\]

\[
F_n = \mu_1 - (\alpha_{1n} g_{n-1} + \alpha_{2n} g_n).
\]  

And the third and the fourth boundary conditions should also be modified accordingly.

4.2 Error Estimation. Suppose $u(x)$ is sufficiently smooth in $[a,b]$, by Taylor expanding $u_{i-1}, u_{i+1}, u''_{i-1}, u''_{i+1}$ in difference equation (47) at $x_i$, we obtain

\[
t_i = \sum_{j=2}^{6} \xi_j u^{(j)}(x_i) + O(h_i^5),
\]

(58)

where
\( \xi_2 = \alpha_{ii} + \alpha_{i1} + \alpha_{i1} - \frac{1}{2} (h_i + h_{i+1}), \)
\( \xi_3 = -\alpha_{i1} h_i + \alpha_{i1} h_{i+1} + \frac{1}{6} (h_i^3 - h_{i+1}^3), \)
\( \xi_4 = \frac{1}{2} \alpha_{i1} h_i^2 + \frac{1}{2} \alpha_{i1} h_{i+1}^2 - \frac{1}{24} (h_i^4 + h_{i+1}^4), \)
\( \xi_5 = \frac{1}{6} \alpha_{i1} h_i^3 + \frac{1}{6} \alpha_{i1} h_{i+1}^3 + \frac{1}{120} (h_i^5 + h_{i+1}^5), \)
\( \xi_6 = \frac{1}{24} \alpha_{i1} h_i^4 + \frac{1}{24} \alpha_{i1} h_{i+1}^4 - \frac{1}{720} (h_i^6 + h_{i+1}^6). \)

(59)

Let \( \tau_1 \rightarrow 0 \) and \( \tau_2 \rightarrow 0 \), we find \( \xi_2 = \xi_3 = \xi_4 = 0 \) and
\( \xi_5 = \frac{1}{360} (h_i + h_{i+1}) (2h_i + h_{i+1}) (h_i + 2h_{i+1}) (h_i + h_{i+1}). \)

(61)

\( \xi_5 = 0 \) for an even splitting, i.e., \( h_i = h, i = 1, 2, \ldots, n \), and \( \xi_6 = (1/240) h^6 \). This indicates that the even splitting has higher accuracy if \( u(x) \) is sufficiently smooth.

4.3. Convergence Analysis. We mainly discuss the convergence of equation (47) and the differential expression of (49) in the sense of \( \|E\|_\infty \).

Lemma 1 (see [14]). If the \( n \)-order matrix \( B \) satisfies one of the following two conditions:

(1) \( B \) is a strictly diagonally dominant matrix

(2) \( B \) is a second half strong diagonally dominant matrix

Then, \( B \) is nonsingular and \( \rho (I - D^{-1} B) < 1 \), where \( D = \text{diag} (B) \) and \( I \) is the identity matrix.

By (14),
\[ \text{AS} = F. \]

(62)

If \( A \) is reversible, combining (50), we obtain
\[ E = A^{-1} T = (B - W)^{-1} T = (I - B^{-1} W)^{-1} B^{-1} T, \]

(63)

where \( E = (e_i) = U - S \), and \( B, W, \) and \( T \) are given by (51)–(53). The reversibility of \( B \) can be proved by 1. From the boundary condition (2), we can get \( e_0 = e_n = 0 \). Thus, discussing the convergence of \( \|E\|_\infty \) is consistent with that of \( \|\text{det}(e_{ik})\|_\infty \). So, in the following convergence discussion, the first and the last row and the first and the last column will be removed from the original matrices of \( A, B, \) and \( W \) to obtain the \( n - 1 \)-order ones, keeping the subscript value unchanged.

If
\[ \|B^{-1}\|_\infty \|W\|_\infty < 1, \]
then
\[ \|E\|_\infty \leq \frac{\|B^{-1}\|_\infty \|T\|_\infty}{1 - \|B^{-1}\|_\infty \|W\|_\infty} \leq \|B^{-1}\|_\infty \|T\|_\infty. \]

(65)

To calculate \( B^{-1} \), the following lemma is needed:

Lemma 2 (see [15]). Let the square matrix \( A \) be an \( n \)-order tridiagonal one, with the following expression:
\[ A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{bmatrix}, \]

(66)

and \( b_i c_i \neq 0, i = 1, 2, \ldots, n - 1, a_i \neq 0, i = 1, 2, \ldots, n; \) then, the expression of \( A^{-1} = \{a_{ij}(-1)\} \) can be written as

\[
\begin{align*}
a_{ij}(-1) &= \begin{cases}
(-1)^{i+j} \left( \prod_{k=i}^{j-1} b_k \right) \frac{\det(A[1, \ldots, i-1] \det(A[j+1, \ldots, n])}{\det(A)}, & i < j, \\
(-1)^{i+j} \left( \prod_{k=j}^{i-1} c_k \right) \frac{\det(A[1, \ldots, i-1] \det(A[j+1, \ldots, n])}{\det(A)}, & i \geq j,
\end{cases}
\end{align*}
\]

(67)

where \( A[i_1, i_2, \ldots, i_k] \) denotes the matrix consisting of the elements of \( i_1, i_2, \ldots, i_k \) rows crossed with the columns in \( A \). Specifically, \( A[1, \ldots, i-1] = 1 \) for \( i = 1 \).

Using (2), \( B^{-1} = (b_{ij}(-1)) \) in (65) can be written as (with \( \sum_{k=1}^{n} h_k = b - a \))
Table 1: The maximum error $\|E\|_\infty$ in solving (72) with the exponential spline difference method.

<table>
<thead>
<tr>
<th></th>
<th>$n = 32$</th>
<th>$n = 64$</th>
<th>$n = 128$</th>
<th>$n = 256$</th>
<th>$n = 512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [7]</td>
<td>1.84e-4</td>
<td>4.61e-5</td>
<td>1.15e-5</td>
<td>2.88e-6</td>
<td>7.21e-7</td>
</tr>
<tr>
<td>Our method ($\tau = \left[ n^{1/4}, -n^{1/4} \right]$)</td>
<td>8.11e-6</td>
<td>2.03e-6</td>
<td>5.09e-7</td>
<td>1.28e-7</td>
<td>4.25e-8</td>
</tr>
</tbody>
</table>

Figure 2: Results of the boundary value problem (72) for $\varepsilon = 0.1$, and $n = 5, 10, \text{and} 20$, respectively.

Table 2: The maximum error $\|E\|_\infty$ and root mean square error for problem (74).

<table>
<thead>
<tr>
<th></th>
<th>$n = 10$</th>
<th>$n = 20$</th>
<th>$n = 40$</th>
<th>$n = 80$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [11]</td>
<td>1.02e-3</td>
<td>2.96e-5</td>
<td>3.69e-6</td>
<td>5.11e-8</td>
</tr>
<tr>
<td>Our method ($\tau = \left[ 10.11, -0.012 \right]$)</td>
<td>8.57e-7</td>
<td>8.18e-7</td>
<td>8.03e-7</td>
<td>7.97e-7</td>
</tr>
</tbody>
</table>

Figure 3: Results of the boundary value problem (74) for $\varepsilon = 0.01$, and $n = 5, 10, \text{and} 20$, respectively.

Figure 4: Results of the boundary value problem (76) for $\varepsilon = 1/32$, and $n = 5, 10, \text{and} 20$, respectively.
then

\[ \|b_{ij}^{-1}\|_{\infty} \leq \frac{1}{4} n(b - a) \leq \frac{(b - a)^2}{4h_{\min}}. \]  

According to (27), we find \(|t_i| = O(h_i^2)|.

\[ \|B^{-1}\|_{\infty} \leq \frac{4}{(b - a)^2} h_{\min}^2. \]  

Then, \( \|E\|_{\infty} = O\left(\frac{h_{\min}^2}{h_{\max}}\right) \). \hfill (71)\]

**5. Example**

**Example 1.** Solve the following singular boundary value problem (see [7]):

\[
\begin{cases}
-\epsilon u'' + p(x)u = g(x), & 0 \leq x \leq 1, \\
u(0) = u(1) = 0.
\end{cases}
\]  

where \( p(x) = 1 + x(1 - x) \) and \( g(x) = 1 + x(1 - x) + \left[2\sqrt{\epsilon} - x - (1 - x)\right]e^{-x/\sqrt{\epsilon}} + [2\sqrt{\epsilon} - x^2(1 - x)]e^{-(1-x)/\sqrt{\epsilon}} \), and its analytical solution is

\[ u(x) = 1 + (x - 1)e^{-x/\sqrt{\epsilon}} - xe^{-(1-x)/\sqrt{\epsilon}}. \]  

Table 1 lists the results of the exponential spline difference method for \( \epsilon = 0.1 \) and that of the method proposed in [7].

Figure 2 shows the results for \( \epsilon = 0.1 \), and \( n = 5 \) and 10, respectively. To show the difference between the exact
solution and the numerical one, we deliberately use fewer split points and simply connect two adjacent solutions with straight lines. A smoother exponential spline function which is closer to the exact solution can be constructed using these obtained numerical solutions.

\[
\begin{align*}
\mathbf{u}(x) &= 1 + \frac{\left(e^{(1-\sqrt{(1+4)/\lambda})/2/\lambda} - 1\right) e^{\left(\eta(1+\sqrt{(1+4)/\lambda})/2/\lambda\right)x} + \left(1 - e^{(1+\sqrt{(1+4)/\lambda})/2/\lambda}\right) e^{\left(1-\sqrt{(1+4)/\lambda})/2/\lambda\right)x}}{\left((1 + \sqrt{(1+4)/\lambda})/2/\lambda\right) - (1 - \sqrt{(1+4)/\lambda})/2/\lambda}. 
\end{align*}
\]

The computational results are shown in Table 2 for \( \varepsilon = 0.01 \) and various values of \( n \) (Figure 3).

Example 3. Consider the convection-dominated equation (see [16–19]) Figure 4:

\[
\begin{align*}
\varepsilon\mathbf{u}'' + \mathbf{u} &= 0, \quad 0 < x < 1, \\
\mathbf{u}(0) &= 0, \mathbf{u}(1) = 1, \\
\mathbf{u}(x) &= \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})} \varepsilon \neq (n\pi)^{-2}. 
\end{align*}
\]

The computational results are shown in Table 3 for \( \varepsilon = 1/2 \) and various values of \( n \).

6. Conclusions

Exponential spline, which is a generalization of polynomial spline, is an ideal function approximation tool due to its excellent curve fitting ability. High accuracy can be achieved in solving second-order ODEs using the exponential spline scheme. The spline difference method is an ideal scheme because it can give not only the numerical results but also the spline function expressions by reusing these numerical results at the same time, whereas it is only suitable for solving certain types of equations and does not have generality. And the selection of the appropriate parameters is also needed for this method, but there is no better guideline for the selecting.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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References


