Research Article

Lower Bound for the Blow-Up Time for the Nonlinear Reaction-Diffusion System in High Dimensions

Baiping Ouyang,1 Wei Fan,2 and Yiwu Lin3

1College of Data Science, Huashang College Guangdong University of Finance & Economics, Guangzhou 511300, China
2The School of Gifted Young, University of Science and Technology of China, Hefei, China
3Department of Applied Mathematics, Guangdong University of Finance, Guangzhou 510521, China

Correspondence should be addressed to Yiwu Lin; 26-062@gduf.edu.cn

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In this paper, we study the blow-up phenomenon for a nonlinear reaction-diffusion system with time-dependent coefficients under nonlinear boundary conditions. Using the technique of a first-order differential inequality and the Sobolev inequalities, we can get the energy expression which satisfies the differential inequality. A lower bound for the blow-up time could be obtained if blow-up does really occur in high dimensions.

1. Introduction

During the past decades, the blow-up phenomena for the solutions to the parabolic problems have been widely concerned. It is important in practice that how to determine the bound of the blow-up time $t^*$ of the solutions about the parabolic equations and systems. Their applications are included in physics, chemistry, astronomy, biology, and population dynamics [1, 2]. Actually, when the blow-up occurs at $t^*$, it is difficult to get the exact value of $t^*$. We mainly focus on estimating its bounds. At present, the studies on the blow-up phenomena of parabolic problems mainly focus on homogeneous Dirichlet boundary condition and homogeneous Neumann and Robin boundary conditions [3–12]. There are also some works under nonlinear boundary conditions [13–15]. Most of these articles are focused on $\mathbb{R}^2$. There are only a few papers dealing with a lower bound for the blow-up time in high dimensions (see [16–18]). Recently, some scholars have started to investigate the blow-up problems with time-dependent coefficients [19–21]. In paper [21], the authors considered the following nonlinear reaction-diffusion system with time-dependent coefficients:

\begin{align}
    u_t &= \Delta u + k_1(t)u^p v^q, \quad (x, t) \in \Omega \times (0, t^*), \\
    v_t &= \Delta v + k_2(t)v^p u^q, \quad (x, t) \in \Omega \times (0, t^*), \\
    u(x, t) &= v(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, t^*), \\
    u(x, 0) &= u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{align}

The authors obtained the lower and upper bounds for the blow-up time when the blow-up occurred. In this paper, we further consider the blow-up phenomena for the following system with time-dependent coefficients under nonlinear boundary conditions in high dimensions:

\begin{align}
    u_t &= \Delta u^m + k_1(t)u^p v^q, \\
    v_t &= \Delta v^l + k_2(t)v^p u^q, \\
    \frac{\partial u}{\partial n} &= g_1(u), \quad \frac{\partial v}{\partial n} = g_2(v), \\
    (x, t) &\in \partial \Omega \times (0, t^*), \\
    u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{align}
We assume that $g_i(\zeta)$ are continuous, and $\alpha, \beta, p, q, r, s, a_1, a_2, m, l, \sigma$, and $\zeta$ satisfy

\[ g_i(\zeta) \leq a_1 \zeta^\alpha, \quad g_2(\zeta) \leq a_2 \zeta^\beta, \]

\[ \zeta > 0, \quad \alpha > 1, \]

\[ \beta > 1, \quad p > 1, \quad q > 0, \quad r > 1, \quad s > 0, \quad a_1 > 0, \quad a_2 > 0, \]

\[ \max \left\{ \frac{\alpha(p+q)\sigma}{\sigma-q} - \frac{2\alpha}{n+1}, 1 \right\} < m \frac{\alpha(p+q)-q}{\sigma-q}, \]

\[ \max \left\{ \frac{\alpha(r+s)\sigma}{\sigma-s} - \frac{2\alpha}{n+1}, 1 \right\} < \frac{\alpha(r+s)-s}{\sigma-s}, \]

(3)

where $\sigma$ is a positive constant to be defined later.

Our goal in this paper is to obtain a lower bound for the blow-up time of the solutions to systems (2) and (3) in $\mathbb{R}^n$ for any $n > 3$. The nonlinear terms $\Delta u^m$ and $\Delta v^l$ and the boundary conditions are difficult to tackle. We cannot get the result by following the method proposed in [21], so we must use a new method to overcome these difficulties. To the best of our knowledge, no results exist in that direction, and we think our result is new and interesting.

In the further discussions, we will use the following Hölder inequality:

\[ \int_\Omega w^{x_1+x_2} \, dx \leq \left( \int_\Omega w^{x_1/n_1} \, dx \right)^{1/n_1} \left( \int_\Omega w^{x_2/n_2} \, dx \right)^{1/n_2}, \]

(4)

where $w$ is a nonnegative function and $x_1, x_2, n_1$, and $n_2$ are positive constants satisfying $(1/n_1) + (1/n_2) = 1$.

We also need the following Sobolev inequality [22]:

\[ \int_\Omega u^{(\sigma+1)n/(n-2)} \, dx \leq C \eta^{-2n/(n-2)} \left( \int_\Omega |\nabla u|^{2(n-2)/n} \, dx \right)^{n/(n-2)}, \]

(5)

\[ \int_\Omega v^{(\sigma+1)n/(n-2)} \, dx \leq C \eta^{-2n/(n-2)} \left( \int_\Omega |\nabla v|^{2(n-2)/n} \, dx \right)^{n/(n-2)}, \]

(6)

with $C = C(n, \Omega)$ which is a Sobolev embedding constant depending on $n$ and $\Omega$.

And the classical (or elementary) inequality is

\[ (a+b)^w \leq a^w + b^w, \]

(7)

where $a$, $b$, and $w$ are positive constants, and $w$ satisfies $0 < w \leq 1$.

2. Lower Bound for the Blow-Up Time

In this part, we define an auxiliary function of the form

\[ \phi(t) = k^2(t) \int_\Omega u^p \, dx + k^2(t) \int_\Omega \phi^q \, dx, \]

(8)

where $\delta = (n-n+2\alpha)/(2\sigma(p+q-1))$, $\rho = (\ln n+2\alpha)/2\sigma(r+s-1)$, and $\sigma > \max\{(a-1)n, (\beta-1)n, q, s\}$.

We establish the following theorem:

**Theorem 1.** Let $u(x, t)$ be the weak solution of problems (1)–(3) in a bounded convex domain $\Omega (\Omega \in \mathbb{R}^n(n > 3))$. Then, the quantity $\phi(t)$ defined in (8) satisfies the integral inequality

\[ \Theta(t^*) \geq \int_\phi \frac{1}{\eta + \eta^{\xi_1} + \eta^{\xi_2} + \eta^{\xi_3} + \eta^{\xi_4}} = S, \]

(9)

which follows that the blow-up time $t^*$ is bounded below. We have

\[ t^* \geq \Theta^{-1}(S), \]

(10)

where $\Theta, \xi_1, \xi_2, \xi_3$, and $\xi_4$ will be defined later.

Now, we prove Theorem 1. For simplicity, assume that the solution is classical of problems (1)–(3). The general case can be done by approximation. Differentiating $\phi(t)$, we have

\[ \phi^\prime(t) = \delta k_{11}(t)k_{11}(t) \int_\Omega u^{p-1} \, dx + \chi k_{22}(t)k_{22}(t) \int_\Omega v^{p-1} \, dx \]

\[ + \chi k_{22}^\prime(t)k_{22}(t) \int_\Omega v^{p-1} \, dx \]

\[ + L \phi(t) + \sigma k_{22}^\prime(t) \int_\Omega u^{p-1} \, dx + \sigma k_{22}^\prime(t) \int_\Omega v^{p-1} \, dx \]

\[ + \sigma k_{22}^\prime(t) \int_\Omega v^{p-1} \, dx \]

(11)

where $L = \max\{\delta k_{11}(t)/k_{11}(t), \chi k_{22}^\prime(t)/k_{22}(t)\}$.

For the second term on the right side of (11), we apply the divergence theorem, the $L^1$ trace embedding, and (3) to get
\[
\int_{\Omega} u^{\sigma-1} \nabla u^m \, dx = \int_{\partial \Omega} u^{\sigma-1} \frac{\partial u^m}{\partial n} \, dA - \int_{\Omega} \nabla u^{\sigma-1} \cdot \nabla u^m \, dx
\]
\[
\leq m \int_{\partial \Omega} u^{\sigma+m-3} \frac{\partial u^m}{\partial n} \, dA - m(\sigma - 1) \int_{\Omega} u^{\sigma+m-3} |\nabla u|^2 \, dx
\]
\[
\leq m \alpha_1 \int_{\partial \Omega} u^{\sigma+m-2} \, dA - m(\sigma - 1) \int_{\Omega} u^{\sigma+m-3} |\nabla u|^2 \, dx
\]
\[
\leq \frac{ma_1 n}{\rho_0} \int_{\Omega} u^{\sigma+m-2} \, dx + \frac{ma_1 (\sigma + m - 2)d}{\rho_0} \int_{\Omega} u^{\sigma+m-3} |\nabla u| \, dx
\]
\[
- \frac{4m(\sigma - 1)}{(\sigma + m - 1)^2} \int_{\Omega} |\nabla u^{\sigma+m-1/2}|^2 \, dx,
\]
where \( \rho_0 = \min_{\partial \Omega} |x \cdot \vec{n}|, \vec{n} \) is the outward normal vector of \( \partial \Omega \) and \( d = \max_{\partial \Omega} |x| \).

For the second term on the right side of (12), using (11), we obtain
\[
\int_{\Omega} u^{\sigma+m-3} |\nabla u| \, dx \leq \frac{1}{2\epsilon_1} \int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx + \frac{2\epsilon_1}{(\sigma + m - 1)^2} \cdot \int_{\Omega} |\nabla u|^{(\sigma+m-1)/2} \, dx,
\]
(13)

where \( \epsilon_1 \) is a positive constant which will be defined later. Using (4), we have
\[
\int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx \leq \left( \int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx \right)^{x_{11}} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{x_{20}}
\]
\[
\leq x_{10} \int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx + x_{20} \int_{\Omega} u^{\sigma} \, dx,
\]
(14)

where \( x_{10} = (m + m - 2)/(m + 2\alpha - 3) \) and \( x_{20} = (\alpha - 1)/(m + 2\alpha - 3) \).

Choosing \( x_{11} = (m + 2\alpha - 2)/(m - 1)n + 2\sigma \) and \( x_{21} = ((m - 1)n + 2\sigma - (m + 2\alpha - 2)(n - 2))/((m - 1)n + 2\sigma) \) and using (4), (5), and (7), we have
\[
\int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx \leq \left( \int_{\Omega} u^{(\sigma+m-1)/n(2-n)} \, dx \right)^{x_{11}} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{x_{21}}
\]
\[
\leq \frac{r_1 nx_{11}}{n - 2} \int_{\Omega} u^{\sigma+m-1} \, dx
\]
\[
+ \frac{r_1 (n - 2 - nx_{11})}{n - 2} \left(1 + \epsilon_2^{-n_{x_{11}+1}/(n-2-nx_{11})}\right)
\]
\[
\cdot \left( \int_{\Omega} u^{\sigma} \, dx \right)^{x_{11}(n-2)}/(n-2-nx_{11}) + \frac{r_1 nx_{11}}{n - 2} \epsilon_2
\]
\[
\cdot \int_{\Omega} |\nabla u|^{(\sigma+m-1)/2} \, dx,
\]
(15)

where \( r_1 = (C^{2\alpha}(n-2)2(n/(n-2))^{n-1})^{x_{11}} \) and \( \epsilon_2 \) is a positive constant which will be defined later.

For the first term on the right side of (15), using (4) and Young’s inequality, we have
\[
\int_{\Omega} u^{\sigma+m-1} \, dx \leq \left( \int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx \right)^{x_{11}} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{x_{22}}
\]
\[
\leq x_{12} \|u\|_\infty \int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx + x_{22} \|u\|_\infty \int_{\Omega} u^{\sigma} \, dx,
\]
(16)

where \( x_{12} = (m - 1)/(m + 2\alpha - 3) \), \( x_{22} = (2\alpha - 2)/(m + 2\alpha - 3) \), and \( \epsilon_3 \) is a positive constant which will be defined later.

Combining (15) and (16), if we choose suitable \( \epsilon_3 \) such that \( r_1 nx_{11} x_{12} \|u\|_\infty/(n - 2) = 1/2 \), we have
\[
\int_{\Omega} u^{\sigma+m+2\alpha-3} \, dx \leq r_2 \int_{\Omega} u^{\sigma} \, dx + r_3 \left( \int_{\Omega} u^{\sigma} \, dx \right)^{(x_{11}(n-2))/(n-2-nx_{11})}
\]
\[
+ r_4 \int_{\Omega} |\nabla u|^{(\sigma+m-1)/2} \, dx,
\]
(17)

where \( r_2 = (2r_1 nx_{11}/(n - 2)) x_{21} \|u\|_\infty^{-((n-1)/(2\alpha-2))} \), \( r_3 = (2r_1 (n - 2 - nx_{11}))/((n - 2)(1 + \epsilon_2^{nx_{11}/(n-2-nx_{11})})) \), and \( r_4 = (2r_1 nx_{11}/(n - 2)) \|u\|_\infty \).

Combining (12), (13), (14), and (17), we have
\[
\int_{\Omega} u^{\sigma-1} \nabla u^m \, dx \leq \left( r_2 r_5 + \frac{ma_1 n x_{11}}{\rho_0} \right) \int_{\Omega} u^{\sigma} \, dx + r_5 r_3
\]
\[
\cdot \left( \int_{\Omega} u^{\sigma} \, dx \right)^{x_{11}(n-2)}/(n-2-nx_{11})
\]
\[
+ r_6 \int_{\Omega} |\nabla u|^{(\sigma+m-1)/2} \, dx,
\]
(18)

where \( r_5 = (ma_1 n x_{11}/\rho_0) + (ma_1 (\sigma + m + 2\alpha - 2)d)/2\epsilon_1 \rho_0 \) and \( r_6 = (ma_1 (\sigma + m + 2\alpha - 2)d)/2\epsilon_1 \rfloor(\sigma + m - 1)^2 - (4m(\sigma - 1))/(\sigma + m - 1)^2 \).

Similarly, for the fourth term on the right side of (11), using the divergence theorem and (3), we have
\[
\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx = \int_\Omega \nabla \cdot \mathbf{v} \, dx \leq \frac{l_0}{\beta_0} \int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \, dx
\]

where \( \frac{l_0}{\beta_0} \) is a positive constant which will be defined later.

For the first term on the right side of (19), we have

\[
\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx \leq \frac{l_0}{\beta_0} \int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \, dx
\]

and using (4), (6), and (7), we have

\[
\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx \leq \left( \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{v} \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \mathbf{v} \cdot \mathbf{v} \, dx \right)^{\frac{1}{2}}
\]

Similarly, we have

\[
\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx \leq \frac{l_0}{\beta_0} \int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \, dx + \frac{2\epsilon_4}{\sigma + l - 1} \int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx
\]

where \( \epsilon_4 \) is a positive constant which will be defined later.

Combining (22) and (23), if we choose suitable \( \epsilon_6 \) such that \( r_1 n_y_{11} y_{12} \epsilon_6 / (n - 2) = 1/2 \), we have

\[
\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx \leq \left( r_{11} r_8 + \frac{l_0 n y_{20}}{\beta_0} \right) \int_\Omega \mathbf{v} \cdot \mathbf{v} \, dx + r_{11} r_9
\]

where \( r_{11} = (l + \beta - 2)/(l + 2\beta - 3) \) and \( y_{20} = (\beta - 1)/(l + 2\beta - 3) \).

For the third term on the right side of (11), using Hölder inequality and Young's inequality, we have

\[
\sigma k_{11}^{-\delta_1}(t) \int_\Omega u^{(\sigma p - 1)\alpha} \, dx \leq \int_\Omega u^{(\sigma p - 1)\alpha} (u^{(\sigma - q)\alpha}) \, dx \leq \sigma k_{11}^{-\delta_1}(t) \int_\Omega u^{(\sigma p - 1)\alpha} (u^{(\sigma - q)\alpha}) \, dx
\]

where \( \sigma k_{11}^{-\delta_1}(t) \) is a positive constant which will be defined later.

For the first term on the right side of (26), using (4), (5), and (7) and taking care of the given condition \( \delta = ((\sigma - q)(nm - n + 2\sigma)) / (2\sigma(p + q - 1)) \), we have

\[
\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial t^2} \, dx \leq \left( \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{v} \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \mathbf{v} \cdot \mathbf{v} \, dx \right)^{\frac{1}{2}}
\]
where $\lambda_1 = (C^{2n}(n-2)(n/(n-2)-1)x_{13}$, $x_{13} = (\sigma(p+q-1)/(n-2))/((\sigma-q)(mn-n+2\sigma))$, and $x_{23} = ((\sigma-q)(mn-n+2\sigma)-\sigma(p+q-1)/(n-2))/((\sigma-q)(mn-n+2\sigma))$.

For the first term on the right side of (27), using (4) and Young’s inequality, we get

$$\int_{\Omega} \nabla u^{(m-1)} dx \leq \chi_{13} \int \nabla u^{(m-1)} dx + \lambda_1 \chi_{13} \int \nabla u^{(m-1)} dx,$$

(28)

where $\chi_{13} = ((m-1)(\sigma-q))/((\sigma-q)(\sigma+q-1))$, $x_{24} = (\sigma(\sigma+q-1)-(m-1)(\sigma-q))/((\sigma(\sigma+q-1))$, and $\epsilon_f$ is a positive constant which will be defined later.

Combining (27) and (28), if we choose suitable $\epsilon_f$ such that $(\lambda_1 nx_{13}^{\epsilon_f}/(n-2)) = 1/2$, we have

$$(\sigma-q)k_1^{(t)}(t) \int \nabla u^{(m-1)} dx \leq K_1(t)k_1^{(t)}(t) \int \nabla u^{(m-1)} dx + K_2(t)(k_1^{(t)}(t) \int \nabla u^{(m-1)} dx)^{\epsilon_f},$$

$$+ \lambda_1 k_1^{(t)}(t) \int \nabla u^{(m-1)} dx,$$

(29)

where $K_1(t) = (2(\sigma-q)\lambda_1 nx_{13}^{\epsilon_f}/(n-2))e^{-(x_{13}/x_{13})} k_1^{(t)}(t)$, $K_2(t) = (2(\sigma-q)\lambda_1(n-2-nx_{13}))/\sigma(e_f(\sigma+q)/\sigma(\sigma+q-1))$.
where \( K(t) = K(t) + \dot{K}(t) + K(t) + K(t) + K(t) \).

Integrating (37) from 0 to \( t \), we have

\[
\Theta(t) = \int_0^t K(\alpha) d\alpha.
\]

(38)

The authors declare that they have no conflicts of interest.

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Data Availability

No data were used to support this study.

Conflicts of Interest

The proof of Theorem 1 is complete.

Combining (13), (18), (25), (30), and (35), we have

\[
\Theta(t) \geq \int_0^t \left( u + \dot{u} + u + \dot{u} + u + \dot{u} \right) d\alpha.
\]

(39)

Considering \( \Theta(t) \geq 1 \) and \( t \leq 0 \), the integration of the right side of (39) exists. It is clear that \( \Theta(t) \) is an increasing function. So, we can get

\[
\int_0^t \left( u + \dot{u} + u + \dot{u} + u + \dot{u} \right) d\alpha = S.
\]

(40)

For the first term on the right side of (32), using (4), we get

\[
\int_0^t \left( u + \dot{u} + u + \dot{u} + u + \dot{u} \right) d\alpha = S.
\]

Combining (32), (33), and (36), we obtain

\[
\int_0^t \left( u + \dot{u} + u + \dot{u} + u + \dot{u} \right) d\alpha = S.
\]
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