

Research Article

Convergence of Antiperiodic Boundary Value Problems for First-Order Integro-Differential Equations

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In this paper, we investigate the convergence of approximate solutions for a class of first-order integro-differential equations with antiperiodic boundary value conditions. By introducing the definitions of the coupled lower and upper solutions which are different from the former ones and establishing some new comparison principles, the results of the existence and uniqueness of solutions of the problem are given. Finally, we obtain the uniform and rapid convergence of the iterative sequences of approximate solutions via the coupled lower and upper solutions and quasilinearization method. In addition, an example is given to illustrate the feasibility of the method.

1. Introduction

In recent decades, the integro-differential equations have developed rapidly because the models described by various of integro-differential equations have appeared in a number of fields such as fluid dynamics, biology, economics, and control theory, for details and examples, we can refer to references [1–7] and cited therein. Meanwhile, the qualitative theory of integral differential equations creates a branch of nonlinear analysis. Boundary value problems for various first-order integral differential equations have been studied by several researchers, and there are some results on the existence of solutions and extremal solutions, the controllability problem controllability of integral boundary value conditions, and antiperiodic boundary value conditions, such as ordinary differential equations [8–12], difference equations [13, 14], fractional differential equations [15–20], impulsive differential equations [9, 14, 21, 22], integro-differential equations, and impulsive functional differential equations [18, 23–26].

However, we found that most of these known results concerned with the existence and uniformly convergence results of solutions and extremal solutions via the method of upper and lower solutions coupled with the monotone

iterative technique (see [27]). It is well known that the method of quasilinearization (QSL) provides a powerful tool for obtaining convergence of approximate solutions of nonlinear problems [28, 29]. The technique of upper and lower solutions coupled with the QSL have been applied successfully to obtain monotone sequences of approximate solutions converging uniformly and quadratically to the unique solution of integro-differential equations with antiperiodic boundary value conditions [30–32]. In terms of applications, it is important to pay attention to the high-order convergence of sequences of approximate solutions. The high-order convergence results of various differential equations can be found in [33–39].

In this paper, we consider the following first-order integro-differential equations with antiperiodic boundary value conditions (APBVP):

$$\begin{cases} x'(t) = f(t, x(t), (Tx)(t)), & t \in J, \\ x(0) = -x(T), \end{cases} \quad (1)$$

where $f \in C(J \times R^2, R)$, $J = [0, T]$, $(Tx)(t) = \int_0^t k(t, s)x(s)ds$, $k \in C(D, R_+)$, $k_0 = \sup\{k(t, s)\}$, $D = \{(t, s) \in J \times J : t \geq s\}$.

The aim of this paper is to investigate the convergence of approximate solutions of the problem. We give the particular definitions of the coupled lower and upper related solutions which are new and establish some new comparison principles in order to discuss the existence and uniqueness of the solutions. Then, by using the method of quasilinearization, we obtain the two monotone sequences of approximate solutions converging to the unique solution of the problem with rate of convergence of order k . Finally, we give an example to illustrate our main results.

2. Comparison Theorems

In this section, we begin with some comparison principles that will be useful in later discussions.

Lemma 1. *Assume that there exist constants $M > 0$, $N > 0$, and $\gamma \geq 0$, such that*

$$(M + k_0NT)T \leq 1. \quad (2)$$

If there exists a function $p \in C^1[J, R]$, such that

$$\begin{cases} p'(t) \leq -M(p(t) + \gamma) - N(Tp)(t), & t \in J, \\ p(0) \leq p(T), \end{cases} \quad (3)$$

then $p(t) \leq 0$ on J .

Proof. Suppose the conclusion is not true, and we consider the following two cases, where $p(0) \leq 0$ and $p(0) > 0$, respectively. \square

Case 1. When $p(0) \leq 0$, there exists a $t^* \in (0, T]$, such that $p(t^*) > 0$. Let $t_* \in [0, t^*]$, such that $p(t_*) = \inf p(t) = -b$, $b \geq 0$. By equation (3), we have

$$p'(t) \leq b[M + k_0TN]. \quad (4)$$

Integrating inequality (4) from t_* to t^* , we obtain

$$0 < p(t^*) \leq p(t_*) + b \int_{t_*}^{t^*} \{M + k_0TN\} dt. \quad (5)$$

Thus,

$$b < b \int_0^T \{M + k_0TN\} dt, \quad (6)$$

which contradicts (2), therefore $p(t) \leq 0$.

Case 2. When $p(0) > 0$, there are two cases: $p(t) > 0$ for $t \in J$ or there exist \bar{t}, \underline{t} , such that $p(\underline{t}) \leq 0$ and $p(\bar{t}) > 0$ for $\bar{t}, \underline{t} \in J$.

Case 3. When $p(t) > 0$, by equation (3), we have $p'(t) \leq 0$, which contradicts the condition of equation (3).

Case 4. If there exist \bar{t} and \underline{t} , such that $p(\underline{t}) \leq 0$ and $p(\bar{t}) > 0$, we have $q(\bar{t}_*) = \inf p(t) = -b$, where $\bar{t}_* \in (0, T)$, $b \geq 0$. Then, equation (4) holds. Integrating inequality (4) from \bar{t}_* to T , we have

$$0 < p(\bar{t}) \leq p(\bar{t}_*) + b \int_{\bar{t}_*}^T \{M + k_0TN\} dt, \quad (7)$$

which is also a contradiction. The proof of Lemma 1 is completed.

Next, consider the linear APBVP:

$$\begin{cases} x'(t) + M(x(t) + \gamma) + N(Tx)(t) = 0, & t \in J, \\ x(0) = -x(T). \end{cases} \quad (8)$$

Corollary 1. *Assume that $M > N > 0$ and $(M + k_0NT)T \leq 1$, then APBVP (8) has at most one solution.*

Proof. Let x_1, x_2 be any solution of APBVP (8), $x_1 \geq x_2$, and $y(t) = x_1(t) - x_2(t)$, then

$$\begin{cases} y'(t) + My(t) + N(Ty)(t) = 0, \\ y(0) = -y(T). \end{cases} \quad (9)$$

If $y(T) > 0$, then it follows from (9) that $y(0) < 0$. By Lemma 1, we have $y(t) \leq 0$, that is a contradiction. On the contrary, if $y(T) < 0$, we have $y(0) > 0$. By the proof of Lemma 1, we have $y(t) \leq 0$, that is also a contradiction. Therefore, we have $y(T) = y(0) = 0$. Furthermore, by Lemma 1, we have $y(t) \leq 0$, that is, $y(t) = 0$ for $t \in J$. The proof of Corollary 1 is completed.

Similar to the proof of Lemma 1, we have the following lemma. \square

Lemma 2. *Assume that there exist integrable functions $\phi_i(t) < 0$, $i = 1, 2$, such that*

$$\int_0^T \{\phi_1(t) + \phi_2(t)k_0T\} dt \geq -1. \quad (10)$$

If there exist functions $p_i \in C^1[J, R]$, $i = 1, 2$, such that

$$\begin{aligned} p_1'(t) &\leq \phi_1(t)(p_1(t) + \gamma_1) + \phi_2(t)(Tp_1)(t), & \text{for } t \in J, p_1(0) \leq p_2(T), \\ p_2'(t) &\leq \phi_1(t)(p_2(t) + \gamma_2) + \phi_2(t)(Tp_2)(t), & \text{for } t \in J, p_2(0) \leq p_1(T), \end{aligned} \quad (11)$$

where

$$\gamma_1 = \frac{p_1(T) - p_2(0)}{1 - e \int_0^T \phi_1(t) dt}, \quad (12)$$

$$\gamma_2 = \frac{p_2(T) - p_1(0)}{1 - e \int_0^T \phi_2(t) dt}.$$

Then, $p_1(t) \leq 0$ and $p_2(t) \leq 0$ on J .

Proof. We just prove that the case of $p_1(t) \leq 0$. Suppose that the conclusion is not true, we can consider the following two cases, where $p_1(0) \leq 0$ and $p_1(0) > 0$, respectively.

Case 5. When $p_1(0) \leq 0$, by the proof of Lemma 1, we have $\int_0^T \{\phi_1(t) + \phi_2(t)k_0T\}dt < -1$, which contradicts (10).

Case 6. When $p_1(0) > 0$, there are two cases: $p_1(t) > 0$ for $t \in J$ or there exist \bar{t} and \underline{t} , such that $p_1(\underline{t}) \leq 0$ and $p_1(\bar{t}) > 0$ for $\bar{t}, \underline{t} \in J$.

Case 7. When $p_1(t) > 0, t \in J$, if $p_2(0) \leq 0$, by the proof of Lemma 1, we have $p_2(t) \leq 0$, that implies $p_1(0) \leq p_2(T) \leq 0$, which is a contradiction.

If $p_2(0) > 0$, we have $p_2(T) \geq p_1(0) > 0$. Then, there are two cases: $p_2(t) > 0$ for $t \in J$ and there exist \tilde{t} and \hat{t} , such that $p_2(\tilde{t}) \leq 0$ and $p_2(\hat{t}) > 0$, respectively.

Case 8. When $p_2(t) > 0$ for all $t \in J$, we have $p_2'(t) < 0$; hence, $p_2(t)$ is decreasing. By $p(t) > 0$ and equation (11), imply $p_1'(t) < 0$; then, $p_1(t)$ is decreasing and $p_2(0) > p_2(T) \geq p_1(0) > p_1(T) \geq p_2(0)$, which is a contradiction.

Case 9. For another case, we have $p_2(\tilde{t}_*) = \inf p_2(t) = -b$, where $\tilde{t}_* \in (0, T), b \geq 0$. Equation (11) implies that

$$p_2'(t) \leq (-b)[\phi_1(t) + k_0T\phi_2(t)]. \tag{13}$$

Integrating inequality (13) from \tilde{t}_* to T , we have

$$0 < p_2(T) \leq p_2(\tilde{t}_*) - b \int_{\tilde{t}_*}^T \{\phi_1(t) + k_0T\phi_2(t)\}dt. \tag{14}$$

Thus,

$$b < -b \int_0^T \{\phi_1(t) + k_0T\phi_2(t)\}dt, \tag{15}$$

which is also a contradiction.

The proof of Case 4 is analogous to the proof of Lemma 1, and we omit its details here. This completes the proof of Lemma 2.

Remark 1. When $\gamma = 0$ and $\gamma_i = 0, i = 1, 2$, respectively, the conclusion of Lemmas 1 and 2 is also true.

3. Linear APBVP

In this section, we consider the linear APBVP:

$$\begin{cases} x'(t) + Mx(t) + N(Tx)(t) = \sigma(t), & t \in J, \\ x(0) + x(T) = 0. \end{cases} \tag{16}$$

We can get the result of the existence and unique solution of equation (16).

Theorem 1. Assume that $M, N > 0, M > 2NT$, and $(M + k_0NT)T \leq 1$. Then, APBVP (16) possesses a unique solution.

Proof. For any $x \in C(J, R)$, denoting $\|x\| = \max_{t \in J} |x(t)|$. Let

$$\begin{aligned} \omega_0 &= \max_{t \in J} |\sigma(t)|, \\ \omega_1 &= \frac{2(1 - e^{-MT})\omega_0}{M - 2NT}. \end{aligned} \tag{17}$$

We define an operator $S: E \rightarrow C(J, R)$ as follows:

$$\begin{aligned} (Sx)(t) &= \frac{e^{-M(t+T)}}{e^{-MT} + 1} \int_0^T (\sigma(s) - N(Tx)(s))e^{Ms} ds \\ &\quad + e^{-Mt} \int_0^t (\sigma(s) - N(Tx)(s))e^{Ms} ds, \end{aligned} \tag{18}$$

where $E = \{x \in C(J, R): |x| \leq \omega_1, x(0) - x(T) = 0\}$.

It is easy to see that E is a closed, bounded, and convex set. Furthermore, for any $x \in E$, we have

$$\begin{aligned} |(Sx)(t)| &\leq \frac{e^{-M(t+T)}}{e^{-MT} + 1} \int_0^T (|\sigma(s)| + N|(Tx)(s)|)e^{Ms} ds \\ &\quad + e^{-Mt} \int_0^t (|\sigma(s)| + N|(Tx)(s)|)e^{Ms} ds \\ &\leq \frac{e^{-M(t+T)}}{e^{-MT} + 1} \int_0^T (\omega_0 + NT\omega_1)e^{Ms} ds \\ &\quad + e^{-Mt} \int_0^t (\omega_0 + NT\omega_1)e^{Ms} ds \\ &\leq \frac{2(\omega_0 + N\omega_1T)(1 - e^{-MT})}{M} = \omega_1, \end{aligned} \tag{19}$$

which implies that $\|S(x)\| \leq \omega_1$, that is, $S(E) \subset E$ and S is uniformly bounded. Furthermore, for any $t_1, t_2 \in J$, we have

$$\begin{aligned}
 |(Sx)(t_1) - (Sx)(t_2)| &= \left| \frac{e^{-M(t_1+T)}}{e^{-MT} + 1} \int_0^T (|\sigma(s)| - N|(Tx)(s))e^{Ms} ds \right. \\
 &\quad - \frac{e^{-M(t_2+T)}}{e^{-MT} + 1} \int_0^T (|\sigma(s)| - N|(Tx)(s))e^{Ms} ds \\
 &\quad + e^{-Mt_1} \int_0^{t_1} (|\sigma(s)| - N|(Tx)(s))e^{Ms} ds \\
 &\quad \left. - e^{-Mt_2} \int_0^{t_2} (|\sigma(s)| - N|(Tx)(s))e^{Ms} ds \right| \tag{20} \\
 &\leq \frac{|e^{-M(t_1+T)} - e^{-M(t_2+T)}|}{e^{-MT} + 1} \int_0^T (|\sigma(s)| + N|(Tx)(s))e^{Ms} ds \\
 &\quad + |e^{-Mt_1} - e^{-Mt_2}| \int_0^{\max\{t_1, t_2\}} (|\sigma(s)| + N|(Tx)(s))e^{Ms} ds \\
 &\quad + e^{-Mt_2} \int_{\min\{t_1, t_2\}}^{\max\{t_1, t_2\}} (|\sigma(s)| + N|(Tx)(s))e^{Ms} ds.
 \end{aligned}$$

Since σ and x are bounded, thus S is uniformly continuous. According to Ascoli–Arzela’s theorem, there exists the subsequences $\{Sx_n\}$ converging uniformly on J to the continuous functions Sx and $Sx \in E$, then, we can see that S is compact. Therefore, there exists a solution of APBVP (16) by Schauder’s fixed point theorem. The uniqueness of solutions of APBVP (16) follows from Corollary 1. The proof is completed.

4. Nonlinear APBVP

In this section, we give the existence and uniqueness of the solutions of APBVP (1).

Definition 1. The functions $v, w \in C'(J, R)$ are said to be a pair of coupled lower and upper solutions for APBVP (1) if the following inequalities

$$\begin{cases} v'(t) \leq f(t, v(t), (Tv)(t)) - M\gamma_1, \\ v(0) \leq -w(T), \end{cases} \tag{21}$$

$$\begin{cases} w'(t) \geq f(t, w(t), (Tw)(t)) + M\gamma_2, \\ w(0) \geq -v(T), \end{cases} \tag{22}$$

hold, where $M > 0$, $\gamma_1 = (v(T) + w(0))/(1 - e^{-MT})$, and $\gamma_2 = (-w(T) - v(0))/(1 - e^{-MT})$.

Theorem 2. Assume that the following conditions hold.

(H_1) $v, w \in C'(J, R)$ are a pair of coupled lower and upper solutions of APBVP (1) such that $v \leq w$ on J ;

(H_2) There exist constants $M > 0$ and $N > 0$ such that $M > 2NT \geq 0$, $(M + k_0NT)T \leq 1$, and

$$|f(t, \bar{u}, T\bar{u}) - f(t, u, Tu)| \leq M(\bar{u} - u) + N(T\bar{u} - Tu), \tag{23}$$

while $v \leq u \leq \bar{u} \leq w$ and $Tv \leq Tu \leq T\bar{u} \leq Tw$ for $t \in J$.

Then, APBVP (1) has a unique solution $x \in [v, w]$.

Proof. We construct iterative sequences $\{v_n\}, \{w_n\} \subset C'(J, R)$ as follows, $v_1 = v$ and $w_1 = w$ on J , and for $n > 1$, v_n and w_n are the solutions of

$$\begin{cases} v'_n(t) = f(t, v_{n-1}(t), (Tv_{n-1})(t)) - M[v_n(t) - v_{n-1}(t)] - N[(Tv_n)(t) - (Tv_{n-1})(t)], \\ v_n(0) = -w_n(T), \end{cases} \tag{24}$$

$$\begin{cases} w'_n(t) = f(t, w_{n-1}(t), (Tw_{n-1})(t)) - M[w_n(t) - w_{n-1}(t)] - N[(Tw_n)(t) - (Tw_{n-1})(t)], \\ w_n(0) = -v_n(T). \end{cases} \tag{25}$$

The existence and uniqueness of the solution can be obtained by standard arguments for IVP (24) and (25).

We next prove that $v_1 \leq v_2 \leq w_2 \leq w_1$.

$$\begin{cases} p_1'(t) \leq -M[v_1(t) - v_2(t) + \gamma_1] - N[(Tv_1)(t) - (Tv_2)(t)] = -M(p_1(t) + \gamma_1) - N(Tp_1)(t), \\ p_2'(t) \leq -M[w_2(t) - w_1(t) + \gamma_2] - N[(Tw_2)(t) - (Tw_1)(t)] = -M(p_2(t) + \gamma_2) - N(Tp_2)(t), \end{cases} \quad (26)$$

where $\gamma_1 = (p_1(T) - p_2(0))/(1 - e^{-MT})$ and $\gamma_2 = (p_2(T) - p_1(0))/(1 - e^{-MT})$.

By Lemma 2, we have $p_1 = v_1 - v_2 \leq 0$ and $p_2 = w_2 - w_1 \leq 0$ on J .

Let $p = v_2 - w_2$, by the condition of (H_2) , and we have

$$p'(t) \leq -Mp(t) - N(Tp)(t), \quad p(0) = p(T). \quad (27)$$

By using similar arguments of Lemma 1, we have $p = v_2 - w_2 \leq 0$. Therefore, it is easy to see that these sequences satisfy

$$v_n \leq v_{n+1} \leq w_{n+1} \leq w_n, \quad n \geq 1. \quad (28)$$

Then, we have two monotone sequences which are bounded, and there exist ρ and μ , which satisfy $\lim_{n \rightarrow \infty} v_n = \rho$, $\lim_{n \rightarrow \infty} w_n = \mu$, and $\rho \leq \mu$. Moreover, the convergence is uniform on J .

Set $p = \mu - \rho$, then we obtain

$$\begin{cases} p'(t) \leq -Mp(t) - N(Tp)(t), \\ p(0) = p(T). \end{cases} \quad (29)$$

By Lemma 1, we have $p(t) \leq 0$ for $t \in J$. Hence, $\rho \equiv \mu$ for $t \in J$, and we can conclude $\rho \equiv \mu \equiv x$, in which x is the solution of APBVP (1). The proof of Theorem 2 is completed.

5. Quasilinearization

In this section, we apply the quasilinearization method in order to obtain the result on convergence of the iterative sequences of approximate solutions for APBVP (1).

Consider the Banach space $C(J, R)$ with the usual maximum norm $\|x\|_1 = \max_{t \in J} |x(t)|$. For any $x \in C(J, R)$, we call that a given sequence $\{x_n\}$ converges to x with order of convergence k , if $\{x_n\}$ converges to x in $C(J, R)$ and there exist $n_0 \in \mathbb{N}$ and $k > 0$ such that $\|x_{m+1} - x\|_1 \leq \|x_m - x\|_1^k$ for all $m \geq n_0$.

Theorem 3. Assume that the conditions of $(H_1) - (H_2)$ hold.

(H_3) $(\partial^i f / \partial x^i)$ and $\partial^i f / \partial (Tx)^i$ exist and are continuous for $i = 0, 1, \dots, k$, and

Let $p_1 = v_1 - v_2$ and $p_2 = w_2 - w_1$, by the condition of (H_2) , and we have $p_1(0) \leq p_2(T)$, $p_2(0) \leq p_1(T)$, and

$$\sum_{i=1}^k iM_i |w - v|^{i-1} \leq M, \quad (30)$$

$$\sum_{i=1}^k iN_i |w - v|^{i-1} \leq N,$$

where M_i and N_i are constants with

$$\left\| \frac{\partial^i f}{\partial x^i}(t, u, Tu) \right\| \leq (i!)M_i, \quad (31)$$

$$\left\| \frac{\partial^i f}{\partial (Tx)^i}(t, u, Tu) \right\| \leq (i!)N_i,$$

where $(t, u, Tu) \in \Omega = \{(t, u, Tu): v \leq u \leq w\}$.

Then, there exist monotone sequences $\{v_n\}, \{w_n\}$ of approximate solutions converging to the unique solution of (1) with rate of convergence of order k .

Proof. Let the function

$$\begin{aligned} f(t, u, Tu) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, \alpha, T\alpha) \frac{(u - \alpha)^i}{i!} + \frac{\partial^k f}{\partial x^k}(t, \chi, T\chi) \frac{(u - \alpha)^k}{k!} \\ &+ \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, \alpha, T\alpha) \frac{T(u - \alpha)^i}{i!} \\ &+ \frac{\partial^k f}{\partial (Tx)^k}(t, \chi, T\chi) \frac{T(u - \alpha)^k}{k!} \\ &\equiv g(t, u, Tu; \alpha, T\alpha), \end{aligned} \quad (32)$$

where $v \leq \alpha \leq u \leq w$, $\chi \in [\alpha, u]$. Consider the following linear equation:

$$\begin{cases} u'(t) = g(t, u, Tu; \alpha, T\alpha), & t \in J, \\ u(0) = -u(T). \end{cases} \quad (33)$$

Setting $v_0 = v$, by (H_3) , we have

$$\begin{cases} v_0'(t) \leq f(t, v_0(t), (Tv_0)(t)) - M\gamma_1 \equiv g(t, v_0(t), (Tv_0)(t); v_0(t), (Tv_0)(t)) - M\gamma_1, \\ v_0(0) \leq -w_0(T). \end{cases} \quad (34)$$

Similarly, setting $w_0 = w$, we obtain

$$\begin{cases} w_0'(t) \geq f(t, w_0(t), (Tw_0)(t)) + M\gamma_2 \equiv g(t, w_0(t), (Tw_0)(t); w_0(t), (Tw_0)(t)) + M\gamma_2, \\ w_0(0) \geq -v_0(T). \end{cases} \quad (35)$$

Then, v_0 and w_0 are lower and upper solutions of equation (33), respectively. Furthermore, for $t \in J$ and $v \leq x \leq y \leq w$, we have

$$\begin{aligned} & |g(t, x, Tx; \alpha, T\alpha) - g(t, y, Ty; \alpha, T\alpha)| \\ &= \left| \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, \alpha, T\alpha) \left(\frac{(x-\alpha)^i - (y-\alpha)^i}{i!} \right) \right. \\ &\quad \left. + \frac{\partial^k f}{\partial x^k}(t, \chi, T\chi) \left(\frac{(x-\alpha)^k - (y-\alpha)^k}{k!} \right) \right. \\ &\quad \left. + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, \alpha, T\alpha) \left(\frac{T(x-\alpha)^i - T(y-\alpha)^i}{i!} \right) \right. \\ &\quad \left. + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi, T\chi) \left(\frac{T(x-\alpha)^k - T(y-\alpha)^k}{k!} \right) \right| \\ &= |x-y| \left| \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, \alpha, T\alpha) \left(\frac{1}{i!} \sum_{j=0}^{i-1} (x-\alpha)^{i-1-j} (y-\alpha)^j \right) \right. \\ &\quad \left. + \frac{\partial^k f}{\partial x^k}(t, \chi, T\chi) \left(\frac{(x-\alpha)^{i-1-j} (y-\alpha)^j}{k!} \right) \right. \\ &\quad \left. + T|x-y| \left| \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, \alpha, T\alpha) \left(\frac{1}{i!} \sum_{j=0}^{i-1} T(x-\alpha)^{i-1-j} T(y-\alpha)^j \right) \right. \right. \\ &\quad \left. \left. + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi, T\chi) \left(\frac{T(x-\alpha)^{i-1-j} T(y-\alpha)^j}{k!} \right) \right| \right| \\ &\leq |x-y| \sum_{i=1}^k M_i \left(\sum_{j=0}^{i-1} (y-x)^{i-1} \right) + T|x-y| \sum_{i=1}^k N_i \left(\sum_{j=0}^{i-1} T(y-x)^{i-1} \right) \\ &\leq M|x-y| + NT|x-y|. \end{aligned} \quad (36)$$

Using Theorem 2, we know that problem (33) has a solution in $[v, w]$.

Let v_1 be a solution of the mentioned problem, with $v_1 \in [v, w]$, and we suppose $v = v_0 \leq v_1 \leq \dots \leq v_n \leq w$, where $v_n \in [v_{n-1}, w]$ is the solution of

$$\begin{cases} u'(t) = g(t, u, Tu; v_{n-1}, Tv_{n-1}), & t \in J, \\ u(0) = -u(T). \end{cases} \quad (37)$$

where v_{n-1} and w are lower and upper solutions, respectively, for the following problem:

$$\begin{cases} u'(t) = g(t, u, Tu; v_n, Tv_n), & t \in J, \\ u(0) = -u(T). \end{cases} \quad (38)$$

Similarly, we know that $g(t, u, Tu; v_n, Tv_n)$ satisfies the conditions of Theorem 2, then problem (33) has a solution in $[v_n, w]$. Let v_{n+1} be a solution of the mentioned problem, with $v_{n+1} \in [v_n, w]$. Hence, the constructed sequence $\{v_n\}$ is nondecreasing and bounded. In the same way, we can construct the sequence $\{w_n\}$ which is nonincreasing and bounded. Therefore, we obtain the two monotone sequences converge uniformly.

Let $\lim v_n = \rho$ and $\lim w_n = \mu$, and we can get $\rho \equiv \mu \equiv u$ by using Theorem 2, in which u is the solution of (1).

Now, we show that the convergence of $\{v_n\}$, $\{w_n\}$ to u is of order k . Let

$$\begin{aligned} p_n &= u - v_n \geq 0, \\ q_n &= w_n - u \geq 0. \end{aligned} \quad (39)$$

Firstly, we note that

$$\begin{aligned} u(t) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \frac{(u - v_n)^i}{i!} + \frac{\partial^k f}{\partial x^k}(t, \chi_n, T\chi_n) \frac{(u - v_n)^k}{k!} \\ &+ \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \frac{T(u - v_n)^i}{i!} \\ &+ \frac{\partial^k f}{\partial (Tx)^k}(t, \chi_n, T\chi_n) \frac{T(u - v_n)^k}{k!}, \end{aligned} \quad (40)$$

where $\chi_n \in [v_n, u]$. In sequence,

$$\begin{aligned} p'_{n+1}(t) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \left(\frac{p_n^i - (v_{n+1} - v_n)^i}{i!} \right) + \frac{\partial^k f}{\partial x^k}(t, \chi_n, T\chi_n) \left(\frac{p_n^k - (v_{n+1} - v_n)^k}{k!} \right) \\ &+ \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \left(\frac{Tp_n^i - T(v_{n+1} - v_n)^i}{i!} \right) + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi_n, T\chi_n) \left(\frac{Tp_n^k - T(v_{n+1} - v_n)^k}{k!} \right) \\ &= p_{n+1} \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} p_n^{i-1-j} (v_{n+1} - v_n)^j \right) + \frac{\partial^k f}{\partial x^k}(t, \chi_n, T\chi_n) \left(\frac{p_n^k - (v_{n+1} - v_n)^k}{k!} \right) \\ &+ Tp_{n+1} \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} (Tp_n^{i-1-j} T(v_{n+1} - v_n)^j) \right) + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi_n, T\chi_n) \left(\frac{Tp_n^k - T(v_{n+1} - v_n)^k}{k!} \right). \end{aligned} \quad (41)$$

In view of condition (H_3) , by the continuity of $\partial^i f / \partial x^i$ and $\partial^i f / \partial (Tx)^i$ in Ω , we have

$$p'_{n+1}(t) \leq Ap_{n+1}(t) + BTp_{n+1}(t) + Cp_n^k(t) + DTp_n^k(t), \quad (42)$$

where $C = 2M_k$, $D = 2N_k$, and

$$\begin{aligned} \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} p_n^{i-1-j} (v_{n+1} - v_n)^j \right) &\leq A, \\ \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} Tp_n^{i-1-j} T(v_{n+1} - v_n)^j \right) &\leq B. \end{aligned} \quad (43)$$

Similarly, we have

$$q'_{n+1}(t) \leq Aq_{n+1}(t) + Bq_{n+1}(t) + Cq_n^k(t) + DTq_n^k(t). \quad (44)$$

The boundary conditions are

$$\begin{aligned} p_n(0) &= q_n(T), \\ q_n(0) &= p_n(T). \end{aligned} \quad (45)$$

Let $R_n = p_n + q_n$, and we obtain

$$\begin{cases} R'_{n+1}(t) \leq AR_{n+1}(t) + BTR_{n+1}(t) + C(p_n^k(t) + q_n^k(t)) + DT(p_n^k(t) + q_n^k(t)) \leq (A + k_0B)R_{n+1}(t) + (C + k_0D)(p_n^k(t) + q_n^k(t)), \\ R_{n+1}(0) = R_{n+1}(T). \end{cases} \quad (46)$$

Using Gronwall's inequality for (46), we have

$$R_{n+1}(t) \leq e^{(A+k_0B)t} R_{n+1}(0) + \int_0^t e^{(A+k_0B)(t-s)} (C + k_0D)(p_n^k(s) + q_n^k(s)) ds. \quad (47)$$

Let $t = T$, and we have

$$R_{n+1}(0) \leq e^{(A+k_0B)T} R_{n+1}(0) + \frac{T}{A+k_0B} e^{(A+k_0B)T} \cdot (C + k_0D) \max_{t \in J} (p_n^k(t) + q_n^k(t)), \quad (48)$$

which implies

$$R_{n+1}(0) \leq \frac{e^{(A+k_0B)T} (C + k_0D) \max_{t \in J} (p_n^k(t) + q_n^k(t))}{T(1 - e^{(A+k_0B)T})}, \quad (49)$$

that is,

$$\max_{t \in J} R_{n+1}(t) \leq K (\max_{t \in J} p_n^k(t) + \max_{t \in J} q_n^k(t)), \quad (50)$$

where $K = \frac{e^{(A+k_0B)T} (C + k_0D) \max_{t \in J} R_n^k(t)}{(1 - e^{(A+k_0B)T}) + T/(A+k_0B)e^{(A+k_0B)T} (C + k_0D)}$.

Since

$\max_{t \in J} p_n(t) \leq \max_{t \in J} R_n(t)$ and $\max_{t \in J} q_n(t) \leq \max_{t \in J} R_n(t)$, we get the desired convergence. The proof is completed.

6. An Example

In this section, we will provide an example which demonstrates the application of Theorem 3.

Example 1. Consider the following APBVP:

$$\begin{cases} x'(t) = \frac{1}{20}t(1+x^2(t)) - \frac{1}{10}x - \frac{1}{20} \int_0^t x(s) \cos s ds, & t \in [0, 1], \\ x(0) = -x(1). \end{cases} \quad (51)$$

It is easy to check that $v_0 = -1$ and $w_0 = 1$ are lower and upper solutions of (51), respectively, which satisfies condition (H_1) of Theorem 3. And we can show that

$$\begin{aligned} |f_x| &\leq \frac{1}{5}, \\ |f_y| &\leq \frac{1}{20}, \\ |f_{xx}| &\leq \frac{1}{10}, \\ f_{yy} &= 0, \quad \frac{\partial^i f}{\partial x^i} = 0, \\ \frac{\partial^i f}{\partial (Tx)^i} &= 0, \\ i &= 3, 4, \dots, k. \end{aligned} \quad (52)$$

Setting $M_1 = 1/5$, $M_2 = 1/10$, $N_1 = 1/20$, $N_2 = 0$, $M_i = 0$, $N_i = 0$, $i = 3, 4, \dots, k$, $M = 3/5$, and $N = 1/20$, satisfy conditions (H_2) and (H_3) of Theorem 3. Then, convergence of the iterative sequences of approximate solutions for APBVP (51) are of order $k \geq 2$.

7. Conclusion

In this paper, we discussed the problem of rapid convergence for the first-order integro-differential equations with anti-periodic boundary value conditions. By using the particular definitions of the coupled lower and upper related solutions, which are new and some new comparison principles, we obtained the existence and uniqueness of solution of the problems. Meanwhile, by using the method of quasilinearization, we obtained the monotone sequences of approximate solutions, converging to the unique solution of such problems with the rate of convergence of order k . Finally, we give an example to illustrate our main results.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors read and approved the final manuscript.

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