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Research Article

How to Contract a Vertex Transitive 5-Connected Graph

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M.Kriesell conjectured that there existed b, h such that every 5-connected graph G with at least b vertices can be contracted to a 5-connected graph G_0 such that $0 < |V(G)| - (V(G_0)) < h$. We show that this conjecture holds for vertex transitive 5-connected graphs.

1. Introduction

All graphs considered here are supposed to be simple, finite, and undirected graphs. For a connected graph G, a subset $T \subseteq V(G)$ is called a smallest separator if $|T| = \kappa(G)$ and $G - T$ has at least two components. Let G be a k-connected graph, and let H be a subgraph of G. Let G/H stand for the graph obtained from G by contracting every component of H to a single vertex and replacing each resulting double edges by a single edge. A subgraph H of G is said to be k-contractible if G/H is still k-connected. An edge e is a k-contractible edge if G/e is k-connected; otherwise, we call it a noncontractible edge. Clearly, two end-vertices of a noncontractible edge are contained in some smallest separator. A k-connected graph without a k-contractible edge is said to be a contraction-critical k-connected graph.

Tutte's [1] wheel theorem showed that every 3-connected graph on more than four vertices contains a 3-contractible edge. For $k \geq 4$, Thomassen and Toft [2] showed that there were infinitely many contraction-critical k-regular k-connected graphs. On the other hand, one can find that every 4-connected graph can be reduced to a smaller 4-connected graph by contracting at most two edges. Therefore, Kriesell [3] posted the following conjecture.

Conjecture 1 (see [3]). *There exists $b(k)$, $h(k)$ such that every k-connected graph G with at least $b(k)$ vertices can be contracted to a k-connected graph G_0 such that $0 < |V(G)| - |V(G_0)| < h(k)$.*

Clearly, Conjecture 1 is true for $k \leq 4$. By Kriesell's examples [3], Conjecture 1 fails for $k \geq 6$. Hence, it is still open for $k = 5$.

A smallest separator T of a k-connected graph is said to be trivial if $G - T$ has exactly two components and one of them has exactly one vertex. A 5-connected graph G is essentially a 6-connected graph if every smallest separator of G is trivial. In ([3]), Kriesell proved the following results.

Theorem 1 (see [3]). *Every essentially 6-connected graph G with at least 13 vertices can be contracted to a 5-connected graph H such that $0 < |V(G)| - |V(H)| < 5$.*

In this paper, we will show that Conjecture 1 is true for vertex transitive 5-connected graphs. Clearly, Conjecture 1 holds for 5-connected graphs which contain a contractible edge. Hence, in order to show that Conjecture 1 holds for vertex transitive 5-connected graphs, we have to show that all vertex transitive contraction-critical 5-connected graphs have a small contractible subgraph. So, the key point of this paper is to characterize the local structure of a vertex transitive contraction-critical 5-connected graph and, then, to find the contractible subgraph of it. In the following, for convenience, a vertex transitive contraction-critical 5-connected graph will be called a TCC-5-connected graph. For a contraction-critical 5-connected graph, there are some results on the local structure of it [4–10].

To state our results, we need to introduce some further definitions. Let G be a 5-connected graph which is 5-regular. For any $x \in V(G)$, we say that x has one of the following four types according the graph induced by the neighborhood of x(see Figures 1(a)–1(d)).

(i) Type 1: $G[N(x)] \cong K_2 \cup K_3$

(ii) Type 2: $G[N(x)] \cong C_5$

(iii) Type 3: $G[N(x)] \cong K_2 \cup P_2$

(iv) Type 4: $G[N(x)] \cong P_4$

Moreover, for $i \in \{1, 2, 3, 4\}$, G has *type i* if every vertex of G has type i.

Furthermore, we need to introduce the graph G^* (see Figure 1(e)). One can check that G^* is vertex transitive, and G^* can be reduced to K_6 by contracting yx_1 and xx_2.

First, we have the following results on the local structure of TCC-5-connected graphs.

Theorem 2. *Let G be a TCC-5-connected graph. If $|V(G)| \le 9$, then either $G \cong K_6$ or $G \cong G^*$.*

Theorem 3. *Let G be a TCC-5-connected graph. If $|V(G)| \ge 10$, then G has type 1, type 2, type 3, or type 4.*

Theorem 4. *Let G be a TCC-5-connected graph. If G has type 2, then G is isomorphic to icosahedron.*

Then, we will prove the following main result of the paper.

Theorem 5. *Let G be a 5-connected vertex transitive graph which is neither K_6 nor icosahedron, and then, G can be contracted to a 5-connected G' such that $0 < |V(G)| - |V(G')| < 3$.*

The organization of the paper is as follows. Section 2 contains some preliminary results. In Section 3, we will characterize the local structure of 5-connected TCC-graphs. In Section 4, we will prove Theorem 5.

2. Terminology and Lemma

For terms not defined here, we refer the reader to [11]. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. Let $\mathrm{Aut}(G)$ denote the automorphism group of G, and let $\kappa(G)$ denote the vertex connectivity of G. Let P_n denote a path on n vertices. An edge joining vertices x and y will be written as xy. Let $[xy]$ stand for the new vertex obtained by contracting the edge xy. For $x \in V(G)$, we define $N_G(x) = \{x \mid xy \in E(G)\}$. For $F \subseteq V(G)$, we define $N_G(F) = \cup_{x \in F} N_G(x) - F$. Furthermore, let $G[F]$ denote the subgraph induced by F, and let $G - F$ denote the graph obtained from G by deleting all the vertices of F together with the edges incident with them. Let $\partial(F)$ stands for the set of edge with one end in F and the other end in $G - F$.

Let T be a smallest separator of a noncomplete connected G, and the union of at least one but not of all components of $G - T$ is called a T-fragment. A fragment of G is a T-fragment for some smallest separator T. Let F be a T-fragment, and let $\overline{F} = V(G) - (F \cup T)$. Clearly, $\overline{F} \neq \emptyset$, and $\overline{F}$ is also a T-fragment such that $N_G(F) = T = N_G(\overline{F})$. A fragment with least cardinality is called an atom. For $N_G(x)$, $d_G(x)$, and $N_G(F)$, we often omit the index G if it is clear from the context.

Furthermore, we need some special terminologies for 5-connected graphs. Let A be a fragment of G, and let $S = N(A)$. Let $x \in S$, and $y \in N(x) \cap A$. A vertex z is said to be an admissible vertex of $(x, y; A)$ if both of the following two conditions hold.

$$\begin{gathered} z \in N(x) \cap N(y) \cap S \cap V_5(G), \\ |N(z) \cap A| \ge 2. \end{gathered} \tag{1}$$

A vertex z is said to be an admissible vertex of $(x; A)$, if z is an admissible vertex of $(x, y; A)$ for some $y \in N(x) \cap A$. Let $Ad(x, y; A)$(resp. $Ad(x; A)$) stand for the set of admissible vertices of $(x, y; A)$(resp. $(x; A)$). Let e be an edge of G, and a fragment A is said to be a fragment with respect to e if $V(e) \subseteq N(A)$.

The following properties of fragment are well known (for the proof, see [12]), and we will use them without any further reference.

Lemma 1 (see [12]). *Let F and F' be two distinct fragments of $G; T = N(F), T' = N(F')$. Then, the following statements hold.*

(1) If $F \cap T' \neq \emptyset$, then $|F \cap T'| \ge |\bar{F'} \cap T|, |T' \cap T| \ge |\overline{F} \cap T'|$

(2) If $F \cap T' \neq \emptyset$ and $F \cap T'$ is not a fragment of G, then $\overline{F} \cap \overline{F'} = \emptyset$ and $|F \cap T'| > |\overline{F'} \cap T|, |T' \cap T| > |\overline{F} \cap T'|$

(3) If $F \cap T' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, then both $F \cap T'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cap T') = (T' \cap T) \cup (T' \cap T) \cup (F \cap T')$

Lemma 2 (see [4]). *Let G be a k-connected graph, and A is a fragment of G. Let $B \subseteq N(A)$. If $|N(B) \cap A| < |B|$, then $A = N(B) \cap A$.*

Lemma 3 (see [5]). *Let G be a contraction-critical 5-connected graph, and then, G contains a vertex x such that every edge incident with x is contained in some triangle.*

Lemma 4 (see [6]). *Let G be a contraction-critical 5-connected graph. Let $x \in V(G)$, and A be a fragment such that $x \in N(A)$, $|A| \ge 3$, and $|\overline{A}| \ge 2$. If $|N(x) \cap A| = 1$, then $Ad(x; A) \neq \emptyset$.*

Lemma 5 (see [7]). *Let A be a fragment of a contraction-critical 5-connected graph such that $|A| = 2$, and let t_1, t_2 be two vertices of $N(A)$ such that $|N(t_1) \cap A| = |N(t_2) \cap A| = 1$. Then, either $Ad(t_1; A) \neq \emptyset$ or $Ad(t_2; A) \neq \emptyset$.*

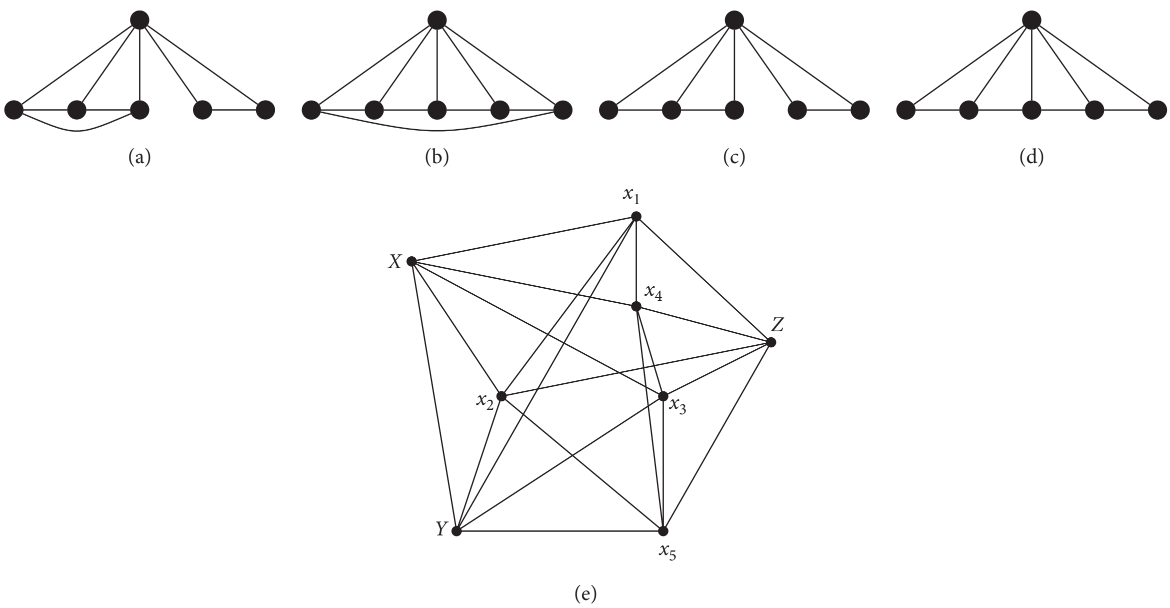

Figure 1: The local structure induced by a neighborhood of a vertex: (a) Type 1, (b) type 2, (c) type 3, (d) type 4, and (e) G^*.

Lemma 6. *Let G be a vertex transitive connected graph, and then, for any two vertices x and y, $G[N(x)] \cong G[N(y)]$.*

Proof. Since G is a vertex transitive graph, there exist $g \in \mathrm{Aut}(G)$ such that $x^g = y$. It follows that $(N(x))^g = N(y)$. Hence, $g|_{N(x)}$ is an isomorphic of $G[N(x)]$ and $G[N(y)]$, where $g|_{N(x)}$ is the restriction of g on $N(x)$. □

Lemma 7. *Let $p \geq 2$ be a prime integer, and let G be a vertex transitive graph with $\kappa(G) = p$; then, G is a p-regular graph.*

Proof. To the contrary, we may assume that $\delta(G) > p$ since $\delta(G) \geq \kappa(G)$. It follows that every atom of G has at least two vertices. Since G is a vertex transitive graph, then every vertex of G is contained in some atom.

First, we show that any two atoms of G are disjoint. Otherwise, let A and B be two distinguished atoms of G such that $A \cap B \neq \varnothing$. By the definition of atom, $A \cap B$ is not a fragment. Lemma 1 assures us that $\overline{A} \cap \overline{B} = \varnothing$ and $|A \cap N(B)| > |\overline{B} \cap N(A)|$. This implies that $|A| > |\overline{B}|$, a contradiction. Thus, any two atoms of G are disjoint.

Let A and C be two atoms of G such that $C \cap N(A) \neq \varnothing$. It follows that $A \cap C = \varnothing$. We show that $C \subseteq N(A)$. Otherwise, suppose $C \cap \overline{A} \neq \varnothing$. If $C \cap \overline{A}$ is a fragment of G, then we see that $|C \cap \overline{A}| < |C|$, since $C \cap N(A) \neq \varnothing$. This contradicts the definition of atom. So, $C \cap \overline{A}$ is not a fragment of G. Lemma 1 assures us that $A \cap \overline{C} = \varnothing$ and $|C \cap N(A)| > |A \cap N(C)| = |A|$. It follows that $|C| > |A|$, a contradiction. Hence, $C \subseteq N(A)$. Therefore, $N(A)$ is the disjoint union of some atom, since any two atoms of G are disjoint and every vertex of G is contained in some atom. This means that $|A|$ is a subdivision of $|N(A)|$, and hence, $|A| = p$. It follows that $N|A| = C$. By symmetry, we see that $N(C) = A$, which implies that $\overline{A} = \varnothing$, a contradiction. □

Lemma 8. *Let G be a TCC-5-connected graph. If G does not contain K_4 as subgraph, then for any $x \in V(G)$, $\Delta(G[N(x)]) \leq 2$.*

Proof. Clearly, Lemma 7 assures us that G is 5-regular, which implies that G has an even order. Suppose that $x \in V(G)$ with $N(x) = \{x_1, \ldots, x_5\}$ such that x_1 adjacent to at least three vertices of $N(x)$. Let $A = \{x\}$, and it follows $N(A) = N(x)$. If $G - A - N(A) = \varnothing$, then G has six vertices. It follows that $G \cong K_6$, which implies G contains K_4, a contradiction. Hence, we may assume that $G - A - N(A) \neq \varnothing$. It follows that A is a fragment of G. By symmetry, we assume that $\{x_2, x_3, x_4\} \subseteq N(x_1)$. Now, we observe that $N(x_1) \cap N(A) = \{x_2, x_3, x_4\}$ and $|N(x_1) \cap \overline{A}| = 1$. Let $N(x_1) \cap \overline{A} = \{t_1\}$. Let $B = \{x, x_1\}$, and then $N(B) = \{x_2, \ldots, x_5, t_1\}$. Now, the fact that $|V(G)|$ is even assures us that $G - B - N(B) \neq \varnothing$. It follows that B is a fragment of G. Furthermore, we see that $N(t_1) \cap B = \{x_1\}$ and $N(x_5) \cap B = \{x\}$.Now, Lemma 5 assures us that either $\mathrm{Ad}(t_1; B) \neq \varnothing$ or $\mathrm{Ad}(x_5; B) \neq \varnothing$. If $\mathrm{Ad}(x_5; B) \neq \varnothing$, then without loss of generality, assume $x_2 \in \mathrm{Ad}(x_5; B)$. Therefore, $G[N(x)]$ is a connected graph. If $\mathrm{Ad}(t_1; B) \neq \varnothing$, then, similarly, we have that $G[N(x)]$ is a connected graph. Now, since G is vertex transitive, the following claims hold. □

Claim 1. For any $t \in V(G)$, $G[N(t)]$ is a connected graph.

Claim 2. For any $t \in V(G)$, $G[N(t)]$ contains a cycle of length 4.

Proof. Since G is a vertex transitive graph, we only show that $N(x_1)$ has a cycle of length 4. By Claim 1, we see that for $y \in N(B)$, $N(y) \cap N(B) \neq \varnothing$. On the other hand, we observe that $G[\{x_2, x_3, x_4\}] \cong \overline{K}_3$, since G does not contain K_4. This implies that every member of $\{x_2, x_3, x_4\}$ is either adjacent to t_1 or x_5. It follows that $|N(t_1) \cap \{x_2, x_3, x_4\}| \geq 2$ or

$|N(x_5) \cap \{x_2, x_3, x_4\}| \geq 2$. By symmetry, we may assume that $|N(t_1) \cap \{x_2, x_3, x_4\}| \geq 2$. It follows that $G[N(x_1)]$ contains a cycle of length 4. Hence, for any $t \in V(G)$, $G[N(t)]$ has a cycle of length 4.

Now, we are ready to complete the proof of Lemma 8. By Claim 2, we see that $|N(x_i) \cap N(B)| \geq 2$ for $i \in \{2, 3, 4\}$. This implies that $\{t_1, x_5\} \subseteq N(x_2) \cap N(x_3) \cap N(x_4)$. Now, we see that every vertex of $N(B)$ is adjacent to exactly one vertex of $\overline{A}$. If $|\overline{B}| = 1$, say $\overline{B} = \{t\}$, then $G[N(t)] \cong K_{2,3}$ and $G[N(x_2)] \cong C_5$, a contradiction. If $|\overline{B}| = 2$, we can observe that $\overline{B}$ has a vertex with a degree of at most 4, a contradiction. Hence, we may assume that $|\overline{B}| = 3$. Now, Lemma 4 shows that $\mathrm{Ad}(x_2; \overline{B}) \neq \emptyset$, which implies $|N(z) \cap \overline{A}| \geq 2$ for some $z \in N(B)$, a contradiction. □

Lemma 9. *Let G be a TCC-5-connected graph. If G has type 4, then G is essentially 6-connected.*

Proof. Since G has type 4, we see that for any $x \in V(G)$, $\Delta(G[N(x)]) \leq 2$. □

Claim 3. If A is a fragment of G, then $|A| \neq 2$.

Proof. Suppose $A = \{x, y\}$. If $xy \in E(G)$, then x has three neighbors in $G[N(y)]$, a contradiction. So, we may assume $xy \notin E(G)$. It follows that $G[N(A)] \cong P_5$. Let $x_1x_2x_3x_4x_5$ be the path of $G[N(A)]$. It follows that $|N(x_2) \cap \overline{A}| = |N(x_3) \cap \overline{A}| = |N(x_4) \cap \overline{A}| = 1$. If $|\overline{A}| \geq 3$, then Lemma 4 implies that $\mathrm{Ad}(x_3, \overline{A}) \neq \emptyset$. Hence, either $x_2 \in \mathrm{Ad}(x_3, \overline{A})$ or $x_4 \in \mathrm{Ad}(x_3, \overline{A})$. This is a contradiction, since $|N(x_2) \cap \overline{A}| = |N(x_4) \cap \overline{A}| = 1$. Hence, we may assume that $|\overline{A}| \leq 2$. If $|\overline{A}| = 1$, then we see that $d(x_1) < 5$, a contradiction. So, we may assume $|\overline{A}| = 2$. It follows that $|V(G)| = 9$, which contradicts the fact that G has an even order. Hence, Claim 3 holds. □

Claim 4. If A is a fragment of G, then $|\overline{A}| \neq 3$.

Proof. We first show that $G[A]$ is a connected graph. Otherwise, let A_1 be a component of $G[A]$ such that A_1 has exactly one vertex. It follows that $A_2 = A - A_1$ is a fragment of cardinality 2, a contradiction. Next, we show that $G[A]$ is a path. Suppose $G[A]$ is a cycle, then a simple calculation shows that $|\partial(A)| = 9$. This implies that one vertex of $N(A)$, say w, has exactly one neighbor in A. Now, we find that $A - N(w)$ is a fragment of cardinality 2, a contradiction.

Let xyz be the path of $G[A]$, and let $N(A) = \{x_1, \ldots, x_5\}$. Without the loss of generality, let $N(y) = \{x, x_1, x_2, x_3, z\}$.

Subclaim 1. $|N(x) \cap \{x_1, x_2, x_3\}| = 2$ and $|N(z) \cap \{x_1, x_2, x_3\}| = 2$.

Proof. Notice that G has type 4; we find that $|N(x) \cap \{x_1, x_2, x_3\}| \leq 2$. If $|N(x) \cap \{x_1, x_2, x_3\}| \leq 1$, then we find that $d(x) \leq 4$, a contradiction. Hence, $|N(x) \cap \{x_1, x_2, x_3\}| = 2$. By symmetry, $|N(z) \cap \{x_1, x_2, x_3\}| = 2$.

Without the loss of generality, we may assume that $\{x_1, x_2\} \subseteq N(x)$. Now, if $\{x_1, x_2\} \subseteq N(z)$, then xx_1zx_2x is a cycle of $G[N(y)]$, a contradiction. Therefore, $\{x_2, x_3\} \subseteq N(z)$, which implies that $\{x_4, x_5\} \subseteq N(x) \cap N(z)$.

Subclaim 2. $G[\{x_1, x_2, x_3\}] \cong \overline{K_3}$.

Proof. If $x_1x_2 \in E(G)$, then $N(y)$ has a triangle, a contradiction. It follows that $x_1x_2 \notin E(G)$. Similarly, we have $x_2x_3 \notin E(G)$. Furthermore, if $x_1x_3 \notin E(G)$, then we find that there is a cycle of length four in $N(y)$, a contradiction. Thus, $x_1x_3 \notin E(G)$. It follows that $G[\{x_1, x_2, x_3\}] \cong \overline{K_3}$.

Now, we are ready to complete the proof of Claim 4. Focusing on x_2, we find that $N(x_2) \cap N(A) \neq \emptyset$ since $G[N(x_2)]$ is connected. By Subclaim 2, we may assume that $x_4 \in N(x_2)$. Now, we find that there is a cycle of length four in $N(x_2)$, a contradiction. □

Claim 5. For every smallest separator T, $G - T$ has exactly two components.

Proof. Otherwise, assume that $G - T$ has at least three components. Let A_1, A_2, and A_3 be three connected components of $G - T$.

Subclaim 3. For any $y \in T$, $|N(y) \cap T| = 2$, and $|N(y) \cap A_i| = 1, i \in \{1, 2, 3\}$.

Proof. Let $y \in T$, let $N(y) = \{y_1, \ldots, y_5\}$. Without the loss of generality, we may assume that $y_i \in A_i, i \in \{1, 2, 3\}$. Now, we find that $N(y) \cap T \neq \emptyset$, since G has type 4. Suppose $y_4 \in N(y) \cap T$. If $N(y) \cap T = \{y_4\}$, then the fact that $G[N(y)]$ is connected shows that y_4 has three neighbors in $G[N(y)]$, which contradicts the fact that G has type 4. So, we have $N(y) \cap T = \{y_4, y_5\}$. It follows that $|N(y) \cap A_i| = 1, i \in \{1, 2, 3\}$ and $|N(y) \cap T| = 2$.

By Subclaim 3, $\delta(G[T]) = 2$, which implies that $G[T]$ is a cycle of length 5. Hence, we see that $|A_i| \neq 1, i \in \{1, 2, 3\}$. Furthermore, by Subclaims 3 and 4, $|A_i| \neq 3$ for each $i = 1, 2, 3$. Focusing on A_1, we find that $\overline{A}_1 = A_2 \cup A_3$, which implies that $|\overline{A}_1| \geq 6$. Recall that $|N(x) \cap A_1| = 1$, and Lemma 4 shows that $\mathrm{Ad}(x; A_1) \neq \emptyset$. Without the loss of generality, assume that $y \in \mathrm{Ad}(x; A_1)$. This implies that $|N(y) \cap A_1| \geq 2$, which contradicts Subclaim 3. Hence, Claim 5 holds.

Next, we assume that G is not essentially 6-connected. It follows that there is a fragment B such that $|B| \geq 2$ and $|\overline{B}| \geq 2$. Let $\mathscr{B} = \{B \mid B$ is a fragment such that $|B| \geq 2$ and $|\overline{B}| \geq 2\}$, and let $t = \min\{|B| \mid B \in \mathscr{B}\}$. By Claims 3 and 4, we see that $t \geq 4$. Let $\mathscr{B}_1 = \{B \mid B \in \mathscr{B}$ and $|B| = t\}$. Let $A \in \mathscr{B}_1$, and let $y \in N(A)$. Now, since G is vertex transitive, every vertex of G is contained in some member of $\mathscr{B}_1$. Therefore, there exist $B \in \mathscr{B}_1$ such that $y \in B$. Next, we will analyse the local structure of A and B. □

Claim 6. If $A \cap B \neq \emptyset$, then $\overline{A} \cap \overline{B} \neq \emptyset$.

Proof. Suppose $A \cap B \neq \varnothing$ and $\overline{A} \cap \overline{B} = \varnothing$. Now, Lemma 1 assures us that $|A \cap N(B)| \geq |\overline{B} \cap N(A)|$. It follows that

$$\begin{aligned} |A| &= |A \cap B| + |A \cap N(B)| + |A \cap \overline{B}| \\ &\geq |A \cap B| + |\overline{B} \cap N(A)| + |A \cap \overline{B}| \\ &> |\overline{B} \cap N(A)| + |A \cap \overline{B}| = |\overline{B}|. \end{aligned} \tag{2}$$

This contradicts the choice of A. □

Claim 7. $A \cap \overline{B} \neq \varnothing$ if and only if $\overline{A} \cap B \neq \varnothing$.

Proof. Suppose $A \cap \overline{B} \neq \varnothing$. Now, Lemma 1 assures us that $|A \cap N(B)| \geq |B \cap N(A)|$. If $\overline{A} \cap B = \varnothing$, then we see that

$$\begin{aligned} |A| &= |A \cap B| + |A \cap N(B)| + |A \cap \overline{B}| \\ &\geq |A \cap B| + |B \cap N(A)| + |A \cap \overline{B}| \\ &> |A \cap B| + |B \cap N(A)| = |B|. \end{aligned} \tag{3}$$

This contradicts the choice of A. Hence, we see that $\overline{A} \cap B \neq \varnothing$. By symmetry, we see that $\overline{A} \cap B \neq \varnothing$ implies $A \cap \overline{B} \neq \varnothing$. □

Claim 8. $A \cap B = \varnothing$.

Proof. Suppose $A \cap B \neq \varnothing$. By Claim 6, we know that $\overline{A} \cap \overline{B} \neq \varnothing$. Hence, $A \cap B$ is a fragment of G. By the choice of A, we know that $|A \cap B| = 1$. Furthermore, since $\overline{A} \cap \overline{B} \neq \varnothing$, Lemma 1 assures us that $|\overline{B} \cap N(A)| \geq |A \cap N(B)|$.

If $A \cap \overline{B} = \varnothing$, then Claim 7 assures us that $\overline{A} \cap B = \varnothing$. Furthermore, $|A| = |B| \geq 4$ implies that $|A \cap N(B)| = |B \cap N(A)| \geq 3$. Hence, we find that $|N(A)| \geq |B \cap N(A)| + |\overline{B} \cap N(A)| \geq |B \cap N(A)| + |A \cap N(B)| \geq 6$, a contradiction.

So, we may assume $A \cap \overline{B} \neq \varnothing$. Then, Claim 7 assures us that $\overline{A} \cap B \neq \varnothing$. Hence, both $A \cap \overline{B}$ and $B \cap \overline{A}$ are fragments of G. By the choice of A and B, we know that $|A \cap \overline{B}| = |B \cap \overline{A}| = 1$. It follows that $|A \cap N(B)| = |B \cap N(A)|$.

Since $\overline{A} \cap \overline{B} \neq \varnothing$, Lemma 1 assure us that $|\overline{B} \cap N(A)| \geq |A \cap N(B)|$. If $|A \cap N(B)| \geq 3$, then $|N(A)| \geq |B \cap N(A)| + |\overline{B} \cap N(A)| \geq 6$, a contradiction.

Therefore, let $|A \cap N(B)| = 2$. It follows that $|A \cap N(B)| = |B \cap N(A)| = 2$. Since $\overline{A} \cap \overline{B} \neq \varnothing$ and $A \cap B \neq \varnothing$, Lemma 1 assures us that $|\overline{A} \cap N(B)| = |\overline{B} \cap N(A)| = 2$. This implies that $|N(B) \cap N(A)| = 1$. Let $N(B) \cap N(A) = \{t\}$. Now, $N(t) \cap A \cap B \neq \varnothing$, since $A \cap B$ is a fragment. Similarly, we find that $N(t) \cap A \cap \overline{B} \neq \varnothing$, $N(t) \cap \overline{A} \cap \overline{B} \neq \varnothing$, and $N(t) \cap \overline{A} \cap B \neq \varnothing$. Now, we find that $G[N(t)]$ has at least two components, a contradiction. □

Claim 9. $A \cap \overline{B} = \varnothing$ and $B \cap \overline{A} = \varnothing$.

Proof. Suppose $A \cap \overline{B} \neq \varnothing$. By Claim 7, we see that $\overline{A} \cap B \neq \varnothing$. Hence, both $A \cap \overline{B}$ and $\overline{A} \cap B$ are fragments of G. By the choice of A, we see that $|A \cap \overline{B}| = |\overline{A} \cap B| = 1$. It follows that $|A \cap N(B)| = |B \cap N(A)| \geq 3$.

If $\overline{A} \cap \overline{B} \neq \varnothing$, then $|\overline{B} \cap N(A)| \geq |A \cap N(B)|$. Hence, we see that $|N(A)| \geq |B \cap N(A)| + |\overline{B} \cap N(A)| \geq 6$, a contradiction.

Hence, we may assume that $\overline{A} \cap \overline{B} = \varnothing$. Then, by the choice of B, we know that $|\overline{B}| \geq |B|$. It follows that $|\overline{B} \cap N(A)| \geq 3$. Hence, we find that $|N(A)| \geq |B \cap N(A)| + |\overline{B} \cap N(A)| \geq 6$, a contradiction.

Now, we are ready to complete the proof of the Lemma. By Claims 8 and 9, we find that $A = A \cap N(B)$ and $B = B \cap N(A)$. Now, we find that $|\overline{B} \cap N(A)| \leq |N(A)| - |B \cap N(A)| \leq 1$, since $|B \cap N(A)| = |B| \geq 4$. It follows that $|\overline{B} \cap N(A)| < |A \cap N(B)|$. Now, Lemma 1 implies that $\overline{A} \cap \overline{B} = \varnothing$. It follows that $|\overline{B}| = |\overline{B} \cap N(A)| \leq 1$, a contradiction. □

3. The Local Structure of TCC-5-Connected Graphs

In this section, since Lemma 7 holds, all TCC-5-connected graphs were supposed to be 5-regular and have an even order.

Theorem 6. *Let G be a TCC-5-connected graph. If $|V(G)| \leq 9$, then either $G \cong K_6$ or $G \cong G^*$.*

Proof. Recall that G has an even order. It follows that either $|V(G)| = 6$ or $|V(G)| = 8$. If $|V(G)| = 6$, then $G \cong K_6$. So, we may assume $|V(G)| = 8$. It follows that G has a fragment of cardinality 2. Let $A = \{x, y\}$ be a fragment of G. Let $N(A) = \{x_1, x_2, x_3, x_4, x_5\}$, and let $\overline{A} = \{z\}$. □

Claim 10. $xy \in E(G)$.

Proof. Otherwise, we find that $G[N(A)] \cong C_5$. It follows that $G[N(x)] \cong C_5$. On the other hand, $G[N(x_1)] \cong \overline{K_3} \times \overline{K_2}$. It follows that $G[N(x)] \cong G[N(x_1)]$, a contradiction.

Let $N(x) = \{x_1, x_2, x_3, x_4, y\}$. By symmetry, we may assume that $N(y) = \{x_1, x_2, x_3, x_5, x\}$. We find that y has at least three neighbors in $G[N(x)]$. Hence, Lemma 8 implies that G contains K_4. It follows that $G[N(x)]$ contains a triangle. □

Claim 11. $x_4x_5 \in E(G)$.

Proof. Suppose $x_4x_5 \notin E(G)$. Notice that for $i \in \{4, 5\}$, $|N(x_i) \cap (A \cup \overline{A})| = 2$, we see that $\{x_4, x_5\} \subseteq N(x_1) \cap N(x_2) \cap N(x_3)$. Therefore, $G[\{x_1, x_2, x_3\}] \cong \overline{K_3}$, which implies that $G[N(x)]$ does not contain a triangle, a contradiction.

Now, we observe that $G[N(A)] - x_4x_5$ is 2-regular. Hence, $G[N(A)] - x_4x_5$ is a cycle of length 5. Now, by symmetry, we may assume that $\{x_2, x_3\} \subseteq N(x_4)$ and $\{x_1, x_2\} \subseteq N(x_5)$. It follows that $x_1x_3 \in E(G)$ since $G[N(A)] - x_4x_5$ is a cycle of length 5. Therefore, we have $G \cong G^*$. □

Lemma 10. *Let G be a TCC-5-connected graph with $|V(G)| \geq 10$. If G contains K_4 as a subgraph, then G has type 1.*

Proof. Since G is a vertex transitive graph, we know that every vertex of G is contained in a K_4. Let x be a vertex of G, and let $N(x) = \{x_1, \ldots, x_5\}$. Without the loss of generality, let $G[\{x, x_1, x_2, x_3\}] \cong K_4$. □

Claim 12. $|N(x_i) \cap \{x_1, x_2, x_3\}| \leq 1, i \in \{4, 5\}$.

Proof. We only show that $|N(x_4) \cap \{x_1, x_2, x_3\}| \leq 1$, and the other one can be handled similarly. Otherwise, by symmetry, we may assume $\{x_1, x_2\} \subseteq N(x_4)$. Let $N(x_1) = \{x, x_2, x_3, x_4, x_1\}$. Let $A = \{x, x_1\}$, and it follows that $N(A) = \{x_2, x_3, x_4, x_5, t_1\}$. Furthermore, recall that $|V(G)| \geq 10$, $G - A - N(A) \neq \varnothing$. If $t_1 = x_5$, then $N(A)$ is a separator of order 4, a contradiction. Thus, $t_1 \neq x_5$. Therefore, A is a fragment of G. Furthermore, since $|V(G)| \geq 10$, we see that $|\overline{A}| \geq 3$. Let $B = \{x, x_2\}$, and let $N(x_2) = \{x, x_2, x_3, x_4, t_2\}$, where $t_2 \in N(x_2) \cap \overline{A}$. Similarly, B is a fragment of G such that $N(B) = \{x_1, x_3, x_4, x_5, t_2\}$. Furthermore, we have $|\overline{A}| = |\overline{B}|$.

Notice that $t_2 \in \overline{A}$ and $t_1 \in N(A)$, we see that $t_1 \neq t_2$. Now, we see that $A \cap B = \{x\}$, $A \cap N(B) = \{x_1\}$, $B \cap N(A) = \{x_2\}$, $N(A) \cap N(B) = \{x_3, x_4, x_5\}$, $\overline{A} \cap N(B) = \{t_2\}$, and $\overline{B} \cap N(A) = \{t_1\}$.

Now, since $|\overline{A}| = |\overline{B}| \geq 3$, we find that $|\overline{A} \cap \overline{B}| \geq 2$. Let $C = A \cup B$. Clearly, C is a fragment with $\overline{C} = \overline{A} \cap \overline{B}$. Notice that $N(t_1) \cap C = \{x_1\}$, $N(t_2) \cap C = \{x_2\}$, and $N(x_5) \cap C = \{x\}$. Now, Lemma 4 implies that $\mathrm{Ad}(t_1; C) \cup \mathrm{Ad}(t_2; C) \cup \mathrm{Ad}(x_5; C) \subseteq \{x_3, x_4\}$. It follows that either x_3 or x_4 has two neighbors in $N(C)$. By symmetry, let x_3 have two neighbors in $N(C)$. It follows that $d(x_3) \geq 6$, a contradiction. Hence, Claim 12 holds. □

Claim 13. $N(x_i) \cap \{x_1, x_2, x_3\} = \varnothing, i \in \{4, 5\}$.

Proof. We only show that $N(x_4) \cap \{x_1, x_2, x_3\} = \varnothing$, and the other one can be handled similarly. Otherwise, by symmetry, we may assume that $x_4x_1 \in E(G)$. Now, by Claim 12, $N(x_4) \cap \{x_2, x_3\} = \varnothing$. Let $A = \{x, x_1\}$ and $N(x_1) = \{x, x_2, x_3, x_4, t_1\}$. Clearly, $N(A) = \{x_2, x_3, x_4, x_5, t_1\}$ and $G - A - N(A) \neq \varnothing$. Now, since G is 5-connected, we observe that $t_1 \neq x_5$. Therefore, A is a fragment of G.

Subclaim 4. $N(x_5) \cap \{x_2, x_3\} = \varnothing$.

Proof. Suppose $x_2x_5 \in E(G)$, and then, $\{x, x_2\}$ is fragments of G. Furthermore, we see that $G[N(x)]$ is a connected graph, and this implies that for any $t \in V(G)$, $G[N(t)]$ is a connected graph. Let $B = \{x, x_2\}$, and it follows $N(B) = \{x_1, x_3, x_4, x_5, t_2\}$, where $t_2 \in N(x_2) - \{x, x_1, x_3, x_5\}$. Now, since G is 5-connected, we see that $t_2 \neq x_4$. We observe that $A \cap B = \{x\}$, $A \cap N(B) = \{x_1\}$, $B \cap N(A) = \{x_2\}$, $N(A) \cap N(B) = \{x_3, x_4, x_5\}$, $\overline{A} \cap N(B) = \{t_2\}$, and $\overline{B} \cap N(A) = \{t_1\}$. Furthermore, we see that $A \cap \overline{B} = \varnothing$ and $B \cap \overline{A} = \varnothing$. Notice that $G[N(x)]$ is connected, and we see that for every vertex t of G, $G[N(t)]$ is connected.

Now, since $A \cap \overline{B} = \varnothing$ and $\overline{B} \cap N(A) = \{t_1\}$, the fact $|\overline{A}| = |\overline{B}| \geq 3$ shows that $|\overline{A} \cap \overline{B}| \geq 2$. Furthermore, $\overline{A} \cap \overline{B}$ is a fragment.

If $|N(x_3) \cap \overline{A} \cap \overline{B}| \geq 2$, then $G[N(x_3)]$ has at least two components, a contradiction. Therefore, $|N(x_3) \cap (\overline{A} \cap \overline{B})| = 1$ and $N(x_3) \cap N(\overline{A} \cap \overline{B}) \neq \varnothing$.

On the other hand, by Claim 12, $N(x_4) \cap \{x_1, x_2, x_3\} = \{x_1\}$, and $N(x_5) \cap \{x_1, x_2, x_3\} = \{x_2\}$. This fact implies that either $x_3t_1 \in E(G)$ or $x_3t_2 \in E(G)$.

By symmetry, we may assume $x_3t_1 \in E(G)$. Now, since $N(x_3) \cap \overline{A} \cap \overline{B} \neq \varnothing$, we see that $G[N(x_3)]$ has only one vertex of degree 3. On the other hand, we find that $G[N(x)]$ has two vertex of degree 3, and this implies that $G[N(x_3)] \cong G[N(x)]$, a contradiction. This contradiction shows that $x_2x_5 \notin E(G)$. By symmetry, $x_3x_5 \notin E(G)$. Hence, Subclaim 4 holds.

Subclaim 5. $x_4x_5 \notin E(G)$.

Proof. Suppose $x_4x_5 \notin E(G)$. Let $P\prime$ be a graph which is got from the path $x_3x_2x_1x_4x_5$ by adding the edge x_1x_3. Clearly, $G[N(x)] \cong P\prime$. Now, since $G[N(x)] \cong G[N(x_1)]$, we find that $x_4t_1 \in E(G)$. This implies that $|N(x_4) \cap \overline{A}| = 1$. Let $N(x_4) \cap \overline{A} = \{t_4\}$. Furthermore, $G[N(x_4)]$ has a triangle, since $G[N(x)]$ has a triangle. Therefore, $G[\{t_1, t_4, x_5\}]$ is a triangle. It follows that $G[N(x_4)]$ has a Hamilton cycle. This implies that $G[N(x)] \cong G[N(x_4)]$, a contradiction. Thus, Subclaim 5 holds.

By Subclaims 4 and 5, $G[N(x)]$ has two components, and one of them has exactly one vertex. If $|N(x_4) \cap \overline{A}| = 3$, then $G[N(x)] \cong G[N(x_4)]$, a contradiction. So, assume that $|N(x_4) \cap \overline{A}| \leq 2$, which implies that $N(x_4) \cap N(A) \neq \varnothing$. By Claim 12 and Subclaim 5, $N(x_4) \cap N(A) = \{t_1\}$. Now, we see that K_4 which contains x_4 is contained in $N(A) \cup \overline{A}$. Hence, $G[N(x_4)]$ is a connected graph, a contradiction. Thus, Claim 13 holds.

By Claim 13 and Lemma 3, $x_4x_5 \in E(G)$, and hence, x has type 1. Therefore, G has type 1. □

Theorem 7. *Let G be a TCC-5-connected graph. If $|V(G)| \geq 10$, then G has type 1, type 2, type 3, or type 4.*

Proof. If G contains K_4, then Lemma 10 assures us that G has type 1. So, we may assume that G does not contain K_4. Hence, Lemma 8 assures us that for any $x \in V(G)$, $\Delta(G[N(x)]) \leq 2$. Now, Lemma 3 assures us that G has either type 2 or type 3 or type 4. □

Theorem 8. *Let G be a TCC-5-connected graph. If G has type 2, then G is isomorphic to icosahedron.*

Proof. Let $N(x) = \{x_1, \ldots, x_5\}$, and let $x_1x_2 \cdots x_5x_1$ be the cycle of $G[N(x)]$. Furthermore, let $N(x_3) = \{x_2, x, x_4, y_1, y_2\}$. Since G has type 2, we may assume that $x_2xx_4y_2y_1x_2$ is a cycle of $G[N(x_3)]$. Let $N(x_2) = \{x_1, x, x_3, y_1, y_3\}$, and then $y_3 \neq y_2$ and $y_3 \notin \{x_1, \ldots, x_5\}$. Now, $x_1xx_3y_1y_3x_1$ is a cycle of $G[N(x_2)]$. Let $N(x_4) = \{x_3, x, x_5, y_4, y_2\}$. If $y_4 = y_3$, then, since G has type

2, $\{x_1, x_2, x_4, x_5, y_1, y_2\} \subseteq N(y_3)$. This implies that $d(y_3) \geq 6$, a contradiction. Hence, we have $y_4 \neq y_3$. Since G does not contain K_4, we see that $y_4 \notin \{x_1, x_2, y_1\}$. Now, we observe that $x_3 x x_5 y_4 y_2 x_3$ is the cycle of $G[N(x_4)]$. Let $N(y_2) = \{x_3, x_4, y_4, z, y_1\}$, then, similarly, we can show that $z \notin \{x_1, \ldots, x_5, y_1, y_2, y_3, y_4\}$ and $\{y_1, y_2, y_3, y_4\} \subseteq N(z)$. Let $N(z) = \{y_1, y_2, y_3, y_4, w\}$, and then $w \notin \{x_1, \ldots, x_5, y_1, y_2, y_3, y_4\}$ and $N(w) = \{y_1, x_1, x_5, y_4, z\}$. It follows that G is icosahedron. □

4. Proof of Theorem 5

Since G is 5-regular, we see that $|V(G)|$ is even. If G has a contractible edge, then we are done. Therefore, in the rest of the paper, we may assume that G is a contraction-critical 5-connected graph. Hence, by Theorem 1, we can assume that $|V(G)| \geq 10$. By Theorem 2, we see that G has type 1, type 2, type 3, or type 4. Next, we complete the proof of Theorem 5 by showing that the following lemmas are true.

Lemma 11. *Let G be a TCC-5-connected-graph. Let $x \in V(G)$, abc be a path of $G[N(x)]$, and $G_0 = G/\{xa, bc\}$. If $G[N(x) - \{a, b, c\}] \cong K_2$, then $\kappa(G_0) \geq 4$.*

Proof. Suppose $\kappa(G_1) \leq 3$. Let T_1 be a smallest separator of G_1, and let A_1 be a T_1-fragment. Clearly, $|T_1| = 3$ and $\{[xa], [bc]\} \subseteq T_1$. Let $T = T_1 \cup \{x, a, b, c\} - \{[xa], [bc]\}$. Clearly, $|T| = 5$ and $\{x, a, b, c\} \subseteq T$. Furthermore, $A = A_1$ is a fragment of G such that $N(A) = T$. Since $G[N(x)] - \{a, b, c\}$ is a complete graph, either $N(x) \cap A = \emptyset$ or $N(x) \cap \overline{A} = \emptyset$, a contradiction. Hence the lemma holds. □

Lemma 12. *Let G be a TCC-5-connected graph such that $|V(G)| \geq 10$. If G has type 1, then G can be contracted to a 5-connected H such that $0 < |V(G)| - |V(H)| < 3$.*

Proof. By the definition of type 1, we know that G contains K_4 as a subgraph. Since G is vertex transitive graph, every vertex of G is contained in some K_4. Let x be a vertex of G, and let $N(x) = \{x_1, \ldots, x_5\}$. Furthermore, without the loss of generality, suppose $G[\{x, x_1, x_2, x_3\}] \cong K_4$.

Since G has type 1, we may let $N(x_1) = \{x, x_2, x_3, y_1, w_1\}$, $N(x_2) = \{x, x_1, x_3, y_2, w_2\}$, and $N(x_3) = \{x, x_1, x_2, y_3, w_3\}$. Clearly, $x_4, x_5, y_1, w_1, y_2, w_2, y_3$, and w_3 are all different to each other since G has type 1.

Let $G_1 = G/\{xx_1, x_2x_3\}$, and let $G_2 = G/\{xx_2, x_1x_3\}$. Now, we see that $\delta(G_1) \geq 5$ and $\delta(G_2) \geq 5$, since $x_4, x_5, y_1, w_1, y_2, w_2, y_3$, and w_3 are all different to each other.

If either $\kappa(G_1) = 5$ or $\kappa(G_2) = 5$, then we are done. So, by Lemma 11, we may assume that $\kappa(G_1) = 4$ and $\kappa(G_2) = 4$. Let T_1 be a smallest separator of G_1, and let A_1 be a T_1-fragment. Since $\delta(G_1) \geq 5$, we see that $|A_1| \geq 2$ and $|\overline{A_1}| \geq 2$. Furthermore, we can observe that $T_1 \cap \{[xx_1], [x_2x_3]\} \neq \emptyset$. □

Claim 14. $\{[xx_1], [x_2x_3]\} - T_1 \neq \emptyset$.

Proof. Suppose $\{[xx_1], [x_2x_3]\} \subseteq T_1$. Let $T = T_1 \cup \{x, x_1, x_2, x_3\} - \{[xx_1], [x_2x_3]\}$. It follows that $|T| = 6$ and $\{x, x_1, x_2, x_3\} \subseteq T$. Furthermore, $G - T = A_1 \cup \overline{A_1}$. Recall that $x_4x_5 \in E(G)$; then, either $N(x) \cap A_1 = \emptyset$ or $N(x) \cap \overline{A_1} = \emptyset$. Without loss of generality, we may assume that $N(x) \cap A_1 = \emptyset$. Then, $A = A_1$ is a fragment of G such $N(A) = T - \{x\}$. It follows that $|\overline{A}| \geq |\overline{A_1}| + 1 \geq 3$. Since $N(x_1) \cap A \neq \emptyset$, we see that $\{y_1, w_1\} \subseteq A \cup N(A)$. It follows that $N(x_1) \cap \overline{A} = \{x\}$. Similarly, $N(x_2) \cap \overline{A} = \{x\}$. Hence, $N(\{x_1, x_2\}) \cap \overline{A} = \{x\}$. Now, Lemma 2 assures us that $|\overline{A}| = 1$, a contradiction.

By Claim 14, without the loss of generality, let $[x_2x_3] \in T_1$ and $[xx_1] \in A$.

Let $T = T_1 \cup \{x_2, x_3\} - \{[x_2x_3]\}$, and then, $|T| = 5$ and $\{x_2, x_3\} \subseteq T$. Furthermore, $A = (A_1 - [xx_1]) \cup \{x, x_1\}$ is a T-fragment and $\overline{A} = \overline{A_1}$. Clearly, $|A| \geq 3$ and $|\overline{A}| \geq 2$.

Similarly, we may assume G has a fragment B such that $\{x, x_2\} \subseteq B$ and $\{x_1, x_3\} \subseteq N(B)$. Furthermore, we may assume that $|B| \geq 3$ and $|\overline{B}| \geq 2$.

Focusing on A and B, we see that $x \in A \cap B$, $x_2 \in N(A) \cap B$, $x_1 \in N(B) \cap A$, and $x_3 \in N(A) \cap N(B)$. If $N(x_3) \cap (B \cap \overline{A}) \neq \emptyset$, then, since $y_3w_3 \in E(G)$, we see that $N(x_3) \cap \overline{B} = \emptyset$, a contradiction. Hence, we may assume $N(x_3) \cap (B \cap \overline{A}) = \emptyset$. By symmetry, let $N(x_3) \cap (A \cap \overline{B}) = \emptyset$. □

Claim 15. $B \cap \overline{A} = \emptyset$ and $\overline{B} \cap A = \emptyset$.

Proof. Suppose $B \cap \overline{A} \neq \emptyset$. Since $N(x_3) \cap (B \cap \overline{A}) = \emptyset$, Lemma 1 assures us that $\overline{B} \cap A = \emptyset$ and $|N(A) \cap B| > |N(B) \cap A|$. If $\overline{B} \cap \overline{A} \neq \emptyset$, then $A \cap B$ is a fragment. On the other hand, we find that $N(x_1) \cap A \cap B = N(x_3) \cap A \cap B = \{x\}$. Now, Lemma 2 assures us that $A \cap B = \{x\}$ and $|N(B) \cap A| \geq 2$. Thus, $|N(A) \cap B| \geq |N(B) \cap A| + 1 \geq 2 + 1 = 3$. It follows that $|N(A) \cap \overline{B}| \leq 1$. Now, Lemma 1 assures us that $\overline{B} \cap \overline{A} = \emptyset$, a contradiction. Hence, $\overline{B} \cap \overline{A} = \emptyset$. Now, we find that $\overline{B} = \overline{B} \cap N(A)$. This implies that $|\overline{B} \cap N(A)| = |\overline{B}| \geq 2$. Since $A \cap B \neq \emptyset$ and $B \cap \overline{A} \neq \emptyset$, Lemma 1 implies that $|N(B) \cap A| \geq 2$ and $|N(B) \cap \overline{A}| \geq 2$. Hence, we see that $|N(B) \cap A| = |N(B) \cap \overline{A}| = 2$, which implies that $(B \cap \overline{A})$ is a fragment of G. It follows that $N(x_3) \cap (B \cap \overline{A}) \neq \emptyset$, a contradiction. Hence, we have $B \cap \overline{A} = \emptyset$, and, similarly, $\overline{B} \cap A = \emptyset$.

If $\overline{B} \cap \overline{A} \neq \emptyset$, then Lemma 1 assures us that $A \cap B$ is a fragment of G. Since every vertex of G has type 1, we see that $N(x_1) \cap A \cap B = N(x_3) \cap A \cap B = \{x\}$. Now, Lemma 2 assure us that $A \cap B = \{x\}$. This implies that $|N(B) \cap A| \geq 2$ and $|N(A) \cap B| \geq 2$. Now, Lemma 1 assures us that $|N(B) \cap A| = |N(A) \cap \overline{B}| = |N(A) \cap B| = |N(B) \cap \overline{A}| = 2$. Thus, we may assume that $N(B) \cap A = \{x_1, x_4\}, N(A) \cap \overline{B} = \{y_1, w_1\}, N(A) \cap B = \{x_2, x_5\}$, and $N(B) \cap \overline{A} = \{y_1, w_2\}$. Since $\{x_1, x_2, x_3\} \cap N(x_5) = \emptyset$, we see that $d(x_5) = 4$, a contradiction. Hence, we may assume that $\overline{B} \cap \overline{A} = \emptyset$. It follows that $\overline{B} = \overline{B} \cap N(A)$ and $\overline{A} = \overline{A} \cap N(B)$. Furthermore, $|\overline{B} \cap N(A)| = |\overline{B}| \geq 2$ and $|\overline{A} \cap N(B)| = |\overline{A}| \geq 2$. Now, Lemma 1 assures us that $|N(B) \cap A| = |N(A) \cap \overline{B}| = |N(A) \cap B| = |N(B) \cap \overline{A}| = 2$. Hence, we see that $A \cap B$ is a fragment of G. Now, similarly, we see that $d(x_4)=4$, a contradiction. Hence, we see that either $\kappa(G_1)\geq 5$ or $\kappa(G_2)\geq 5$. □

Lemma 13. *Let G be a TCC-5-connected graph such that $|V(G)|\geq 10$. If G has type 3, then G can be contracted to a 5-connected H such that $0<|V(G)|-|V(H)|<3$.*

Proof. Clearly, G does not contain K_4. Suppose G has a fragment of cardinality two, say $A=\{x,y\}$. Since G is 5-regular, we see that $|N(x)\cap N(y)|=3$. Hence, we see that $\Delta(G[N(y)])\geq 3$. This contradicts Lemma 8. Hence, every fragment of G contains either one vertex or at least three vertices. Let x be a vertex of G such that $N(x)=\{x_1,\ldots,x_5\}$. Let $x_1x_2x_3$ be a path of $G[N(x)]$. Furthermore, let $N(x_1)=\{x,x_2,y_1,y_2,y_3\}$, $N(x_2)=\{x,x_1,x_3,w_1,w_2\}$, and $N(x_3)=\{x,x_2,z_1,z_2,z_3\}$. Since G has type 3, we see that $\{y_1,y_2,y_3\}\cap\{x_4,x_5,w_1,w_2\}=\varnothing$ and $\{z_1,z_z,z_3\}\cap\{x_4,x_5,w_1,w_2\}=\varnothing$.

Let $G_1=G/\{xx_1,x_2x_3\}$ and $G_2=G/\{xx_3,x_1x_2\}$. By Lemma 11, we have $\kappa(G_1)\geq 4$ and $\kappa(G_2)\geq 4$. If either G_1 or G_2 is 5-connected, then we are done. So we may assume $\kappa(G_1)=4$ and $\kappa(G_2)=4$.

Clearly, $\delta(G_1)\geq 5$ and $\delta(G_2)\geq 5$. For $i\in\{1,2\}$, let T_i be a smallest separator of G_i and A_i be a T_i-fragment. Since $\delta(G_1)\geq 5$ and $\delta(G_2)\geq 5$, we see that every component of $G-T_i$ has at least two vertices, where $i\in\{1,2\}$. Furthermore, $T_1\cap\{[xx_1],[x_2x_3]\}\neq\varnothing$ and $T_2\cap\{[xx_3],[x_1x_2]\}\neq\varnothing$. Let $T_i'=(T_i\cup\cup_{[ab]\in T_i}\{a,b\})-\cup_{[ab]\in T_i}\{[a,b]\}$, where $i\in\{1,2\}$. It follows that $T_i'\cap\{x_1,x,x_2,x_3\}\neq\varnothing, i\in\{1,2\}$. Clearly, either $|T_i'\cap\{x_1,x,x_2,x_3\}|=2$ or $|T_i'\cap\{x_1,x,x_2,x_3\}|=4$, $i\in\{1,2\}$. □

Claim 16. For a smallest separator T of G, the following holds.

(1) $\{x,x_1,x_2\}-T\neq\varnothing$ and $\{x,x_2,x_3\}-T\neq\varnothing$

(2) If either $\{x_1,x_2,x_3\}\subseteq T$ or $\{x_1,x,x_3\}\subseteq T$, then one component of $G-T$ has exactly one vertex

Proof

(1) By symmetry, we only show that $\{x,x_1,x_2\}-T\neq\varnothing$, and the other one can be handled similarly. Suppose $\{x,x_1,x_2\}-T\neq\varnothing$, which implies that $\{x,x_1,x_2\}\subseteq T$. Let A be a T-fragment. Since $G[N(x)]-\{x_1,x_2,x_3\}\cong K_2$, we see that $x_3\notin T$. Hence, without the loss of generality, let $x_3\in A$. Notice the fact that $x_1x_3\notin E(G)$, and we see that A has at least two vertices. Now, the fact that $G[N(x)]-\{x_1,x_2,x_3\}\cong K_2$ assure us that $N(x)\cap A=\{x_3\}$. Similarly, we have $N(x_2)\cap A=\{x_3\}$. Hence, $N(\{x,x_2\})\cap A=\{x_3\}$. Now Lemma 2 assures us that $A=\{x_3\}$, a contradiction.

(2) Suppose $\{x_1,x_2,x_3\}\subseteq T$. Furthermore, suppose every component of $G-T$ has at least two vertices. Let A be a fragment of T. Since $G[N(x)]-\{x_1,x_2,x_3\}\cong K_2$, we see that $x\notin T$. Without the loss of generality, let $x\in A$. Since both A and $\overline{A}$ have at least two vertices, it follows that A and $\overline{A}$ have at least three vertices.

Notice that $N(x_2)\cap A=\{x\}$. If $|N(x_1)\cap A|=1$, then $N(\{x_2,x_1\})\cap A=\{x\}$. Now, Lemma 2 assures us that $A=\{x\}$, a contradiction. So, we may assume that $|N(x_1)\cap A|\geq 2$. Similarly, we see that $|N(x_3)\cap A|\geq 2$. Since $|N(x_1)\cap A|\geq 2$, we see that $y_2\in N(A)$. Hence, by symmetry, we may assume $y_1\in A$ and $y_3\in\overline{A}$. It follows that $N(x_1)\cap A=\{x,y_1\}$ and $N(x_1)\cap\overline{A}=\{y_3\}$. Thus, Lemma 4 assures us that $y_2\in \mathrm{Ad}(x_1,\overline{A})$. Now, we see that $N(y_2)\cap A=\{y_1\}$, since $G[N(y_2)]-\{y_1,x_1,y_3\}\cong K_2$. Notice that $N(\{x_1,x_2,y_2\})\cap A=\{x,y_1\}$, and Lemma 2 assures us that $A=\{x,y_1\}$, a contradiction. Hence, we see that one component of $G-T$ has exactly one vertex. Similarly, we see that the fact $\{x_1,x,x_3\}\subseteq T$ assures us that one component of $G-T$ has exactly one vertex. □

Claim 17. $\{x_1,x,x_2,x_3\}-T_i'\neq\varnothing$, $i\in\{1,2\}$.

Proof. We only show that $\{x_1,x,x_2,x_3\}-T_1'\neq\varnothing$, and the other one can be handled similarly. Suppose $\{x_1,x,x_2,x_3\}\subseteq T_1'$. It follows that $|T_1'|=6$. Let $A\prime$ be a component of $G-T_1'$, $\overline{A\prime}=G-T_1'-A\prime$. As $x_4x_5\in E(G)$, it follows that either $N(x)\cap A\prime=\varnothing$ or $N(x)\cap\overline{A'}=\varnothing$. Similarly, we see that either $N(x_2)\cap A\prime=\varnothing$ or $N(x_2)\cap\overline{A'}=\varnothing$. Without the loss of generality, let $N(x)\cap A\prime=\varnothing$. It follows that $N(x_2)\cap\overline{A'}=\varnothing$, $N(x_2)\cap A\prime\neq\varnothing$, and $N(x)\cap\overline{A'}\neq\varnothing$.

It follows that $T_1'-\{x\}$ is a smallest separator of G. Let $T=T_1'-\{x\}$, and $A=A\prime$ is a $T-$ fragment such that $\overline{A}=\overline{A'}\cup\{x\}$. If $|A|=1$, then either $A=\{w_1\}$ or $A=\{w_2\}$. It follows that $N(x_1)\cap\{w_1,w_2\}\neq\varnothing$, a contradiction. Hence, we have $|A|\geq 3$. Notice that $\overline{A}=\overline{A'}\cup\{x\}$, and we see that $|\overline{A}|\neq 1$. It follows that $|\overline{A}|\geq 3$.

Hence, we see that T is a smallest separator of G such that $T\cap\{x_1,x,x_2,x_3\}=\{x_1,x_2,x_3\}$, but both A and $\overline{A}$ have cardinality at least two, which contradicts Claim 16.

By Claim 17, without the loss of generality, we may assume that $T_1'\cap\{x_1,x,x_2,x_3\}=\{x_1,x\}$. It follows that $|T_1'|=5$. Let B_1 be a T_1'-fragment. Without the loss of generality, let $\{x_2,x_3\}\subseteq B_1$. On the other hand, by Claim 17, either $T_2'\cap\{x_1,x,x_2,x_3\}=\{x_1,x_2\}$ or $T_2'\cap\{x_1,x,x_2,x_3\}=\{x,x_3\}$. Furthermore, since every component of $G-T_i$ has at least two vertices, we see that every component of $G-T_i'$ has at least two vertices, where $i\in\{1,2\}$. This implies that every component of $G-T_i'$ has at least three vertices for each $i=1,2$. We will complete the proof of the lemma according the following two cases. □

Case 1. $T_2'\cap\{x_1,x,x_2,x_3\}=\{x_3,x\}$.

It follows that $|T_2'|=5$. Let B_2 be a T_2'-fragment. Without the loss of generality, suppose $\{x_1,x_2\}\subseteq B_2$.

Now, we see that $x_2\in B_1\cap B_2$, $x_1\in T_1'\cap B_2$, $x_3\in T_2'\cap B_1$, and $x\in T_1'\cap T_2'$.

Claim 18. $\overline{B}_1\cap\overline{B}_2\neq\varnothing$.

Proof. Otherwise, assume that $\overline{B}_1\cap\overline{B}_2=\varnothing$. Notice that $N(x)\cap\overline{B}_1\neq\varnothing$ and $N(x)\cap\overline{B}_2\neq\varnothing$, and we see that $N(x)\cap\overline{B}_1\cap T_2'\neq\varnothing$ and $N(x)\cap\overline{B}_2\cap T_1'\neq\varnothing$ since $G[N(x)]-\{x_1,x_2,x_3\}\cong K_2$. Without the loss of generality, let $x_4 \in \overline{B}_2 \cap T_1'$. Furthermore, we see that $N(x) \cap \overline{B}_1 \cap B_2 = \varnothing$ and $N(x) \cap \overline{B}_2 \cap B_1 = \varnothing$. If $B_1 \cap \overline{B}_2 \neq \varnothing$, then Lemma 1 assures us that $\overline{B}_1 \cap B_2 = \varnothing$. It follows that $|\overline{B}_1 \cap T_2'| \geq |\overline{B}_1| \geq 3$. Now, Lemma 1 assures us that $|\overline{B}_2 \cap T_1'| \geq |\overline{B}_1 \cap T_2'| \geq 3$ and $|B_2 \cap T_1'| \geq |\overline{B}_1 \cap T_2'| \geq 3$. This implies that $|T_1'| \geq 6$, a contradiction.

Thus, we may assume that $B_1 \cap \overline{B}_2 = \varnothing$ and, similarly, $\overline{B}_1 \cap B_2 = \varnothing$. Similar to the argument of the last paragraph, we have $|\overline{B}_1| \geq 3$ and $|\overline{B}_2| \geq 3$. It follows that $|B_2 \cap T_1'| \geq |\overline{B}_1 \cap T_2'| \geq 3$ and $|\overline{B}_2 \cap T_1'| \geq |\overline{B}_1 \cap T_2'| \geq 3$. Hence, $|T_1'| \geq 6$, a contradiction. Hence, Claim 18 holds.

By Claim 18 and Lemma 1, we see that $B_1 \cap B_2$ is a fragment and $N(B_1 \cap B_2) \cap \{x_1, x, x_2, x_3\} = \{x_1, x, x_3\}$. Now, Claim 16 implies that $B_1 \cap B_2 = \{x_2\}$. Therefore, $\{w_1, w_2\} \subseteq B_1 \cup T_1'$. If $|B_1 \cup T_1'| \leq 8$, it follows that $N(x_3) \subseteq B_1 \cup T_1' - \{x_4, x_1, y_1, y_2\}$, which implies that $d(x_3) \leq 4$, a contradiction. So, we may assume that $|B_1 \cup T_1'| \geq 9$. Similarly, we may assume that $|B_2 \cup T_2'| \geq 9$. □

Claim 19. $B_1 \cap \overline{B}_2 = \varnothing$ and $B_2 \cap \overline{B}_1 = \varnothing$.

Proof. Clearly, $\overline{B}_1 \cap \overline{B}_2$ is a fragment. It follows that $N(x) \cap \overline{B}_1 \cap \overline{B}_2 \neq \varnothing$. Hence, $N(x) \cap \overline{B}_1 \cap B_2 = \varnothing$ and $N(x) \cap \overline{B}_2 \cap B_1 = \varnothing$, since $G[N(x)] - \{x_1, x_2, x_3\} \cong K_2$.

Now, if $B_1 \cap \overline{B}_2 \neq \varnothing$, then the fact $N(x) \cap \overline{B}_1 \cap B_2 = \varnothing$ assures us that $B_2 \cap \overline{B}_1 = \varnothing$. Notice that $|B_2 \cup T_2'| \geq 9$ and $B_1 \cap B_2 = \{x_2\}$, and we have $|B_2 \cap T_1'| \geq 3$.

Now, Lemma 1 assures us that $|B_1 \cap T_2'| \geq |B_2 \cap T_1'| \geq 3$ and $|\overline{B}_1 \cap T_2'| \geq |B_2 \cap T_1'| \geq 3$. Hence, $|T_2'| \geq 6$, a contradiction. This contradiction implies that $B_1 \cap \overline{B}_2 = \varnothing$. Similarly, $B_2 \cap \overline{B}_1 = \varnothing$.

Now, we are ready to complete the proof of Case 1. Notice that $|B_2 \cup T_2'| \geq 9$, $B_1 \cap B_2 = \{x_2\}$, and $B_2 \cap \overline{B}_1 = \varnothing$, and we see that $|B_2 \cap T_1'| \geq 3$. Similarly, $|B_1 \cap T_2'| \geq 3$. Now, Lemma 1 assures us that $|\overline{B}_1 \cap T_2'| \geq |B_2 \cap T_1'| \geq 3$. It follows that $|T_2'| \geq 6$, a contradiction. □

Case 2. $T_2' \cap \{x_1, x, x_2, x_3\} = \{x_1, x_2\}$.

It follows that $|T_2'| = 5$. Let B_2 be a T_2'-fragment. Without the loss of generality, suppose $\{x, x_3\} \subseteq B_2$. Now, we see that $x_3 \in B_1 \cap B_2$, $x \in T_1' \cap B_2$, $x_2 \in T_2' \cap B_1$, and $x_1 \in T_1' \cap T_2'$.

Claim 20. $\overline{B}_1 \cap \overline{B}_2 \neq \varnothing$.

Proof. Suppose $\overline{B}_1 \cap \overline{B}_2 = \varnothing$. If $B_1 \cap \overline{B}_2 = \varnothing$, then $|\overline{B}_2 \cap T_1'| = |\overline{B}_2| \geq 3$. Now, Lemma 1 assures us that $|B_1 \cap T_2'| \geq |\overline{B}_2 \cap T_1'| \geq 3$. This implies that $|\overline{B}_1 \cap T_2'| \leq 2$, which implies that $|\overline{B}_1 \cap T_2'| \geq |\overline{B}_2 \cap T_1'|$. Now, Lemma 1 assures us that $\overline{B}_1 \cap B_2 = \varnothing$, which implies that $|\overline{B}_1| \leq 2$, a contradiction.

Thus, we may assume that $B_1 \cap \overline{B}_2 \neq \varnothing$. Similarly, $\overline{B}_1 \cap B_2 \neq \varnothing$. Thus, both $B_1 \cap \overline{B}_2$ and $\overline{B}_1 \cap B_2$ are fragments of G. Without loss of generality, we may assume $y_1 \in B_1 \cap \overline{B}_2$. Therefore, $y_3 \in \overline{B}_1 \cap B_2$ and $y_2 \in T_1' \cap T_2'$. Notice that $G[N(y_2)] - \{x_1, y_1, y_3\} \cong K_2$, and then either $|N(y_2) \cap B_1 \cap \overline{B}_2| = 1$ or $|N(y_2) \cap \overline{B}_1 \cap B_2| = 1$. Without the loss of generality, let $|N(y_2) \cap B_1 \cap \overline{B}_2| = 1$. Now, we find that $N(\{x_1, y_2\}) \cap B_1 \cap \overline{B}_2 = \{y_1\}$. Now, Lemma 2 assures us that $B_1 \cap \overline{B}_2 = \{y_1\}$. Thus, $y_1 x_2 \in E(G)$. This contradicts the fact that G has type 3.

Now, we are ready to complete the proof of Case 2. By Claim 20, we have that $B_1 \cap B_2$ is a fragment, and $N(B_1 \cap B_2) \cap \{x_1, x, x_2, x_3\} = \{x, x_1, x_2\}$. This contradicts Claim 16. □

Lemma 14. *Let G be a TCC-5-connected graph such that $|V(G)| \geq 10$. If G has type 4, then G can be reduced to a 5-connected H such that $0 < |V(G)| - |V(H)| < 3$.*

Proof. Let x be a vertex of G such that $N(x) = \{x_1, \ldots, x_5\}$. Let $x_1 x_2 x_3 x_4 x_5$ be a path of G. Now, Lemma 10 assures us that G does not contain K_4. By Lemma 9, every fragment of G has cardinality one or at least four. □

Claim 21. Either $N(x_2) \cap N(x_3) = \{x\}$ or $N(x_3) \cap N(x_4) = \{x\}$.

Proof. Suppose $|N(x_2) \cap N(x_3)| \geq 2$ and $|N(x_3) \cap N(x_4)| \geq 2$. Let $N(x_3) = \{z_1, z_2, x_4, x, x_2\}$ and $N(x_2) = \{w_1, x_1, x, x_2, z_1\}$, where $z_2 x_4 \in E(G)$. If $w_1 x_1 \notin E(G)$, then $G[N(x_1)]$ has at least two components, a contradiction. Thus, $w_1 x_1 \in E(G)$. Let $N(x_1) = \{t_1, t_2, w_1, x_2, x\}$. We may assume $t_1 t_2 w_1 x_2 x$ is the path of $G[N(x_1)]$. Now, we see that $N(x_2) \cap N(w_1) = \{x_1\}$. Let $g \in \mathrm{Aut}(G)$ such that $g(x) = x_1$. It follows that $g|_{N(x)}$ is a map from $N(x)$ to $N(x_1)$. It follows that either $g(x_1) = x$ or $g(x_1) = t_1$.

If $g(x_1) = x$, then $g(x_2) = x_2, g(x_3) = w_1$, $g(x_4) = t_2$, and $g(x_5) = t_1$. Now, since $|N(x_2) \cap N(x_3)| \geq 2$, we see that $|N(x_2) \cap N(w_1)| \geq 2$, a contradiction. So, we may assume that $g(x_1) = t_1$. It follows that $g(x_2) = t_2$, $g(x_3) = w_1, g(x_4) = x_2, g(x_5) = x_1$. Now, since $|N(x_3) \cap N(x_4)| \geq 2$, we see that $|N(x_2) \cap N(w_1)| \geq 2$, a contradiction.

Now, by symmetry, assume that $N(x_2) \cap N(x_3) = \{x\}$. Let $N(x_2) = \{y_1, y_2, x_1, x, x_3\}$ and $N(x_3) = \{z_1, z_2, x_4, x, x_2\}$. Furthermore, let $y_1 y_2 x_1 x x_3$ and $z_1 z_2 x_4 x x_2$ be the paths of $G[N(x_2)]$ and $G[N(x_3)]$, respectively. Hence, we may let $N(x_1) = \{w_1, w_2, y_2, x_2, x\}$, and $w_1 w_2 y_2 x_2 x$ is the path of $G[N(x_1)]$. Furthermore, we have $|N(x_2) \cap N(x_3)| = 1$ and $|N(x_3) \cap N(x_4)| = 2$.

Let $G_0 = G/\{x x_1, x_2 x_3\}$. Lemma 11 assures us that $\kappa(G_0) \geq 4$. Suppose $\kappa(G_0) = 4$. Let T' be a smallest separator of G_0. Clearly, we observe that $\{[x x_1], [x_2 x_3]\} \cap T' \neq \varnothing$. □

Claim 22. $\{[x x_1], [x_2 x_3]\} \subseteq T'$.

Proof. Suppose $[x x_1] \notin T'$. Let $T_0 = T' \cup \{x_2, x_3\} - \{[x_2 x_3]\}$. Then, T_0 is a smallest separator of G. Let A be a fragment of T_0 which contains $\{x, x_1\}$. It follows that $|A| \geq 2$. Now, Lemma 9 shows that $|\overline{A}| = 1$. Clearly, $\overline{A} \subseteq \{N(x_2) \cap N(x_3)\}$. Now, since $x \in A$, we see that $|N(x_2) \cap N(x_3)| = 2$, which is a contradiction. Hence, we may assume that $[x x_1] \in T'$. If $[x_2 x_3] \notin T'$, let $T_0 = T' \cup \{x, x_1\} - \{[x x_1]\}$. Then, T_0 is smallest separator G. Let B be a T_0-fragment which contains $\{x_2, x_3\}$. It follows that $|B| \geq 2$. Now, Lemma 9 shows that $|\overline{B}| = 1$. Clearly,

$\overline{B} \subseteq \{x_4, x_5\}$. Hence, x_1 is adjacent to either x_4 or x_5, a contradiction. Hence, $[x_2x_3] \in T'$.

Let $T_0 = T' \cup \{x, x_1, x_2, x_3\} - \{[xx_1], [x_2x_3]\}$. Let A be a component of $G - T_0$, $A' = G - T_0 - A$. As $x_4x_5 \in E(G)$, $N(x) \cap A = \varnothing$ or $N(x) \cap A' = \varnothing$. Similarly, $N(x_2) \cap A = \varnothing$ or $N(x_2) \cap A' = \varnothing$. Without the loss of generality, let $N(x) \cap A = \varnothing$. Then, $N(x_2) \cap A\prime = \varnothing$ and $N(x_2) \cap A \neq \varnothing$, $N(x) \cap A\prime \neq \varnothing$.

Hence, $T = T_0 - \{x\}$ is a smallest separator of G. Let A be a T–fragment. Clearly, $\overline{A} = A' \cup \{x\}$. This implies that $|\overline{A}| \geq 2$. Now, Lemma 9 shows that $|A| = 1$. So, $A \subseteq N(x_2) \cap N(x_3)$. Recall that $x \in \overline{A}$, and we find that $|N(x_2) \cap N(x_3)| \geq 2$. This contradicts the fact that $N(x_2) \cap N(x_3) = \{x\}$. □

Data Availability

No data were used to support this study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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