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Research Article

Maximum Reciprocal Degree Resistance Distance Index of Unicyclic Graphs

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The reciprocal degree resistance distance index of a connected graph G is defined as $\mathrm{RDR}(G) = \sum_{\{u,v\}\subseteq V(G)} ((d_G(u) + d_G(v))/(r_G(u,v)))$, where $r_G(u,v)$ is the resistance distance between vertices u and v in G. Let $\mathscr{U}_n$ denote the set of unicyclic graphs with n vertices. We study the graph with maximum reciprocal degree resistance distance index among all graphs in $\mathscr{U}_n$ and characterize the corresponding extremal graph.

1. Introduction

Let $G = (V, E)$ be a simple connected graph of order n with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. For any $u \in V(G)$, $d_G(u)$ is the degree of vertex u, and the distance between vertices of u and v, denoted by $d_G(u,v)$, is the length of a shortest path between them. Topological indices are numbers associated with molecular structures which serve for quantitative relationships between chemical structures and properties. The first such index was published by Wiener [1], but the name topological index was invented by Hosoya [2]. Many of them are based on the graph distance [3] and the vertex degree [4]. In addition, several graph invariants are based on both the vertex degree and the graph distance [5].

One of the most intensively studied topological indices is the Wiener index. The Wiener index was introduced by American chemist Wiener in [1], defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v). \tag{1}$$

Another distance-based graph invariant Harary index has been introduced by Plavšić et al. [6] and independently by Ivanciuc et al. [7] in 1993 for the characterization of molecular graphs. The Harary index $H(G)$ of graph G is defined as

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)}. \tag{2}$$

For more results related to Harary index, refer to [8, 9–17].

The resistance distance $r(u,v)$ (if more than one graphs are considered, we write $r_G(u,v)$ in order to avoid confusion) between vertices u and v in G is defined as the effective resistance between the two nodes of the electronic network obtained so that its nodes correspond to the vertices of G and each edges of G is replaced by a resistor of unit resistance, which is compared by the methods of the theory of resistive electrical networks based on Ohm's and Kirchoff's laws.

The Kirchhoff index $\mathrm{Kf}(G)$ of a graph G is defined as [18, 19]

$$\mathrm{Kf}(G) = \sum_{\{u,v\}\subseteq V(G)} r_G(u,v). \tag{3}$$

As a new structure descriptor, the Kirchoff index is well studied (see recent papers [20–31]). In 2017, Chen et al. [32] introduced a new graph invariant reciprocal to Kirchoff index, named Resistance-Harary index:

$$\mathrm{RH}(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{r_G(u,v)}. \tag{4}$$

For more results related to Resistance-Harary index, refer to [32–34].

The first and the second Zagreb indices are defined as

$$\begin{aligned} M_1 &= M_1(G) = \sum_{u\in V(G)} d_G(u)^2 = \sum_{uv\in E(G)} \left(d_G(u) + d_G(v)\right), \\ M_2 &= M_2(G) = \sum_{uv\in E(G)} d_G(u) d_G(v), \end{aligned} \tag{5}$$

respectively. These are the oldest [35, 36] and best studied degree-based topological indices (see the reviews [4], recent papers [37, 38], and the references cited therein).

Dobrynin and Kochtova [39] and Gutman [40] independently proposed a vertex-degree-weighted version of Wiener index called degree distance, which is defined for a connected graph G as

$$\mathrm{DD}(G) = \frac{1}{2} \sum_{\{u,v\}\in V(G)} \left(d_G(u) + d_G(v)\right) d_G(u,v). \tag{6}$$

The reciprocal degree distance [41] is defined as

$$\mathrm{RDD}(G) = \frac{1}{2} \sum_{\{u,v\}\in V(G)} \frac{d_G(u) + d_G(v)}{d_G(u,v)}. \tag{7}$$

Hua and Zhang [42] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [41, 43].

Analogous to the relationship between degree distance and reciprocal degree distance, we introduce here a new graph invariant based on both the vertex degree and the graph distance, named the reciprocal degree resistance distance index:

$$\mathrm{RDR}(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{d_G(u) + d_G(v)}{r_G(u,v)}. \tag{8}$$

A unicyclic graph is a connected graph with n vertices and n edges. Let $\mathcal{U}_n$ denote the set of unicyclic graphs with n vertices. In this paper, we determine the graph with maximum reciprocal degree resistance distance index among all graphs in $\mathcal{U}_n$ and characterize the corresponding extremal graph.

2. Preliminaries

In this section, we will introduce some useful lemmas and three transformations.

Let $C_g = v_1 v_2 \cdots v_g v_1$ be the cycle on $g \geq 3$ vertices. For any two vertices $v_i, v_j \in V(C_g)$ with $i < j$, by Ohm's law, one has

$$r_{C_g}\left(v_i, v_j\right) = \frac{(j-i)(g+i-j)}{g}. \tag{9}$$

Lemma 1 (see [19]). *Let x be a cut vertex of a connected graph G and let a and b be vertices occurring in different components which arise upon deletion of x. Then,*

$$r_G(a,b) = r_G(a,x) + r_G(x,b). \tag{10}$$

2.1. Edge-Lifting Transformation. Let G_1 and G_2 be two graphs with $n_1 \geq 2$ and $n_2 \geq 2$ vertices, respectively. G is the graph obtained from G_1 and G_2 by adding an edge between a vertex u_0 of G_1 to a vertex v_0 of G_2. And G' is the graph obtained by identifying u_0 of G_1 to a vertex v_0 of G_2 and adding a pendant edge to u_0 (v_0). We say that G' is obtained from G by an edge-lifting transformation at $e = \{u_0, v_0\}$ (see Figure 1).

Lemma 2. *If G' can be obtained from G by an edge-lifting transformation, then $RDR(G) < RDR(G')$.*

Proof. Consider G and G' shown in Figure 1. By the definition of $\mathrm{RDR}(G)$,

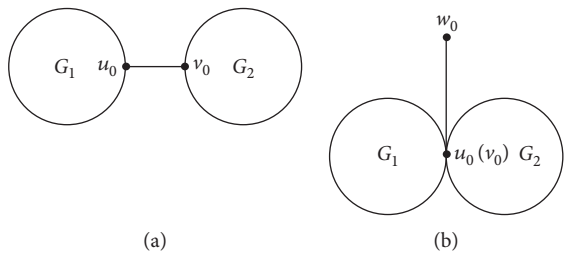

Figure 1: The edge-lifting transformation. (a) G. (b) G'.

$$\begin{aligned}
\mathrm{RDR}(G) = {} & \sum_{x,y\in V(G_1)\setminus\{u_0\}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} + \sum_{x,y\in V(G_2)\setminus\{v_0\}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} \\
& + \sum_{x\in V(G_1)\setminus\{u_0\}} \frac{d_G(u_0)+d_G(x)}{r_G(u_0,x)} + \sum_{x\in V(G_2)\setminus\{v_0\}} \frac{d_G(u_0)+d_G(x)}{r_G(v_0,x)+1} \\
& + \sum_{x\in V(G_1)\setminus\{u_0\}} \frac{d_G(v_0)+d_G(x)}{r_G(u_0,x)+1} + \sum_{x\in V(G_2)\setminus\{v_0\}} \frac{d_G(v_0)+d_G(x)}{r_G(v_0,x)} \\
& + \sum_{x\in V(G_1)\setminus\{u_0\},y\in V(G_2)\setminus\{v_0\}} \frac{d_G(x)+d_G(y)}{r_G(x,u_0)+1+r_G(v_0,y)} + \frac{d_G(u_0)+d_G(v_0)}{r_G(u_0,v_0)}, \\
\mathrm{RDR}(G') = {} & \sum_{x,y\in V(G_1)\setminus\{u_0\}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} + \sum_{x,y\in V(G_2)\setminus\{v_0\}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} \\
& + \sum_{x\in V(G_1)\setminus\{u_0\}} \frac{d_{G'}(u_0)+d_{G'}(x)}{r_{G'}(u_0,x)} + \sum_{x\in V(G_2)\setminus\{v_0\}} \frac{d_{G'}(v_0)+d_{G'}(x)}{r_{G'}(v_0,x)} \\
& + \sum_{x\in V(G_1)\setminus\{u_0\}} \frac{d_{G'}(w_0)+d_{G'}(x)}{r_{G'}(u_0,x)+1} + \sum_{x\in V(G_2)\setminus\{v_0\}} \frac{d_{G'}(w_0)+d_{G'}(x)}{r_{G'}(v_0,x)+1} \\
& + \sum_{x\in V(G_1)\setminus\{u_0\},y\in V(G_2)\setminus\{v_0\}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,u_0)+r_{G'}(v_0,y)} + \frac{d_{G'}(u_0)+d_{G'}(w_0)}{r_{G'}(u_0,w_0)}.
\end{aligned} \tag{11}$$

(i) Note that $d_G(x)+d_G(y)=d_{G'}(x)+d_{G'}(y)$, $r_G(x,y)=r_{G'}(x,y)$ for $x,y\in V(G_1)\setminus\{u_0\}$ or $x,y\in V(G_2)\setminus\{v_0\}$, and $d_G(u_0)+d_G(v_0)=d_{G'}(u_0)+d_{G'}(w_0)$, $r_G(u_0,v_0)=r_{G'}(u_0,w_0)$; then, we have

$$\begin{aligned}
\sum_{x,y\in V(G_1)\setminus\{u_0\}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} &= \sum_{x,y\in V(G_1)\setminus\{u_0\}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}, \\
\sum_{x,y\in V(G_2)\setminus\{v_0\}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} &= \sum_{x,y\in V(G_2)\setminus\{v_0\}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}, \\
\frac{d_G(u_0)+d_G(v_0)}{r_G(u_0,v_0)} &= \frac{d_{G'}(u_0)+d_{G'}(w_0)}{r_{G'}(u_0,w_0)}.
\end{aligned} \tag{12}$$

(ii) Note that $d_G(x)+d_G(y)=d_{G'}(x)+d_{G'}(y)$, $r_G(x,u_0)+r_G(v_0,y)=r_{G'}(x,u_0)+r_{G'}(v_0,y)$ for $x\in V(G_1)\setminus\{u_0\}$, $y\in V(G_2)\setminus\{v_0\}$; then, we have

$$\begin{aligned}
& \sum_{x\in V(G_1)\setminus\{u_0\},y\in V(G_2)\setminus\{v_0\}} \frac{d_G(x)+d_G(y)}{r_G(x,u_0)+1+r_G(v_0,y)} \\
& \quad < \sum_{x\in V(G_1)\setminus\{u_0\},y\in V(G_2)\setminus\{u_0\}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,u_0)+r_{G'}(v_0,y)}.
\end{aligned} \tag{13}$$

(iii) Note that $r_{G'}(u_0,x)=r_G(u_0,x)$ for any $x\in V(G_1)\setminus\{u_0\}$, and $d_{G'}(u_0)=d_G(u_0)+d_G(v_0)-1$; then, we have

$$\left(\sum_{x\in V(G_1)\setminus\{u_0\}}\frac{d_{G'}(u_0)+d_{G'}(x)}{r_{G'}(u_0,x)}-\sum_{x\in V(G_1)\setminus\{u_0\}}\frac{d_G(u_0)+d_G(x)}{r_G(u_0,x)}\right)$$
$$+\left(\sum_{x\in V(G_1)\setminus\{u_0\}}\frac{d_{G'}(w_0)+d_{G'}(x)}{r_{G'}(u_0,x)+1}-\sum_{x\in V(G_1)\setminus\{u_0\}}\frac{d_G(v_0)+d_G(x)}{r_G(u_0,x)+1}\right)$$
$$=\sum_{x\in V(G_1)\setminus\{u_0\}}\frac{d_G(v_0)-1}{r_G(u_0,x)}+\sum_{x\in V(G_1)\setminus\{u_0\}}\frac{1-d_G(v_0)}{r_G(u_0,x)+1}>0. \tag{14}$$

(iv) Note that $r_{G'}(v_0,x)=r_G(v_0,x)$, $d_{G'}(x)=d_G(x)$ for any $x\in V(G_2)$, and $d_{G'}(v_0)=d_G(u_0)+d_G(v_0)-1$; then, we have

$$\left(\sum_{x\in V(G_2)\setminus\{v_0\}}\frac{d_{G'}(v_0)+d_{G'}(x)}{r_{G'}(v_0,x)}-\sum_{x\in V(G_2)\setminus\{v_0\}}\frac{d_G(v_0)+d_G(x)}{r_G(v_0,x)}\right)$$
$$+\left(\sum_{x\in V(G_2)\setminus\{v_0\}}\frac{d_{G'}(w_0)+d_{G'}(x)}{r_{G'}(v_0,x)+1}-\sum_{x\in V(G_2)\setminus\{v_0\}}\frac{d_G(u_0)+d_G(x)}{r_G(v_0,x)+1}\right)$$
$$=\sum_{x\in V(G_2)\setminus\{v_0\}}\frac{d_G(u_0)-1}{r_G(v_0,x)}+\sum_{x\in V(G_2)\setminus\{v_0\}}\frac{1-d_G(u_0)}{r_G(v_0,x)+1}>0. \tag{15}$$

Thus, by (i)–(iv), we get $\mathrm{RDR}(G')-\mathrm{RDR}(G)>0$. □

2.2. Cycle-Lifting Transformation. Let G be a graph as shown in Figure 2. Take a cycle C in G, say $C=v_1v_2\cdots v_pv_1$; G can be viewed as a graph obtained by coalescing C with a number of star subgraphs of G, say $G_1,G_2,\ldots,G_p$, by identifying v_i with the center of G_i for all $i\,(1\le i\le p)$, denoted as $|E(G_i)|=s_i$ and $|V(G_i)|=s_i+1$. Deleting all edges in G_i and joining v_1 to all pendant vertices of $G_i\,(2\le i\le p)$, we obtained a new graph, denoted by G' (see Figure 2). This operation is called a cycle-lifting transformation of G with respect to C.

Lemma 3. *If G' can be obtained from G by a cycle-lifting transformation, then $RDR(G)<RDR(G')$.*

Proof. Consider G and G' shown in Figure 2. By the definition of $\mathrm{RDR}(G)$,

$$\begin{aligned}\mathrm{RDR}(G)=&\sum_{i=1}^{p}\sum_{x,y\in V(G_i)\setminus\{v_i\}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}+\sum_{i=1}^{p}\sum_{x\in V(G_i)\setminus\{v_i\}}\frac{d_G(v_i)+d_G(x)}{r_G(v_i,x)}\\&+\sum_{\substack{i,j=1\\ i\neq j}}^{p}\sum_{\substack{x\in V(G_i)\setminus\{v_i\}\\ y\in V(G_j)\setminus\{v_j\}}}\frac{d_G(x)+d_G(y)}{r_G(x,v_i)+r_G(v_i,v_j)+r_G(v_j,y)}+\sum_{i=1}^{p}\sum_{\substack{x\in V(G_i)\setminus\{v_i\}\\ y\in V(G_j)\setminus\{v_j\}}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\\&+\sum_{x,y\in V(C_p)}\frac{d_G(x)+d_G(y)}{r_G(x,y)},\\ \mathrm{RDR}(G')=&\sum_{i=1}^{p}\sum_{x,y\in V(G_i)\setminus\{v_1\}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}+\sum_{i=1}^{p}\sum_{x\in V(G_i)\setminus\{v_1\}}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}\\&+\sum_{\substack{i,j=1\\ i\neq j}}^{p}\sum_{\substack{x\in V(G_i)\setminus\{v_1\}\\ y\in V(G_j)\setminus\{v_1\}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,v_1)+r_{G'}(v_1,y)}+\sum_{i=1}^{p}\sum_{\substack{x\in V(C_p)\setminus\{v_1\}\\ y\in V(G_i)\setminus\{v_1\}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}\\&+\sum_{x,y\in V(C_p)}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}.\end{aligned} \tag{16}$$

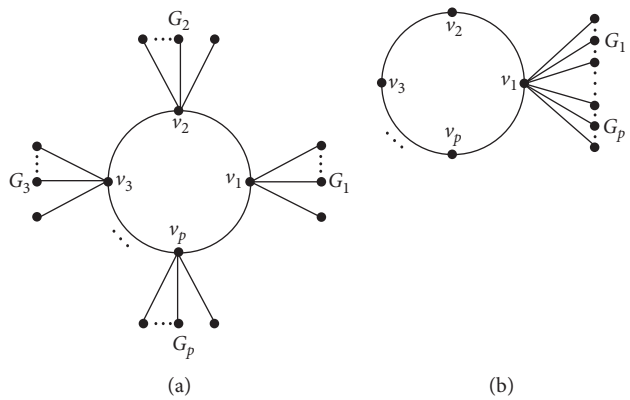

Figure 2: The cycle-lifting transformation. (a) G. (b) G'.

(i) Note that $d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y)$, $r_G(x, y) = r_{G'}(x, y)$ for $x, y \in V(G_i)\backslash\{v_i\}$ in G, and $x, y \in V(G_i)\backslash\{v_1\}$ in G'; then, we have

$$\sum_{i=1}^{p} \sum_{x,y\in V(G_i)\backslash\{v_i\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{i=1}^{p} \sum_{x,y\in V(G_i)\backslash\{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)}. \tag{17}$$

(ii) Let $r_G(v_i, v_j) = r_{G'}(v_i, v_j) = r_t$; when $|j - i| = t$, then $r_t = r_{p-t}$. $d_G(v_i) = s_i + 2$ for any $v_i \in V(C_p)$ in G and $d_{G'}(v_1) = s_1 + \cdots + s_p + 2$ for $v_1 \in V(C_p)$ in G'; then, we have

$$\sum_{x,y\in V(C_p)} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{i=2}^{p} \frac{d_G(v_1) + d_G(v_i)}{r_G(v_1, v_i)} + \sum_{i=3}^{p} \frac{d_G(v_2) + d_G(v_i)}{r_G(v_2, v_i)} + \cdots + \frac{d_G(v_{p-1}) + d_G(v_p)}{r_G(v_{p-1}, v_p)}. \tag{18}$$

Thus,

$$\begin{aligned}
\sum_{i=2}^{p} \frac{d_G(v_1) + d_G(v_i)}{r_G(v_1, v_i)} &= \frac{s_1 + s_2 + 4}{r_1} + \frac{s_1 + s_3 + 4}{r_2} + \cdots + \frac{s_1 + s_p + 4}{r_{p-1}}, \\
\sum_{i=3}^{p} \frac{d_G(v_2) + d_G(v_i)}{r_G(v_2, v_i)} &= \frac{s_2 + s_3 + 4}{r_1} + \cdots + \frac{s_2 + s_p + 4}{r_{p-2}}, \\
&\cdots \\
\frac{d_G(v_{p-1}) + d_G(v_p)}{r_G(v_{p-1}, v_p)} &= \frac{s_{p-1} + s_p + 4}{r_1}.
\end{aligned} \tag{19}$$

Thus,

$$\sum_{x,y\in V(C_p)} \frac{d_G(x)+d_G(y)}{r_G(x,y)} = \frac{(s_1+\cdots+s_{p-1})+(s_2+\cdots+s_p)+4(p-1)}{r_1} + \frac{(s_1+\cdots+s_{p-2})+(s_3+\cdots+s_p)+4(p-2)}{r_2} + \cdots + \frac{s_1+s_p+4}{r_{p-1}}. \tag{20}$$

Similarly,

$$\sum_{x,y\in V(C_p)} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} = \frac{(s_1+\cdots+s_p)+4(p-1)}{r_1} + \frac{(s_1+\cdots+s_p)+4(p-2)}{r_2} + \cdots + \frac{s_1+\cdots+s_p+4}{r_{p-1}}. \tag{21}$$

Then,

$$\sum_{x,y\in V(C_p)} \frac{d_G(x)+d_G(y)}{r_G(x,y)} - \sum_{x,y\in V(C_p)} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} = \frac{s_2+\cdots+s_{p-1}}{r_1} + \frac{s_3+\cdots+s_{p-2}}{r_2} + \cdots + \frac{-(s_2+\cdots+s_{p-1})}{r_{p-1}} = 0. \tag{22}$$

(iii) Note that $d_G(x)+d_G(y)=d_{G'}(x)+d_{G'}(y)$, $r_G(x,v_i)+r_G(v_j,y)=r_{G'}(x,v_1)+r_{G'}(v_1,y)$ for $x\in V(G_i)\backslash\{v_i\}$, $y\in V(G_j)\backslash\{v_j\}$ in G and for $x\in V(G_i)\backslash\{v_1\}$, $y\in V(G_j)\backslash\{v_1\}$ in G'; then, we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{p} \sum_{\substack{x\in V(G_i)\backslash\{v_i\}\\ y\in V(G_j)\backslash\{v_j\}}} \frac{d_G(x)+d_G(y)}{r_G(x,v_i)+r_G(v_i,v_j)+r_G(v_j,y)} < \sum_{\substack{i,j=1\\ i\neq j}}^{p} \sum_{\substack{x\in V(G_i)\backslash\{v_1\}\\ y\in V(G_j)\backslash\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,v_1)+r_{G'}(v_1,y)}. \tag{23}$$

(iv) Note that $r_G(v_i,x)=r_{G'}(v_1,x)=1$, $d_G(v_i)+d_G(x)=s_i+3$ for $x\in V(G_i)\backslash\{v_i\}$ in G, and $d_{G'}(v_1)+d_{G'}(x)=s_1+\cdots+s_p+3$ for $x\in V(G_i)\backslash\{v_1\}$ in G'; then, we have

$$\sum_{i=1}^{p} \sum_{x\in V(G_i)\backslash\{v_i\}} \frac{d_G(v_i)+d_G(x)}{r_G(v_i,x)} = s_1(s_1+3)+s_2(s_2+3)+\cdots+s_p(s_p+3) = \sum_{i=1}^{p} s_i(s_i+3),$$

$$\sum_{i=1}^{p} \sum_{x\in V(G_i)\backslash\{v_1\}} \frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)} = s_1(s_1+\cdots+s_p+3)+\cdots+s_p(s_1+\cdots+s_p+3) = \sum_{i=1}^{p} s_i(s_1+\cdots+s_p+3). \tag{24}$$

Thus, we have

$$\sum_{i=1}^{p} \sum_{x\in V(G_i)\backslash\{v_1\}} \frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)} - \sum_{i=1}^{p} \sum_{x\in V(G_i)\backslash\{v_i\}} \frac{d_G(v_i)+d_G(x)}{r_G(v_i,x)} = \sum_{\substack{i,j=1\\ i\neq j}}^{p} s_i\cdot s_j. \tag{25}$$

(v) Note that $d_G(x)+d_G(y)=3+s_i$, $d_{G'}(x)+d_{G'}(y)=3$, $r_G(x,y)=r_{G'}(x,y)$ for any $x\in V(C_p)\backslash\{v_i\}$, $y\in V(G_i)\backslash\{v_i\}$ in G and for $x\in V(C_p)\backslash\{v_1\}$, $y\in V(G_i)\backslash\{v_1\}$ in G'; when $s_{i+t}>s_p$ $(1\le t\le p-1)$, then $s_{i+t}=s_{i+t-p}$; we have

$$\sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_i\} \\ y\in V(G_i)\setminus\{v_i\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} = \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_1)\setminus\{v_1\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} + \sum_{\substack{x\in V(C_p)\setminus\{v_2\} \\ y\in V(G_1)\setminus\{v_2\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} + \cdots + \sum_{\substack{x\in V(C_p)\setminus\{v_p\} \\ y\in V(G_p)\setminus\{v_p\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)}. \tag{26}$$

Thus,

$$\begin{aligned}
\sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_1)\setminus\{v_1\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} &= \frac{s_1(3+s_2)}{r_1+1} + \frac{s_1(3+s_3)}{r_2+1} + \cdots + \frac{s_1(3+s_p)}{r_{p-1}+1}, \\
\sum_{\substack{x\in V(C_p)\setminus\{v_2\} \\ y\in V(G_1)\setminus\{v_2\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} &= \frac{s_2(3+s_3)}{r_1+1} + \frac{s_2(3+s_4)}{r_2+1} + \cdots + \frac{s_2(3+s_1)}{r_{p-1}+1}, \\
&\cdots \\
\sum_{\substack{x\in V(C_p)\setminus\{v_p\} \\ y\in V(G_p)\setminus\{v_p\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} &= \frac{s_p(3+s_1)}{r_1+1} + \frac{s_p(3+s_2)}{r_2+1} + \cdots + \frac{s_p(3+s_{p-1})}{r_{p-1}+1},
\end{aligned} \tag{27}$$

and we have

$$\sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_j\} \\ y\in V(G_p)\setminus\{v_j\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} = \sum_{i=1}^{p} \frac{s_i(3+s_{(i+1)\mathrm{mod}p})}{r_1+1} + \sum_{i=1}^{p} \frac{s_i(3+s_{(i+2)\mathrm{mod}p})}{r_2+1} + \cdots + \sum_{i=1}^{p} \frac{s_i(3+s_{(i+p-1)\mathrm{mod}p})}{r_{p-1}+1}. \tag{28}$$

Thus,

$$\sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_i\} \\ y\in V(G_p)\setminus\{v_i\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} = \sum_{\substack{i,j=1 \\ i\neq j}}^{p} \frac{s_i\cdot(3+s_j)}{r_G(v_i,v_j)+1}. \tag{29}$$

Similarly,

$$\begin{aligned}
\sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} &= \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} + \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} \\
&\quad + \cdots + \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}, \\
\sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} &= \frac{3s_1}{r_1+1} + \frac{3s_1}{r_2+1} + \cdots + \frac{3s_1}{r_{p-1}+1}, \\
\sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} &= \frac{3s_2}{r_1+1} + \frac{3s_2}{r_2+1} + \cdots + \frac{3s_2}{r_{p-1}+1}, \\
&\cdots \\
\sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} &= \frac{3s_p}{r_1+1} + \frac{3s_p}{r_2+1} + \cdots + \frac{3s_p}{r_{p-1}+1}.
\end{aligned} \tag{30}$$

Thus, we have

$$\begin{aligned}
&\sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} \\
&= \sum_{i=1}^{p} \frac{3s_i}{r_1+1} + \sum_{i=1}^{p} \frac{3s_i}{r_2+1} + \cdots + \sum_{i=1}^{p} \frac{3s_i}{r_{p-1}+1}.
\end{aligned} \tag{31}$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_1\} \\ y\in V(G_i)\setminus\{v_1\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} \\
&- \sum_{i=1}^{p} \sum_{\substack{x\in V(C_p)\setminus\{v_i\} \\ y\in V(G_i)\setminus\{v_i\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} = -\sum_{\substack{i,j=1 \\ i\neq j}}^{p} \frac{s_i\cdot s_j}{r_G(v_i,v_j)+1}.
\end{aligned} \tag{32}$$

Thus, by (i)–(v), we get $\mathrm{RDR}(G') - \mathrm{RDR}(G) > 0$. □

2.3. Cycle-Shrinking Transformation. Denote by S_n^p the unicyclic graph obtained from cycle C_p by attaching $n-p$ pendant edges to a vertex v_1 of C_p (see Figure 3). Let $G = S_n^p\,(p \geq 4)$; deleting the edges $v_2v_3, \ldots, v_{p-1}v_p$ and adding the edges $v_2v_p, v_3v_1, \ldots, v_{p-1}v_1$, we obtain the graph $G' = S_n^3$. This operation is called cycle-shrinking transformation. Denote by $W\,(W'$, resp) the set of pendant vertices of $G\,(G'$, resp). Let $|W| = k$, $|W'| = k'$. It is clear that $p + k = n$ and $3 + k' = n$. Then, $k' = k + (p-3)$; thus, W' can be partitioned into two subsets. One has k vertices, which is naturally corresponding to W. So, we also denote it by W; another has $p-3$ vertices, denoted by $\widetilde{W}$.

Lemma 4. *Let G be an unicyclic graph of order n and a cycle C_p with $p \geq 4$; if G' can be obtained from G by a cycle-shrinking transformation, then $RDR(G) < RDR(G')$.*

Proof. Consider G and G' shown in Figure 3. By the definition of $\mathrm{RDR}(G)$,

$$\begin{aligned}
\mathrm{RDR}(G) = {} & \sum_{x,y\in W} \frac{d_G(x)+d_G(y)}{r_G(x,y)} + \sum_{x\in W} \frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)} + \sum_{\substack{x\in W\\ y\in\{v_2,v_p\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} \\
& + \sum_{\substack{x\in W\\ y\in V(C_p)\setminus\{v_1,v_2,v_p\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} + \sum_{x\in\{v_2,v_p\}} \frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)} \\
& + \sum_{x\in V(C_p)\setminus\{v_1,v_2,v_p\}} \frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)} + \sum_{\substack{x\in V(C_p)\setminus\{v_1,v_2,v_p\}\\ y\in\{v_2,v_p\}}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} \\
& + \sum_{x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}} \frac{d_G(x)+d_G(y)}{r_G(x,y)} + \frac{d_G(v_2)+d_G(v_p)}{r_G(v_2,v_p)},
\end{aligned} \tag{33}$$

$$\begin{aligned}
\mathrm{RDR}(G') = {} & \sum_{x,y\in W} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} + \sum_{x\in W} \frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)} + \sum_{\substack{x\in W\\ y\in\{v_2,v_p\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} \\
& + \sum_{x\in W, y\in\widetilde{W}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} + \sum_{x\in\{v_2,v_p\}} \frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)} + \sum_{x\in\widetilde{W}} \frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)} \\
& + \sum_{\substack{x\in W\\ y\in\{v_2,v_p\}}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} + \sum_{x,y\in\widetilde{W}} \frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)} + \frac{d_{G'}(v_2)+d_{G'}(v_p)}{r_{G'}(v_2,v_p)}.
\end{aligned}$$

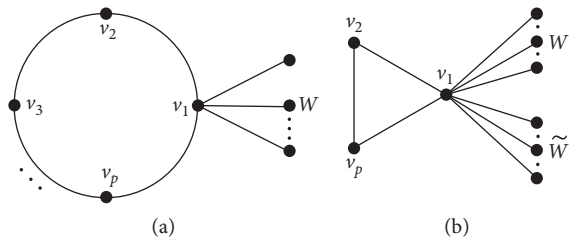

Figure 3: The cycle-shrinking transformation. (a) G. (b) G'.

(i) Note that $d_G(x)+d_G(y)=d_{G'}(x)+d_{G'}(y)$, $r_G(x,y)=r_{G'}(x,y)$ for any $x,y\in W$; then, we have

$$\sum_{x,y\in W}\frac{d_G(x)+d_G(y)}{r_G(x,y)}=\sum_{x,y\in W}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}. \tag{34}$$

(ii) Note that $d_G(x)+d_G(y)=d_{G'}(x)+d_{G'}(y)=3$, $r_G(x,y)=(p-1)/p+1\geq r_{G'}(x,y)=(2/3)+1$ for any $x\in W$, $y\in\{v_2,v_p\}$ in G and $x\in W$, $y\in\{v_2,v_p\}$ in G'; then, we have

$$\sum_{\substack{x\in W\\ y\in\{v_2,v_p\}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{\substack{x\in W\\ y\in\{v_2,v_p\}}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\geq 0. \tag{35}$$

(iii) Note that $d_G(v_1)+d_G(x)=k+3$, $d_{G'}(v_1)+d_{G'}(x)=k+p$, $r_G(v_1,x)=r_{G'}(v_1,x)=1$, for $x\in W$ in G and G', respectively; then, we have

$$\begin{aligned}&\sum_{x\in W}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}-\sum_{x\in W}\frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)}\\&\quad=k(k+p)-k(k+3)=k(p-3).\end{aligned} \tag{36}$$

(iv) Note that $d_G(x)+d_G(y)=3, r_G(x,y)=r_G(v_i,v_1)+1\geq 2\,(3\leq i\leq p-1)$ for $x\in W, y\in V(C_p)\backslash\{v_1,v_2,v_p\}$ and $d_{G'}(x)+d_{G'}(y)=2, r_{G'}(x,y)=2$ for $x\in W, y\in\widetilde{W}$; then, we have

$$\begin{aligned}&\sum_{\substack{x\in W\\ y\in\widetilde{W}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{\substack{x\in W\\ y\in V(C_p)\backslash\{v_1,v_2,v_p\}}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\\&\quad\geq k(p-3)-\frac{3k(p-3)}{2}=-\frac{k(p-3)}{2}.\end{aligned} \tag{37}$$

Combing (iii) − (iv), we have

$$\begin{aligned}&\left(\sum_{x\in W}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}-\sum_{x\in W}\frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)}\right)\\&\quad+\left(\sum_{\substack{x\in W\\ y\in\widetilde{W}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}\right.\\&\quad\left.-\sum_{\substack{x\in W\\ y\in V(C_p)\backslash\{v_1,v_2,v_p\}}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\right)\\&\quad\geq\frac{k(p-3)}{2}\geq 0.\end{aligned} \tag{38}$$

(v) Note that $d_G(v_1)+d_G(x)=k+4, r_G(v_1,x)=(p-1)/p$, and $d_{G'}(v_1)+d_{G'}(x)=k+p+1$, $r_{G'}(v_1,x)=2/3$, for $x\in\{v_2,v_p\}$ in G and G', respectively; then, we have

$$\begin{aligned}&\sum_{x\in\{v_2,v_p\}}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}-\sum_{x\in\{v_2,v_p\}}\frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)}\\&\quad=3(k+p+1)-\frac{2p(k+4)}{p-1}=\frac{(k+3p+1)(p-3)}{p-1}.\end{aligned} \tag{39}$$

(vi) Note that $d_G(v_1)+d_G(x)=k+4, r_G(v_1,x)=r_G(v_1,v_i)\geq 1\,(3\leq i\leq p-1)$ for $x\in V(C_p)\backslash\{v_1,v_2,v_p\}$ and $d_{G'}(v_1)+d_{G'}(x)=k+p, r_{G'}(v_1,x)=1$ for $x\in\widetilde{W}$; then, we have

$$\sum_{x\in\widetilde{W}}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}-\sum_{x\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)}$$

$$\geq(p-3)(k+p)-(p-3)(k+4)=(p-3)(p-4). \tag{40}$$

(vii) Note that $d_G(x)+d_G(y)=3$ for $x\in V(C_p)\setminus\{v_1,v_2,v_p\}$, $y\in\{v_2,v_p\}$ and $d_{G'}(x)+d_{G'}(y)=4$, $r_{G'}(x,y)=r_{G'}(v_2,v_1)+1=(2/3)+1=(5/3)$ for $x\in\widetilde{W}$, $y\in\{v_2,v_p\}$; by the symmetry between v_2 and v_p, we have

$$\sum_{\substack{x\in\widetilde{W}\\ y\in\{v_2,v_p\}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{\substack{x\in V(C_p)\setminus\{v_1,v_2,v_p\}\\ y\in[v_2,v_p]}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}$$

$$=2\sum_{i=3}^{p-1}\frac{3}{r_{G'}(v_2,v_1)+1}-2\sum_{i=3}^{p-1}\frac{4}{r_G(v_2,v_i)}$$

$$=\frac{18(p-3)}{5}-\left(\frac{8p}{p-1}+2\sum_{i=4}^{p-1}\frac{4}{r_G(v_2,v_i)}\right). \tag{41}$$

Since $r_G(v_2,v_i)\geq r_G(v_2,v_4)=((2(p-2))/p)(4\leq i\leq p-1)$, if $((2(p-2))/p)\geq(5/3)$, then $p\geq 12$.

Then,

$$2\sum_{i=4}^{p-1}\frac{4}{r_G(v_2,v_i)}\Bigg)\leq\frac{24(p-4)}{5}. \tag{42}$$

Thus,

$$\sum_{\substack{x\in\widetilde{W}\\ y\in\{v_2,v_p\}}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{\substack{x\in V(C_p)\setminus\{v_1,v_2,v_p\}\\ y\in[v_2,v_p]}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}$$

$$\geq\frac{18(p-3)}{5}-\left(\frac{8p}{p-1}+\frac{24(p-4)}{5}\right). \tag{43}$$

(viii) Note that $d_G(x)+d_G(y)=4$, $r_G(x,y)=r_G(v_i,v_j)(3\leq i<j\leq p-1)$ for $x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}$ and $d_{G'}(x)+d_{G'}(y)=2$, $r_{G'}(x,y)=2$ for $x,y\in\widetilde{W}$; then, we have

$$\sum_{x,y\in\widetilde{W}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}$$

$$=\frac{(p-3)(p-4)}{2}-\sum_{i,j=3\,i<j}^{p-1}\frac{4}{r_G(v_i,v_j)}. \tag{44}$$

Then,

$$\sum_{i,j=3\,i<j}^{p-1}\frac{4}{r_G(v_i,v_j)}=\sum_{i=4}^{p}\frac{d_G(v_3)+d_G(v_i)}{r_G(v_3,v_i)}+\sum_{i=5}^{p}\frac{d_G(v_4)+d_G(v_i)}{r_G(v_4,v_i)}$$

$$+\cdots+\frac{d_G(v_{p-2})+d_G(v_{p-1})}{r_G(v_{p-2},v_{p-1})}. \tag{45}$$

When $j-i=t\,(i<j)$, $r_G(v_i,v_j)=r_t=r_{p-t}$; thus,

$$\begin{aligned}
\sum_{i=4}^{p}\frac{d_G(v_3)+d_G(v_i)}{r_G(v_3,v_i)}&=\frac{4}{r_1}+\frac{4}{r_2}+\cdots+\frac{4}{r_{p-4}},\\
\sum_{i=5}^{p}\frac{d_G(v_4)+d_G(v_i)}{r_G(v_4,v_i)}&=\frac{4}{r_1}+\cdots+\frac{4}{r_{p-5}},\\
&\cdots\\
\frac{d_G(v_{p-2})+d_G(v_{p-1})}{r_G(v_{p-2},v_{p-1})}&=\frac{4}{r_1},
\end{aligned} \tag{46}$$

and

$$\sum_{x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}=\frac{4(p-4)}{r_1}+\frac{4(p-5)}{r_2}$$

$$+\cdots+\frac{4}{r_{p-4}}. \tag{47}$$

Since $r_1=((p-1)/p)$, $r_t\geq r_2=(2(p-2))/p\,(2\leq t\leq p-4)$, if $(2(p-2))/p\geq(5/3)$, then $p\geq 12$.

Then,

$$\sum_{x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\leq\frac{4p(p-4)}{p-1}$$

$$+\frac{6(p-4)(p-5)}{5}. \tag{48}$$

Thus,

$$\sum_{x,y\in\widetilde{W}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}$$
$$\geq\frac{(p-3)(p-4)}{2}-\left(\frac{4p(p-4)}{p-1}+\frac{6(p-4)(p-5)}{5}\right). \tag{49}$$

When $p\geq 12$, combing (v) − (viii), we have

$$\begin{aligned}&\left(\sum_{x\in\{v_2,v_p\}}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}-\sum_{x\in\{v_2,v_p\}}\frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)}\right)\\&+\left(\sum_{x\in\widetilde{W}}\frac{d_{G'}(v_1)+d_{G'}(x)}{r_{G'}(v_1,x)}-\sum_{x\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(v_1)+d_G(x)}{r_G(v_1,x)}\right)\\&+\left(\sum_{x\in\widetilde{W}\,y\in\{v_2,v_p\}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{\substack{x\in V(C_p)\setminus\{v_1,v_2,v_p\}\\ y\in\{v_2,v_p\}}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\right)\\&+\left(\sum_{x,y\in\widetilde{W}}\frac{d_{G'}(x)+d_{G'}(y)}{r_{G'}(x,y)}-\sum_{x,y\in V(C_p)\setminus\{v_1,v_2,v_p\}}\frac{d_G(x)+d_G(y)}{r_G(x,y)}\right)\\&\geq\frac{(k+3p+1)(p-3)}{p-1}+\frac{3(p-3)(p-4)}{2}+\frac{18(p-3)}{5}-\frac{4p(p-2)}{p-1}-\frac{6(p-1)(p-4)}{5}.\\&=\frac{3p^3-22p^2+33p-54+k(10p-30)}{10(p-1)}.\end{aligned} \tag{50}$$

Let $f(p)=3p^3-22p^2+33p-54$, $p\in[12,+\infty)$; this function is strictly increasing in this interval; thus, $f(p)\geq f(12)=2358>0$.

Hence,

$$\frac{3p^3-22p^2+33p-54+k(10p-30)}{10(p-1)}>0. \tag{51}$$

(ix) Note that $d_G(x)+d_G(y)=4$, $r_G(v_2,v_p)=((2(p-2))/p)$ for $x\in\{v_2,v_p\}$ in G and $d_{G'}(x)+d_{G'}(y)=4$, $r_{G'}(v_2,v_p)=(2/3)$ for $x\in\{v_2,v_p\}$ in G'; then, we have

$$\frac{d_{G'}(v_2)+d_{G'}(v_p)}{r_{G'}(v_2,v_p)}-\frac{d_G(v_2)+d_G(v_p)}{r_G(v_2,v_p)}=6-\frac{2p}{p-2}=\frac{4p-12}{p-2}>0. \tag{52}$$

Thus, by (i)–(ix), when $p\geq 12$, we get $\mathrm{RDR}(G')-\mathrm{RDR}(G)>0$.

When $4\leq p\leq 11$, $U_n(G_0)=\{S_n^4,S_n^5,S_n^6,S_n^7,S_n^8,S_n^9,S_n^{10},S_n^{11}\}$. By comparison of direct calculation, we can obtain $\mathrm{RDR}(S_n^3)>\mathrm{RDR}(S_n^p)$.

The proof is completed. □

3. Maximum Reciprocal Degree Resistance Distance Index of Unicyclic Graphs

Let G be a connected graph with exactly one cycle, say $C_p = v_1 v_2 \cdots v_p v_1$. G can be viewed as a graph obtained by coalescing C_p with a number of trees, denoted by $G_1, G_2, \ldots, G_p$, by identifying v_i with some vertex of G_i for all $i\,(1 \le i \le p)$. Denote S_n^3 by the unicyclic graph obtained from cycle C_3 by attaching $n-3$ pendant edges to a vertex of C_3. The next theorem discusses the maximum reciprocal degree resistance distance index in $\mathcal{U}_n$, which is the main result of our work.

Theorem 1. *Let G be a unicyclic graph of order n; then, $RDR(G) \le RDR(S_n^3)$, and the equality holds if and only if $G \cong S_n^3$.*

Proof. Let G be a unicyclic graph of order n such that $\xi(G)$ is as big as possible. By Lemmas 2–4, we have $\mathrm{RDR}(G) \le \mathrm{RDR}(S_n^3)$; the equality holds if and only if $G \cong S_n^3$. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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